DUALISABILITY VERSUS RESIDUAL CHARACTER: A THEOREM AND A COUNTEREXAMPLE

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Abstract. We show that a finite algebra must be inherently non-dualisable if the variety that it generates is both residually large and congruence meet-semidistributive. We also give the first example of a finite dualisable algebra that generates a variety that is residually large.

There is no obvious connection between the dualisability of a finite algebra and the residual character of the variety it generates. Certainly, there are many non-dualisable algebras that generate a residually small variety: every finite algebra that does not have a near-unanimity term but generates a congruence-distributive variety [4, 12].

Nevertheless, there are many large classes of algebras for which it turns out that every finite member that generates a residually large variety is non-dualisable. As examples, there are the classes of groups [19, 8], commutative rings with identity [3, 8], bands [11, 9, 15], flat graph algebras [14, 13], p-semilattices [7] and closure semilattices [6, 13]. The weight of these examples led the first two authors to the following rash conjecture: 'Every finite algebra that generates a residually large variety is non-dualisable' [18].

This paper partially vindicates that conjecture. We show that a finite algebra must be inherently non-dualisable if the variety that it generates is both residually large and congruence meet-semidistributive (Corollary 3.3). In particular, the conjecture is true for every finite algebra with a semilattice reduct (Corollary 3.2).

This paper also provides the first counterexample to the conjecture. In Section 4, we present a finite algebra that is dualisable and yet generates a variety that is residually large. Our counterexample is a term-reduct of a four-element ring, and the variety it generates is congruence permutable.

1. A semilattice-based example

In this section, we study one particular three-element algebra, and show the relationship between a proof that it generates a residually large variety and a proof that it is inherently non-dualisable. This example provides some insight into the impetus for the main theorem, which is proved in Section 3.

Roughly speaking, a finite algebra $\mathbf{A}$ is *inherently non-dualisable* if there is no natural representation for the quasivariety $\mathbb{ISP}(\mathbf{B})$, whenever $\mathbf{B}$ is a finite algebra such that $\mathbf{A} \in \mathbb{ISP}(\mathbf{B})$. For a precise definition of inherent non-dualisability (indeed,

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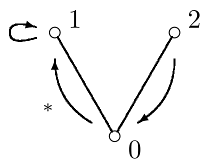


Figure 1. The flat unar $\mathbf{V} = \langle \{0,1,2\}; \wedge, {}^* \rangle$

for a complete introduction to the theory of natural dualities), we refer the reader to the text by Clark and Davey [1]. For the proof of our main theorem, all the duality theory that we shall really need is contained in the following general theorem of Davey, Idziak, Lampe and McNulty [5].

Inherent Non-dualisability Theorem 1.1. [5, Theorem 3] *Let* $\mathbf{A}$ *be a finite algebra, let* κ *be an infinite cardinal and let* $\varphi : \omega \to \omega$. *Assume there is a subalgebra* $\mathbf{C}$ *of* $\mathbf{A}^Z$, *for some set* Z, *and a subset* C_0 *of* C *of cardinality at least* κ *such that*

(i) *for each* $k \in \omega$ *and each congruence* γ *on* $\mathbf{C}$ *of index at most* k, *the equivalence relation* $\gamma{\restriction}_{C_0}$ *has a unique block of size greater than* $\varphi(k)$,
(ii) *the algebra* $\mathbf{C}$ *does not contain the element* g *of* A^Z *given by* $g(z) := c_z(z)$, *where* c_z *is any element of the unique infinite block of* $\ker(\pi_z){\restriction}_{C_0}$.

Then $\mathbf{A}$ *is inherently non-*κ*-dualisable.*

When applying this theorem, we use the following notation. Let A and Z be non-empty sets. For all $n > 0$, all distinct $z_1, \dots, z_n \in Z$ and all $a, b_1, \dots, b_n \in A$, define $a^{b_1 \dots b_n}_{z_1 \dots z_n} \in A^Z$ by

$$a^{b_1 \dots b_n}_{z_1 \dots z_n}(z) = \begin{cases} b_i & \text{if } z = z_i, \text{ for some } i \in \{1, \dots, n\}, \\ a & \text{otherwise,} \end{cases}$$

for all $z \in Z$. For each $a \in A$, let $\underline{a}$ denote the constant map in A^Z with value a.

Recall that a variety is **residually large** if there is no bound on the sizes of its subdirectly irreducible members. Our example is a **flat unar**, that is, a flat semilattice enriched with a single unary operation.

Example 1.2. *Define the flat unar* $\mathbf{V}$ *as in Figure 1. Then* $\mathrm{Var}(\mathbf{V})$ *is residually large.*

Proof. Let Z be a non-empty set. We shall construct a subdirectly irreducible algebra in $\mathrm{Var}(\mathbf{V})$ of size at least $|Z|$.

We can define θ_0 to be the congruence on $\mathbf{V}^Z$ whose only non-trivial block is $\{0,1\}^Z \backslash \{\underline{1}\}$. Now let θ be a congruence on $\mathbf{V}^Z$ that contains θ_0 and is maximal with respect to separating $\underline{1}$ from $\{0,1\}^Z \backslash \{\underline{1}\}$. Then the congruence θ is completely meet-irreducible, and therefore $\mathbf{V}^Z/\theta$ is subdirectly irreducible.

For any congruence γ on $\mathbf{V}^Z$ and for all $s, t \in Z$ with $s \neq t$, we have

$$\begin{aligned} 1^2_s \equiv_\gamma 1^2_t &\implies 1^2_s = 1^2_s \wedge 1^2_s \equiv_\gamma 1^2_s \wedge 1^2_t = 1^{00}_{st} \\ &\implies (1^2_s)^* \equiv_\gamma (1^{00}_{st})^* \\ &\implies 1^0_s \equiv_\gamma \underline{1}. \end{aligned} \tag{RL$_{\mathbf{V}}$}$$

Since θ separates $\underline{1}$ from $\{0,1\}^Z \backslash \{\underline{1}\}$, it follows that $1^2_s/\theta \neq 1^2_t/\theta$, for all distinct $s, t \in Z$. Thus $|\mathbf{V}^Z/\theta| \geqslant |Z|$. □

Example 1.3. *Define the flat unar* $\mathbf{V}$ *as in Figure 1. Then* $\mathbf{V}$ *is inherently non-*κ*-dualisable, for every cardinal* κ.

Proof. We use the Inherent Non-dualisability Theorem, 1.1. Let κ be an infinite cardinal and define the map $\varphi : \omega \to \omega$ by $\varphi(k) := k$. Now let Z be a set of cardinality κ and fix an element $0 \in Z$. We define two subsets of V^Z:

$$C_0 := \{\, 1^{00}_{0z} \mid z \in Z\backslash\{0\} \,\} \quad \text{and} \quad C_1 := \{\, 1^2_z \mid z \in Z\backslash\{0\} \,\}.$$

So $|C_0| = \kappa$. Now define $\mathbf{C}$ to be the subalgebra of $\mathbf{V}^Z$ generated by $C_0 \cup C_1$. It remains to prove that conditions (i) and (ii) of the Inherent Non-dualisability Theorem are satisfied.

Condition (i) *holds.*

Let $k \in \omega$ and let γ be a congruence on $\mathbf{C}$ of index at most k. Assume that S and U are disjoint subsets of $Z\backslash\{0\}$, each of size greater than $\varphi(k)$, such that

- the set $\{\, 1^{00}_{0s} \mid s \in S \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$, and
- the set $\{\, 1^{00}_{0u} \mid u \in U \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$.

We shall prove that $\{\, 1^{00}_{0z} \mid z \in S \cup U \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$. It will then follow that $\gamma{\restriction}_{C_0}$ has a unique block of size greater than $\varphi(k)$, proving (i).

We are assuming that γ has index at most k and that $|S|, |U| > \varphi(k) = k$. Thus there are distinct $s, t \in S$ and distinct $u, v \in U$ such that

$$1^2_s \equiv_\gamma 1^2_t \quad \text{and} \quad 1^2_u \equiv_\gamma 1^2_v$$

in $\mathbf{C}$. Note that the calculation $(\mathrm{RL})_{\mathbf{V}}$ in the previous proof applies to any congruence γ on any subalgebra of $\mathbf{V}^Z$ that contains 1^2_s and 1^2_t. Hence we can use $(\mathrm{RL})_{\mathbf{V}}$ to conclude that $1^0_s \equiv_\gamma \underline{1}$ and, by symmetry, that $1^0_t \equiv_\gamma \underline{1}$. Thus $\underline{1} \equiv_\gamma 1^0_s \wedge 1^0_t = 1^{00}_{st}$.

Since $u, v \in U$, we have $1^{00}_{0u} \equiv_\gamma 1^{00}_{0v}$, by assumption. Thus

$$1^{00}_{0u} \equiv_\gamma 1^{00}_{0u} \wedge 1^{00}_{0v} = 1^{000}_{0uv} = \underline{1} \wedge 1^{000}_{0uv} \equiv_\gamma 1^{00}_{st} \wedge 1^{000}_{0uv} = 1^{00000}_{0stuv}.$$

Using the symmetry in our assumptions on S and U, we have $1^{00}_{0u} \equiv_\gamma 1^{00000}_{0stuv} \equiv_\gamma 1^{00}_{0s}$. Hence $\{\, 1^{00}_{0z} \mid z \in S \cup U \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$, whence (i) holds.

Condition (ii) *holds.*

The element of V^Z defined by condition (ii) is $g := 1^0_0$. Define

$$D := \{\, f \in V^Z \mid f(0) = 1 \text{ or } \big(\exists z \in Z\backslash\{0\}\big)\ f(z) = f(0) = 0 \,\}.$$

It is easy to check that D is a subuniverse of $\mathbf{V}^Z$, with $C_0 \cup C_1 \subseteq D$ and $g \notin D$. Thus $g \notin \mathrm{sg}_{\mathbf{V}^Z}(C_0 \cup C_1) = C$, proving (ii). ☐

Remark 1.4. Our proof of the inherent non-dualisability of $\mathbf{V}$ allowed us to reuse the congruence calculation $(\mathrm{RL})_{\mathbf{V}}$. For this to be possible, it was necessary that the calculation $(\mathrm{RL})_{\mathbf{V}}$ applied to any congruence γ on an appropriate subalgebra of $\mathbf{V}^Z$, not just to the particular congruence θ. We also needed to ensure that our subalgebra $\mathbf{C}$ of $\mathbf{V}^Z$ contained enough elements from $\{\, 1^2_z \mid z \in Z \,\}$.

As a first choice for C_0, we could have tried to use the elements occurring at the end of $(\mathrm{RL})_{\mathbf{V}}$, namely $\{\, 1^0_z \mid z \in Z \,\}$. The proof that (i) holds is easier with this choice. But the element g from (ii) would be $\underline{1}$, which would belong to $\mathbf{C}$ and cause (ii) to fail. The elements of C_0 were obtained by modifying the elements in $\{\, 1^0_z \mid z \in Z \,\}$; these elements are effectively 'tagged' with an extra 0 at a new coordinate 0. Our proof that (i) still holds for these 'tagged' elements relies heavily on the semilattice operation of **V**. Our proof of (ii) is very specific to **V**.

2. A general RL-configuration

For us to be able to take a congruence calculation from a residual-largeness proof and reuse it in an inherent-non-dualisability proof, we need the calculation to be of a special type. In this section, we present a configuration of McKenzie [16] that can be used to witness every instance of residual largeness for a large class of finite algebras. This configuration will give us a reusable congruence calculation.

First, we give a few definitions. Consider an algebra $\mathbf{A}$ and a subset S of A. There is a unique congruence θ_S on $\mathbf{A}$ that is maximal with respect to $s \not\equiv_{\theta_S} a$, for all $s \in S$ and $a \in A\backslash S$. We call θ_S the **syntactic congruence on A determined by** S. It is easy to check that

$$\theta_S = \big\{\, (a,b) \in A^2 \;\big|\; \big(\forall h \in \mathrm{Pol}_1(\mathbf{A})\big)\ h(a) \in S \iff h(b) \in S \,\big\},$$

where $\mathrm{Pol}_1(\mathbf{A})$ denotes the set of all unary polynomials of $\mathbf{A}$. (More generally, there is a largest congruence inside every equivalence relation on an algebra. These congruences, which have long been useful in general algebra, have only recently inherited the name 'syntactic congruence' [2] from semigroup theory, where they are used to study languages.)

Let $n > 0$. We will denote the ith coordinate of an n-tuple $\vec{a} \in A^n$ by a_i, so that $\vec{a} = (a_1, \dots, a_n)$. For an equivalence relation θ on A and tuples $\vec{a}, \vec{b} \in A^n$, we write $\vec{a} \equiv_\theta \vec{b}$ to mean that $a_i \equiv_\theta b_i$, for all $i \in \{1, \dots, n\}$.

Now let θ be any congruence on $\mathbf{A}$. The congruence θ is **non-abelian** if there exists an $(m+n)$-ary term function τ of $\mathbf{A}$, for some $m, n > 0$, and tuples $\vec{a}, \vec{b} \in A^m$ and $\vec{c}, \vec{d} \in A^n$ such that

$$\vec{a} \equiv_\theta \vec{b}, \quad \vec{c} \equiv_\theta \vec{d}, \quad \tau(\vec{a}, \vec{c}) = \tau(\vec{a}, \vec{d}) \quad \text{but} \quad \tau(\vec{b}, \vec{c}) \neq \tau(\vec{b}, \vec{d}).$$

For example, if θ is non-trivial and $\mathbf{A}$ has a meet-semilattice operation $\wedge$, then there is $c = a < b = d$ in $\mathbf{A}$ such that $a \equiv_\theta b$, and we have $a \wedge c = a \wedge d$ but $b \wedge c \neq b \wedge d$. Thus, on an algebra with a semilattice reduct, every non-trivial congruence is non-abelian. The **monolith** of a subdirectly irreducible algebra is its least non-trivial congruence.

The following is a slight refinement of a result due to McKenzie [16].

Theorem 2.1. *Let* $\mathbf{A}$ *be a finite algebra. There is no bound on the cardinalities of the subdirectly irreducible algebras in* $\mathrm{Var}(\mathbf{A})$ *with a non-abelian monolith if and only if there exist*

1. *a finite algebra* $\mathbf{B} \in \mathbb{ISP}(\mathbf{A})$,
2. *an idempotent unary polynomial* e *of* $\mathbf{B}$ *and distinct elements* $0, 1 \in e(B)$,
3. *a binary polynomial* $\wedge$ *of* $\mathbf{B}$,
4. *a congruence* α *on* $\mathbf{B}$, *and*
5. *an* $(n+1)$*-ary polynomial* p *of* $\mathbf{B}$, *for some* $n > 0$, *and elements* $a, b \in B$ *and tuples* $\vec{c}, \vec{d} \in B^n$ *with* $a \equiv_\alpha b$ *and* $\vec{c} \equiv_\alpha \vec{d}$

such that

6. $x = x \wedge x = x \wedge 1 = 1 \wedge x$, *for all* $x \in e(B)$,
7. $x \equiv_\theta x \wedge 0$, *for every* $x \in e(B)\backslash\{1\}$, *where* θ *is the syntactic congruence on* $\mathbf{B}$ *determined by* $e^{-1}(1)$,

8. $\alpha \cap \mathrm{Cg}_{\mathbf{B}}(0,1) \subseteq \theta$, *and*
9. $e \circ p(a,\vec{c}) = e \circ p(b,\vec{d}) = 1$ *but* $e \circ p(b,\vec{c}) \neq 1$.

Proof. Nearly all of the work has already been done for us by McKenzie: we use the equivalence of conditions $\neg(1)$ and $\neg(5)$ in his Theorem 3.1 [16]. Translated into our notation, he proved that there is no bound on the cardinalities of the subdirectly irreducible members of $\mathrm{Var}(\mathbf{A})$ with a non-abelian monolith if and only if there exist

1′. a finite algebra $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$,
2. an idempotent unary polynomial e of $\mathbf{B}$ and distinct elements $0, 1 \in e(B)$,
3. a binary polynomial $\wedge$ of $\mathbf{B}$,
4. a congruence α on $\mathbf{B}$, and
5′. an $(m+n)$-ary polynomial p of $\mathbf{B}$, for some $m, n > 0$, and tuples $\vec{a}, \vec{b} \in B^m$ and $\vec{c}, \vec{d} \in B^n$ with $\vec{a} \equiv_\alpha \vec{b}$ and $\vec{c} \equiv_\alpha \vec{d}$

such that

6′. $e(B)$ is closed under $\wedge$, and $x = x \wedge x = x \wedge 1 = 1 \wedge x$, for all $x \in e(B)$,
7. $x \equiv_\theta x \wedge 0$, for every $x \in e(B)\backslash\{1\}$, where θ is the syntactic congruence on $\mathbf{B}$ determined by $e^{-1}(1)$,
8. $\alpha \cap \mathrm{Cg}_{\mathbf{B}}(0,1) \subseteq \theta$, and
9′. $e \circ p(\vec{a},\vec{c}) = e \circ p(\vec{b},\vec{d}) = 1$ but $e \circ p(\vec{b},\vec{c}) \neq 1$.

It is easy to check that condition $6'$ can be replaced by the weaker condition 6: if there is a binary polynomial $x \wedge y$ such that conditions 6 and 7 hold, then $6'$ and 7 hold for the binary polynomial $e(x \wedge y)$. It remains to argue that conditions $1'$, $5'$ and $9'$ can be replaced by the stronger conditions 1, 5 and 9.

The fact that $1'$ can be replaced by 1 can be deduced from McKenzie's proof of (5) $\Rightarrow$ (1) [16, 3.1]. This proof proceeds via (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). In the proof of (5) $\Rightarrow$ (4), the only use of condition (5) is at the bottom of page 215, where it is applied with $\mathbf{B}$ a finite algebra in $\mathbb{ISP}(\mathbf{A})$. Hence McKenzie has actually proved that, if a failure of (5) exists, then there is one in which $\mathbf{B} \in \mathbb{ISP}(\mathbf{A})$.

Finally, to prove that $5'$ and $9'$ can be replaced by 5 and 9, we apply the following claim with $S := e^{-1}(1)$.

Let $\mathbf{B}$ *be an algebra, let* α *be a congruence on* $\mathbf{B}$ *and let* $S \subseteq B$. *Assume that there exist an* $(m+n)$*-ary polynomial* p *of* $\mathbf{B}$, *for some* $m, n > 0$, *and tuples* $\vec{a}, \vec{b} \in B^m$ *and* $\vec{c}, \vec{d} \in B^n$ *such that*

$$\vec{a} \equiv_\alpha \vec{b}, \quad \vec{c} \equiv_\alpha \vec{d}, \quad p(\vec{a},\vec{c}) \in S, \quad p(\vec{b},\vec{d}) \in S \quad \textit{and} \quad p(\vec{b},\vec{c}) \notin S.$$

Then there exist such a polynomial and tuples with $m = 1$.

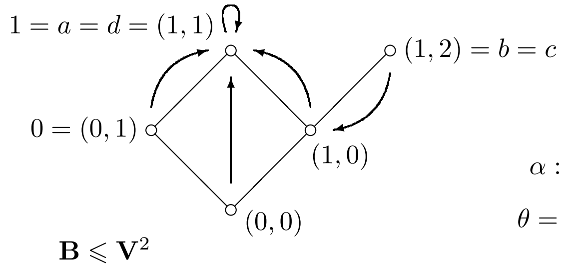
To prove this claim, we start by defining, for each $i \in \{0, 1, \ldots, m\}$, the assertion:

$$p(b_1, \ldots, b_i, a_{i+1}, \ldots, a_m, \vec{c}) \notin S. \tag{$(*)_i$}$$

We are assuming that $(*)_0$ is false and that $(*)_m$ is true.

Let ℓ be the smallest integer such that $(*)_\ell$ is true. Then $0 < \ell \leqslant m$. Define $n' := n + m - \ell$ and define the $(n'+1)$-ary polynomial p' of $\mathbf{B}$ by

$$p'(x, \vec{y}) := p(b_1, \ldots, b_{\ell-1}, x, \vec{y}).$$



$$e(x) := x \wedge (1,1)$$

$$p(x,y) := (x \wedge y)^*$$

$$\alpha := \{\, (0,0)(0,1) \mid (1,1)(1,0)(1,2) \,\}$$

$$\theta = \{\, (0,0)(0,1)(1,0) \mid (1,1) \mid (1,2) \,\}$$

FIGURE 2. The RL-configuration for the flat unar $\mathbf{V}$

Now define

$$a' := a_\ell \in B, \qquad \vec{c}' := (a_{\ell+1}, \dots, a_m, \vec{c}) \in B^{n'},$$

$$b' := b_\ell \in B, \qquad \vec{d}' := (b_{\ell+1}, \dots, b_m, \vec{d}) \in B^{n'}.$$

Then $a' \equiv_\alpha b'$ and $\vec{c}' \equiv_\alpha \vec{d}'$. We have

$$p'(a', \vec{c}') = p(b_1, \dots, b_{\ell-1}, a_\ell, a_{\ell+1}, \dots, a_m, \vec{c}) \in S, \qquad \text{by } \neg(*)_{\ell-1},$$

$$p'(b', \vec{d}') = p(b_1, \dots, b_{\ell-1}, b_\ell, b_{\ell+1}, \dots, b_m, \vec{d}) = p(\vec{b}, \vec{d}) \in S, \qquad \text{by assumption, and}$$

$$p'(b', \vec{c}') = p(b_1, \dots, b_{\ell-1}, b_\ell, a_{\ell+1}, \dots, a_m, \vec{c}) \notin S, \qquad \text{by } (*)_\ell.$$

So the claim holds, which finishes the proof of the theorem. □

Figure 2 illustrates the RL-configuration of the previous theorem (namely, conditions 1–9) for the flat unar $\mathbf{V}$ of Example 1.2. In the next section, we show that any finite algebra $\mathbf{A}$ that has the RL-configuration must be inherently non-dualisable. Our proof reuses a congruence calculation from a residual-largeness proof for $\mathrm{Var}(\mathbf{A})$. So we will first present this residual-largeness proof, which is drawn from McKenzie's paper [16, 2.2 and 2.3].

Theorem 2.2. *Let* $\mathbf{A}$ *be a finite algebra and assume that there exist*

$$\mathbf{B},\ e,\ 0,\ 1,\ \wedge,\ \alpha,\ p,\ a,\ b,\ \vec{c},\ \vec{d},\ \theta$$

satisfying conditions 1–9 of Theorem 2.1. Then $\mathrm{Var}(\mathbf{A})$ *is residually large.*

Proof. Let Z be a non-empty set, and define $\mathbf{C}$ to be the subalgebra of $\mathbf{B}^Z$ with the underlying set

$$C := \{\, f \in B^Z \mid (\forall s, t \in Z)\ f(s) \equiv_\alpha f(t) \,\}.$$

Each constant map in B^Z belongs to C. So there are polynomials e, $\wedge$ and p of $\mathbf{C}$ that can be defined coordinate-wise from the polynomials e, $\wedge$ and p of $\mathbf{B}$. As e is idempotent on $\mathbf{B}$, its extension to $\mathbf{C}$ is also idempotent.

Now define $\hat{\theta}$ to be the syntactic congruence on $\mathbf{C}$ determined by $e^{-1}(\underline{1})$, and define $\mathbf{D} := \mathbf{C}/\hat{\theta} \in \mathrm{Var}(\mathbf{A})$. We split the rest of the proof into three parts. The third part contains the reusable congruence calculation.

Claim (i): *For all $f \in e(C)\backslash\{\underline{1}\}$, we have $f \equiv_{\hat{\theta}} f \wedge \underline{0}$.*

Let h be a unary polynomial of $\mathbf{C}$. Since $\hat{\theta}$ is the syntactic congruence determined by $e^{-1}(\underline{1})$, it suffices to prove that $e \circ h(f) = \underline{1} \iff e \circ h(f \wedge \underline{0}) = \underline{1}$, for all $f \in e(C)\backslash\{\underline{1}\}$.

We have $h(x) = \tau^{\mathbf{C}}(x, g_1, \dots, g_k)$, for some term τ and $g_1, \dots, g_k \in C$. For each $z \in Z$, we can define the polynomial h_z of $\mathbf{B}$ by $h_z(x) := \tau^{\mathbf{B}}(x, g_1(z), \dots, g_k(z))$. This gives us $h_z(f(z)) = h(f)(z)$, for all $z \in Z$ and $f \in C$. The definition of $\mathbf{C}$ ensures that we have $h_y(f(y)) \equiv_\alpha h_z(f(z))$, for all $y, z \in Z$ and $f \in C$.

Now let $f \in e(C)\backslash\{\underline{1}\}$ and assume that $e \circ h(f) = \underline{1}$. Choose any $z \in Z$ and fix some $y \in Z$ with $f(y) \neq 1$. Then $e \circ h_y(f(y)) = e \circ h(f)(y) = 1$. So it follows from condition 7 that $e \circ h_y(f(y) \wedge 0) = 1$. We now have

$$\begin{aligned} e \circ h(f \wedge \underline{0})(z) &= e \circ h_z((f \wedge \underline{0})(z)) \\ &\overset{\alpha}{\equiv} e \circ h_y((f \wedge \underline{0})(y)) = e \circ h_y(f(y) \wedge 0) = 1. \end{aligned}$$

Setting $\beta := \mathrm{Cg}_{\mathbf{B}}(0,1)$, we also have

$$\begin{aligned} e \circ h(f \wedge \underline{0})(z) &= e \circ h_z(f(z) \wedge 0) \\ &\overset{\beta}{\equiv} e \circ h_z(f(z) \wedge 1) = e \circ h_z(f(z)) = e \circ h(f)(z) = 1, \end{aligned}$$

by condition 6. Since $\alpha \cap \beta \subseteq \theta$, by condition 8, the previous two calculations give us $e \circ h(f \wedge \underline{0})(z) \equiv_\theta 1$, whence $e \circ h(f \wedge \underline{0})(z) = 1$.

We have shown that $e \circ h(f) = \underline{1} \implies e \circ h(f \wedge \underline{0}) = \underline{1}$, for all $f \in e(C)\backslash\{\underline{1}\}$. The proof of the reverse implication is similar.

Claim (ii): *The algebra $\mathbf{D}$ is subdirectly irreducible.*

Let $\gamma \in \mathrm{Con}(\mathbf{C})$ with $\gamma > \hat{\theta}$. We can prove that $\hat{\theta}$ is completely meet-irreducible by showing that $\underline{0} \equiv_\gamma \underline{1}$. There exist $f, g \in C$ with $f \equiv_\gamma g$ but $f \not\equiv_{\hat{\theta}} g$. So we can assume that there is a unary polynomial h of $\mathbf{C}$ such that $e \circ h(f) = \underline{1}$ and $e \circ h(g) \neq \underline{1}$. Using claim (i) and condition 6, we get

$$\underline{1} = e \circ h(f) \equiv_\gamma e \circ h(g) \equiv_{\hat{\theta}} e \circ h(g) \wedge \underline{0} \equiv_\gamma e \circ h(f) \wedge \underline{0} = \underline{1} \wedge \underline{0} = \underline{0}.$$

It now follows that $\mathbf{D}$ is subdirectly irreducible.

Claim (iii): *The size of $\mathbf{D}$ is at least $|Z|$.*

Using condition 9, we first define two elements of $\mathbf{B}$:

$$q := e \circ p(a, \vec{d}) \quad \text{and} \quad r := e \circ p(b, \vec{c}) \neq 1.$$

As $a \equiv_\alpha b$ and $\vec{c} \equiv_\alpha \vec{d}$, we have $a_z^b \in C$ and $(c_i)_z^{d_i} \in C$, for all $z \in Z$ and $i \leqslant n$. Let $s, t \in Z$ with $s \neq t$, and suppose that $a_s^b \equiv_{\hat{\theta}} a_t^b$ in $\mathbf{C}$. Then condition 9 gives us

$$\begin{aligned} \underline{1} &= e \circ p\big(a_s^b, (c_1)_s^{d_1}, \dots, (c_n)_s^{d_n}\big) \\ &\overset{\hat{\theta}}{\equiv} e \circ p\big(a_t^b, (c_1)_s^{d_1}, \dots, (c_n)_s^{d_n}\big) = 1_{st}^{qr}, \end{aligned} \tag{RL}$$

and therefore $\underline{1} \equiv_{\hat{\theta}} 1_{st}^{qr}$. But $e(1_{st}^{qr}) = 1_{st}^{qr} \neq \underline{1} = e(\underline{1})$. Since $\hat{\theta}$ is the syntactic congruence on $\mathbf{C}$ determined by $e^{-1}(\underline{1})$, we obtain a contradiction. We have shown that $a_s^b/\hat{\theta} \neq a_t^b/\hat{\theta}$, and so $|D| = |C/\hat{\theta}| \geqslant |Z|$. □

3. The main theorem

In this section, we prove that a finite algebra must be inherently non-dualisable if the variety it generates is residually large and congruence meet-semidistributive.

Theorem 3.1. *Let* $\mathbf{A}$ *be a finite algebra and assume that there is no bound on the cardinalities of the subdirectly irreducible algebras in* $\mathrm{Var}(\mathbf{A})$ *with a non-abelian monolith. Then* $\mathbf{A}$ *is inherently non-*κ*-dualisable, for every cardinal* κ*.*

Proof. By Theorem 2.1, there exist

$$\mathbf{B},\ e,\ 0,\ 1,\ \wedge,\ \alpha,\ p,\ a,\ b,\ \vec{c},\ \vec{d},\ \theta$$

satisfying conditions 1–9 of that theorem. Since $\mathbf{B}$ is a finite algebra in $\mathbb{ISP}(\mathbf{A})$, it is sufficient to prove that $\mathbf{B}$ is inherently non-κ-dualisable, for all κ.

We use the Inherent Non-dualisability Theorem, 1.1. Let κ be an infinite cardinal and define $\varphi : \omega \to \omega$ by $\varphi(k) := k$. Let Z be any set of cardinality κ and fix some $0 \in Z$. Using condition 9, we can define

$$q := e \circ p(a, \vec{d}) \quad \text{and} \quad r := e \circ p(b, \vec{c}) \neq 1$$

in $\mathbf{B}$. Now define the sets $C_0, C_1 \subseteq B^Z$ by

$$C_0 := \{\, 1_{0z}^{0r} \mid z \in Z\backslash\{0\} \,\},$$
$$C_1 := \{\, f \in B^Z \mid f(z) \equiv_\alpha f(0), \text{ for all } z \in Z, \text{ and } f^{-1}(f(0)) \text{ is cofinite in } Z \,\},$$

and define the algebra

$$\mathbf{C} := \mathbf{sg}_{\mathbf{B}^Z}(C_0 \cup C_1).$$

We shall check conditions (i) and (ii) of the Inherent Non-dualisability Theorem.

Condition (i) *holds.*

Let $\gamma \in \mathrm{Con}(\mathbf{C})$ such that γ has index at most $k \in \omega\backslash\{0\}$. Assume that S and U are disjoint subsets of $Z\backslash\{0\}$, each of size greater than $\varphi(k)$, such that

- the set $\{\, 1_{0s}^{0r} \mid s \in S \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$, and
- the set $\{\, 1_{0u}^{0r} \mid u \in U \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$.

We shall prove that $\{\, 1_{0z}^{0r} \mid z \in S \cup U \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$. It will then follow that $\gamma{\restriction}_{C_0}$ has a unique block of size greater than $\varphi(k)$, as required.

We are assuming that γ has index at most $k = \varphi(k) < |S|, |U|$. Thus there are distinct $s, t \in S$ and distinct $u, v \in U$ such that

$$a_s^b \equiv_\gamma a_t^b \quad \text{and} \quad a_u^b \equiv_\gamma a_v^b$$

in $\mathbf{C}$.

Each constant map in B^Z belongs to $C_1 \subseteq C$. So there are polynomials e, $\wedge$ and p of $\mathbf{C}$ that can be defined coordinate-wise from the polynomials e, $\wedge$ and p of $\mathbf{B}$. We can now use condition 9 to obtain

$$\begin{aligned} \underline{1} &= e \circ p\big(a_s^b, (c_1)_s^{d_1}, \ldots, (c_n)_s^{d_n}\big) \\ &\overset{\gamma}{\equiv} e \circ p\big(a_t^b, (c_1)_s^{d_1}, \ldots, (c_n)_s^{d_n}\big) = 1_{st}^{qr}, \end{aligned}$$

and therefore $\underline{1} \equiv_\gamma 1_{st}^{qr}$. (This is calculation (RL) from the proof of Theorem 2.2.)

By condition 6, the binary polynomial $\wedge$ of $\mathbf{B}$ is a meet-semilattice operation on each of the sets $\{0,1\}$, $\{q,1\}$ and $\{r,1\}$, with $0 < 1$, $q \leqslant 1$ and $r < 1$. We will use this fact often throughout the rest of the proof.

As $r = e \circ p(b, \vec{c}) \equiv_\alpha e \circ p(a, \vec{c}) = 1$ in $\mathbf{B}$, we have $1^r_t \in C_1 \subseteq C$. So

$$\underline{1} \equiv_\gamma 1^{qr}_{st} = 1^{qr}_{st} \wedge 1^r_t \equiv_\gamma \underline{1} \wedge 1^r_t = 1^r_t.$$

Using the symmetry between s and t, we get $1^r_t \equiv_\gamma \underline{1} \equiv_\gamma 1^r_s$. This implies that

$$\underline{1} \equiv_\gamma 1^r_t = 1^r_t \wedge 1^r_t \equiv_\gamma 1^r_s \wedge 1^r_t = 1^{rr}_{st},$$

and so $\underline{1} \equiv_\gamma 1^{rr}_{st}$.

Since $u, v \in U$, we have $1^{0r}_{0u} \equiv_\gamma 1^{0r}_{0v}$, by assumption. Thus

$$\begin{aligned} 1^{0r}_{0u} = 1^{0r}_{0u} \wedge 1^{0r}_{0u} \equiv_\gamma 1^{0r}_{0u} \wedge 1^{0r}_{0v} = 1^{0rr}_{0uv} = \underline{1} \wedge 1^{0rr}_{0uv} \\ \equiv_\gamma 1^{rr}_{st} \wedge 1^{0rr}_{0uv} = 1^{0rrrr}_{0stuv}. \end{aligned}$$

Using the symmetry in our assumptions on S and U, we have $1^{0r}_{0u} \equiv_\gamma 1^{0rrrr}_{0stuv} \equiv_\gamma 1^{0r}_{0s}$. Hence $\{\, 1^{0r}_{0z} \mid z \in S \cup U \,\}$ is contained in a block of $\gamma{\restriction}_{C_0}$, whence (i) holds.

Condition (ii) *holds.*

The element of B^Z defined by (ii) is $g := 1^0_0$. Suppose, by way of contradiction, that $g \in C$. Then $1^0_0 \in \mathrm{sg}_{\mathbf{B}^Z}(C_0 \cup C_1)$. Thus there exist distinct $z_1, \dots, z_k \in Z \backslash \{0\}$ and $f_1, \dots, f_\ell \in C_1$, for some $k, \ell > 0$, and a $(k+\ell)$-ary term τ such that

$$1^0_0 = \tau\big(1^{0\,r}_{0z_1}, \dots, 1^{0\,r}_{0z_k}, f_1, \dots, f_\ell\big) \tag{$*$}$$

in $\mathbf{B}^Z$.

For each $i \in \{1, \dots, k\}$, define the tuple $\vec{v}_i := \big(f_1(z_i), \dots, f_\ell(z_i)\big) \in B^\ell$. The definition of C_1 ensures that the tuples $\vec{v}_1, \dots, \vec{v}_k$ are pairwise in α. By evaluating equation $(*)$ at the coordinates $z_1, \dots, z_k$, we have

$$\begin{aligned} \tau(r, 1, 1, \dots, 1, 1, \vec{v}_1) &= 1, && (z_1) \\ \tau(1, r, 1, \dots, 1, 1, \vec{v}_2) &= 1, && (z_2) \\ \vdots \qquad & \quad \vdots && \vdots \\ \tau(1, 1, 1, \dots, 1, r, \vec{v}_k) &= 1 && (z_k) \end{aligned}$$

in $\mathbf{B}$.

Each $f_1, \dots, f_\ell \in C_1$ agrees almost everywhere on Z with its value at 0. Thus we can find a coordinate $z_{k+1} \in Z \backslash \{0, z_1, \dots, z_k\}$ such that

$$\vec{v}_{k+1} := \big(f_1(z_{k+1}), \dots, f_\ell(z_{k+1})\big) = \big(f_1(0), \dots, f_\ell(0)\big).$$

Again, the tuples $\vec{v}_1, \dots, \vec{v}_{k+1}$ are pairwise in α.

Now, by evaluating equation $(*)$ at the coordinates z_{k+1} and 0, we get

$$\begin{aligned} \tau(1, 1, 1, \dots, 1, 1, \vec{v}_{k+1}) &= 1, && (z_{k+1}) \\ \tau(0, 0, 0, \dots, 0, 0, \vec{v}_{k+1}) &= 0 && (0) \end{aligned}$$

in $\mathbf{B}$. We shall obtain a contradiction by deducing from equations (z_1) to (z_{k+1}) that $e \circ \tau(0, \dots, 0, \vec{v}_{k+1}) = 1$.

We argue by induction, with the first step being equation (z_1). Let $i \in \{1, \dots, k\}$ and assume that

$$e \circ \tau(0, \dots, 0, \overset{i}{r}, 1, \dots, 1, \vec{v}_i) = 1.$$

(Recall that τ has arity $k + \ell$. We write an input for τ as a string of elements of B appended with an ℓ-tuple. Starting from the labelled position in the string, determine the elements in positions $1, \dots, k$. Ignore any other elements of the string. For example, if $i = 1$ in the equation above, then the actual input string starts with r and there are no 0's.)

By condition 7, we have $r \equiv_\theta r \wedge 0$. So we can deduce from the previous equation that

$$e \circ \tau(0, \dots, 0, r \overset{i}{\wedge} 0, 1, \dots, 1, \vec{v}_i) \equiv_\theta 1.$$

As θ is the syntactic congruence on $\mathbf{B}$ determined by $e^{-1}(1)$, this implies that

$$e \circ \tau(0, \dots, 0, r \overset{i}{\wedge} 0, 1, \dots, 1, \vec{v}_i) = 1.$$

We have $r = e \circ p(b, \vec{c}) \equiv_\alpha e \circ p(a, \vec{c}) = 1$ and $\vec{v}_i \equiv_\alpha \vec{v}_{i+1}$. Thus

$$\begin{aligned} 1 &= e \circ \tau(0, \dots, 0, r \overset{i}{\wedge} 0, 1, 1, \dots, 1, \vec{v}_i) \\ &\overset{\alpha}{\equiv} e \circ \tau(0, \dots, 0, 1 \overset{i}{\wedge} 0, r, 1, \dots, 1, \vec{v}_{i+1}) \\ &= e \circ \tau(0, \dots, 0, 0, \overset{i+1}{r}, 1, \dots, 1, \vec{v}_{i+1}). \end{aligned}$$

(If $i = k$, then the above input for τ actually consists only of 0's and the ℓ-tuple $\vec{v}_{k+1}$.) On the other hand, if we set $\beta := \mathrm{Cg}_{\mathbf{B}}(0, 1)$, then equation (z_{i+1}) gives us

$$\begin{aligned} 1 &= e \circ \tau(1, \dots, 1, \overset{i+1}{r}, 1, \dots, 1, \vec{v}_{i+1}) \\ &\overset{\beta}{\equiv} e \circ \tau(0, \dots, 0, \overset{i+1}{r}, 1, \dots, 1, \vec{v}_{i+1}). \end{aligned}$$

By condition 8, we have $\alpha \cap \beta \subseteq \theta$. So, as θ is the syntactic congruence determined by $e^{-1}(1)$, the previous two calculations imply that

$$e \circ \tau(0, \dots, 0, \overset{i+1}{r}, 1, \dots, 1, \vec{v}_{i+1}) = 1.$$

It now follows by induction that $e \circ \tau(0, \dots, 0, \vec{v}_{k+1}) = 1$. Since $0 \in e(B)\backslash\{1\}$, this contradicts equation (0). Thus $g \notin C$, and so condition (ii) holds. ☐

We have seen that, on an algebra with a semilattice reduct, every non-trivial congruence is non-abelian. Thus we immediately obtain the following corollary.

Corollary 3.2. *Let* $\mathbf{A}$ *be a finite algebra that has a semilattice reduct. If* $\mathrm{Var}(\mathbf{A})$ *is residually large, then the algebra* $\mathbf{A}$ *is inherently non-*κ*-dualisable, for every cardinal* κ.

A variety $\mathcal{V}$ is **congruence meet-semidistributive** if the congruence lattice of each algebra in $\mathcal{V}$ satisfies

$$x \wedge y \approx x \wedge z \implies x \wedge y \approx x \wedge (y \vee z).$$

It follows from Hobby and McKenzie's theory of tame congruences [10, 9.10] that the following conditions are equivalent for each finite algebra $\mathbf{A}$:

- the variety $\mathrm{Var}(\mathbf{A})$ is congruence meet-semidistributive;
- for all $\mathbf{B} \in \mathrm{Var}(\mathbf{A})$, every non-trivial congruence on $\mathbf{B}$ is non-abelian.

So each finite algebra with a semilattice reduct generates a variety that is congruence meet-semidistributive. (Indeed, any variety that has a semilattice term is congruence meet-semidistributive, by Papert [17]). Thus we obtain a more general corollary of the previous theorem.

Corollary 3.3. *Let* $\mathbf{A}$ *be a finite algebra such that* $\mathrm{Var}(\mathbf{A})$ *is congruence meet-semidistributive. If* $\mathrm{Var}(\mathbf{A})$ *is residually large, then the algebra* $\mathbf{A}$ *is inherently non-*κ*-dualisable, for every cardinal* κ.

4. The counterexample

In this section, we exhibit a four-element algebra that is dualisable but generates a variety that is residually large, thus refuting the conjecture that 'every finite algebra that generates a residually large variety is non-dualisable' [18].

We will need only one general theorem from duality theory to establish our counterexample. We shall set up the background for this theorem very briefly. Again, we refer to the Clark–Davey text [1] for details.

A finite algebra $\mathbf{A}$ is *dualisable* if there exists an especially natural representation for the quasivariety $\mathbb{ISP}(\mathbf{A})$. Such a representation is built from a set P of finitary partial operations on A. We require that P is **compatible** with the algebra $\mathbf{A}$:

- every term function of $\mathbf{A}$ preserves each partial operation in P;
- more precisely, for all $m, n \geqslant 0$, each n-ary term function τ of $\mathbf{A}$, each m-ary partial operation $p \in P$, and all $\vec{a}_1, \dots, \vec{a}_m \in A^n$ with $(\vec{a}_1, \dots, \vec{a}_m) \in \mathrm{dom}(p)$, we must have
$$\big(\tau(\vec{a}_1), \dots, \tau(\vec{a}_m)\big) \in \mathrm{dom}(p) \quad \text{and} \quad \tau\big(p(\vec{a}_1, \dots, \vec{a}_m)\big) = p\big(\tau(\vec{a}_1), \dots, \tau(\vec{a}_m)\big),$$
where $\mathrm{dom}(p)$ and p are extended coordinate-wise to A^n.

If P *dualises* the algebra $\mathbf{A}$, then there is a natural dual equivalence between the quasivariety $\mathbb{ISP}(\mathbf{A})$ and a special category of compact topological partial algebras of type P.

We can now state the general theorem from duality theory that we use. It follows immediately from the IC Lemma [1, 2.2.5] and the Duality Compactness Theorem [1, 2.2.11]. (The initials IC stand for 'Interpolation Condition'.)

IC Duality Theorem 4.1. *Let $\mathbf{A}$ be a finite algebra and let P be a finite set of partial operations on A that is compatible with $\mathbf{A}$. Define the partial algebra $\boldsymbol{\mathcal{A}} := \langle A; P\rangle$ and assume that the following condition holds.*

(IC) *For every $n > 0$, every $\boldsymbol{\mathcal{X}} \leqslant \boldsymbol{\mathcal{A}}^n$ and every homomorphism $\psi : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}$, there exists a term function $\tau : A^n \to A$ of $\mathbf{A}$ such that $\tau{\restriction}_X = \psi$.*

Then P dualises $\mathbf{A}$.

The following definition sets up our counterexample. The specified algebra $\mathbf{C}$, congruence μ and partial operation q will be fixed throughout this section.

Definition 4.2. Our counterexample is a term-reduct of the ring (with identity) of integers modulo four, $\mathbf{Z}_4 = \langle \{0,1,2,3\}; +, \cdot, 0, 1\rangle$. We define the algebra
$$\mathbf{C} := \langle \{0,1,2,3\}; +, \diamond, 0, 1\rangle,$$
where the binary operation $\diamond$ is given by $x \diamond y := (x \cdot y)^2$. Note that the unary term functions $x \mapsto -x$ and $x \mapsto x^2$ of the ring $\mathbf{Z}_4$ are also term functions of the algebra $\mathbf{C}$.

The equivalence relation μ on $\{0,1,2,3\}$, given by
$$a \equiv_\mu b \iff a - b \in \{0, 2\},$$
is a congruence on the ring $\mathbf{Z}_4$ and therefore also a congruence on $\mathbf{C}$. Since the four-element cyclic group is a reduct of $\mathbf{C}$, it follows that $\mathrm{Con}(\mathbf{C}) = \{0_C, \mu, 1_C\}$.

Now we define a ternary partial operation q on C by
$$\mathrm{dom}(q) := \{\, (a,b,c) \in C^3 \mid a \equiv_\mu b \,\} \quad \text{and} \quad q(a,b,c) := a - b + c,$$
for all $(a,b,c) \in \mathrm{dom}(q)$. So q is a restriction of a Mal'cev term function of $\mathbf{C}$.

Lemma 4.3. *The partial operation q is compatible with the algebra* $\mathbf{C}$.

Proof. The domain of q is equal to $\mu \times C$, and so forms a subalgebra of $\mathbf{C}^3$. It is easy to check that $+$, 0 and 1 preserve q. We shall show that $\diamond$ preserves q.

First note that

$$a \equiv_\mu b \iff (a-b)^2 = 0 \iff 2(a-b) = 0 \iff a^2 = b^2,$$

for all $a, b \in C$. This implies that

$$q(a,b,c)^2 = \big((a-b)+c\big)^2 = (a-b)^2 + 2(a-b)c + c^2 = c^2,$$

for all $(a,b,c) \in \operatorname{dom}(q)$.

Now let $(a,b,c), (d,e,f) \in \operatorname{dom}(q)$. Since $\operatorname{dom}(q)$ forms a subalgebra of $\mathbf{C}^3$, we know that $(a \diamond d, b \diamond e, c \diamond f) \in \operatorname{dom}(q)$. We have $a^2 = b^2$ and $d^2 = e^2$, and so

$$q(a,b,c) \diamond q(d,e,f) = c^2 f^2 = a^2 d^2 - b^2 e^2 + c^2 f^2 = q(a \diamond d, b \diamond e, c \diamond f).$$

Thus $\diamond$ preserves q. It now follows that every term function of $\mathbf{C}$ preserves q. □

We will prove that $\operatorname{Var}(\mathbf{C})$ is residually large by applying standard results from commutator theory; see the text by Freese and McKenzie [8].

Lemma 4.4. *The variety* $\operatorname{Var}(\mathbf{C})$ *is residually large and congruence permutable.*

Proof. Since $\mathbf{C}$ has a Mal'cev term, the variety $\operatorname{Var}(\mathbf{C})$ is congruence permutable (and hence congruence modular). So, by Theorem 10.14 of [8], it will follow that $\operatorname{Var}(\mathbf{C})$ is residually large once we have proved that $\mu \leqslant [1_C, 1_C]$ and $[\mu, 1_C] < \mu$.

By the definition of the commutator, the congruence 1_C centralises itself modulo $\delta := [1_C, 1_C]$. Since $0 \diamond 0 = 0 \diamond 1$, this gives us $0 = 1 \diamond 0 \equiv_\delta 1 \diamond 1 = 1$. As $\mathrm{Cg}_{\mathbf{C}}(0,1) = 1_C$, we obtain that $\mu \leqslant 1_C = \delta = [1_C, 1_C]$.

We can use Proposition 5.7 of [8] to observe that $[\mu, 1_C] = 0_C < \mu$, as q is compatible with $\mathbf{C}$ by the previous lemma. Hence $\operatorname{Var}(\mathbf{C})$ is residually large. □

In order to use the IC Duality Theorem, 4.1, to show that $\mathbf{C}$ is dualisable, we need to understand the term functions of $\mathbf{C}$.

Lemma 4.5. *Let $f : C^n \to C$, for some $n > 0$. Then the following are equivalent*:

(i) *f is a term function of* $\mathbf{C}$;
(ii) *f preserves the partial operation q*;
(iii) *there are functions $f_1 : \{0,1\}^n \to C$ and $f_2 : \{0,2\}^n \to \{0,2\}$ such that*
  (a) *f_2 preserves $+$, and*
  (b) *$f(\vec{a} + \vec{b}) = f_1(\vec{a}) + f_2(\vec{b})$, for all $\vec{a} \in \{0,1\}^n$ and $\vec{b} \in \{0,2\}^n$.*

Proof. We have (i) $\Rightarrow$ (ii), by Lemma 4.3. To see that (ii) $\Rightarrow$ (iii), assume that f preserves q. Let $\vec{a} \in \{0,1\}^n$ and $\vec{b} \in \{0,2\}^n$. Then we have $\vec{b} \equiv_\mu \hat{0}$, where $\hat{0} := (0, \dots, 0) \in C^n$. Since f preserves q, it follows that

$$f(\vec{a} + \vec{b}) = f(\vec{b} - \hat{0} + \vec{a}) = f(\vec{b}) - f(\hat{0}) + f(\vec{a}),$$

with $f(\vec{b}) \equiv_\mu f(\hat{0})$. So we can define $f_1 : \{0,1\}^n \to C$ and $f_2 : \{0,2\}^n \to \{0,2\}$ by

$$f_1(\vec{a}) := f(\vec{a}) \qquad \text{and} \qquad f_2(\vec{b}) := f(\vec{b}) - f(\hat{0}),$$

and (iii)(b) holds. To see that f_2 preserves $+$, let $\vec{b}, \vec{c} \in \{0,2\}^n$. Again, since f preserves q, we have

$$f_2(\vec{b} + \vec{c}) = f(\vec{b} + \vec{c}) - f(\hat{0}) = \big(f(\vec{b}) - f(\hat{0}) + f(\vec{c})\big) - f(\hat{0}) = f_2(\vec{b}) + f_2(\vec{c}).$$

So (iii)(a) holds.

It remains to prove that (iii) $\Rightarrow$ (i). So assume that f_1 and f_2 exist as stipulated by (iii). We shall prove that f is a term function of $\mathbf{C}$ via a sequence of claims.

Claim 1. For each function $g : \{0,1\}^n \to \{0,1\}$, *there exists an* n*-ary term function* $\overline{g}$ *of* $\mathbf{C}$ *such that* $\overline{g}{\restriction}_{\{0,1\}^n} = g$ *and* $\overline{g}{\restriction}_{\{0,2\}^n}$ *is constant.*

Let $\oplus$ denote addition modulo two on $\{0,1\}$. The ring $\mathbf{Z}_2 = \langle \{0,1\}; \oplus, \cdot, 0, 1\rangle$ is primal, and so every function $g : \{0,1\}^n \to \{0,1\}$ is a term function of $\mathbf{Z}_2$. We can therefore define the operation $g \mapsto \overline{g}$ inductively on n-ary term functions of $\mathbf{Z}_2$:

- for a projection $\pi_i : \{0,1\}^n \to \{0,1\}$, we define $\overline{\pi_i}(\vec{x}\,) := (x_i)^2$;
- for a constant function $c : \{0,1\}^n \to \{0,1\}$, we define $\overline{c}(\vec{x}\,) := c(\hat{0})$;
- for $h, k : \{0,1\}^n \to \{0,1\}$, we define $\overline{h \oplus k} := (\overline{h} + \overline{k})^2$ and $\overline{h \cdot k} := \overline{h} \diamond \overline{k}$.

It is now easy to check that the claim holds.

Claim 2. There exists an n*-ary term function* τ_1 *of* $\mathbf{C}$ *such that* $\tau_1{\restriction}_{\{0,1\}^n} = f_1$ *and* $\tau_1{\restriction}_{\{0,2\}^n}$ *is constant.*

We can define functions $g_1, g_2 : \{0,1\}^n \to \{0,1\}$ such that $f_1(\vec{a}\,) = g_1(\vec{a}\,) + 2 \cdot g_2(\vec{a}\,)$, for all $\vec{a} \in \{0,1\}^n$. Now define the term function

$$\tau_1 := \overline{g_1} + \big(\overline{g_2} + \overline{g_2}\big) = \overline{g_1} + 2 \cdot \overline{g_2}$$

of $\mathbf{C}$, using the previous claim.

Claim 3. There exists an n*-ary term function* τ_2 *of* $\mathbf{C}$ *such that* $\tau_2{\restriction}_{\{0,2\}^n} = f_2$ *and* $\tau_2{\restriction}_{\{0,1\}^n}$ *is constant with value* 0.

As $f_2 : \{0,2\}^n \to \{0,2\}$ preserves $+$, there is an n-ary term function σ of $\mathbf{C}$ such that $f_2 = \sigma{\restriction}_{\{0,2\}^n}$. Indeed, we can take

$$\sigma(\vec{x}\,) := \sum\big\{\, x_i \;\big|\; f_2(0,\dots,0,\overset{i}{2},0,\dots,0) = 2 \,\big\}.$$

We now define the term function $\tau_2(\vec{x}\,) := \sigma\big(x_1 - (x_1)^2, \dots, x_n - (x_n)^2\big)$.

Claim 4. The operation f *is a term function of* $\mathbf{C}$.

Define an n-ary term function of $\mathbf{C}$ by $\tau := \tau_1 + \tau_2$. To see that $\tau = f$, let $\vec{c} \in C^n$. Then $\vec{c} = \vec{a} + \vec{b}$, for some $\vec{a} \in \{0,1\}^n$ and $\vec{b} \in \{0,2\}^n$. By construction, we have

$$\tau(\vec{a}\,) = \tau_1(\vec{a}\,) + \tau_2(\vec{a}\,) = f_1(\vec{a}\,), \quad \tau(\vec{b}\,) = \tau_1(\hat{0}) + f_2(\vec{b}\,) \quad \text{and} \quad \tau(\hat{0}) = \tau_1(\hat{0}).$$

Since (i) $\Rightarrow$ (ii), we know that τ preserves q. As $\vec{b} \equiv_\mu \hat{0}$, this gives us

$$\tau(\vec{c}\,) = \tau(\vec{a} + \vec{b}\,) = \tau(\vec{b}\,) - \tau(\hat{0}) + \tau(\vec{a}\,) = f_1(\vec{a}\,) + f_2(\vec{b}\,) = f(\vec{a} + \vec{b}\,) = f(\vec{c}\,),$$

by (iii)(b). Hence f is a term function of $\mathbf{C}$, as required. □

We can now prove that the algebra $\mathbf{C}$ is dualisable, even though the variety that it generates is residually large.

Theorem 4.6. *The partial operation* q *dualises the algebra* $\mathbf{C}$.

Proof. We will use the IC Duality Theorem, 4.1. Let $n > 0$, let X be a non-empty subset of C^n that is closed under q, and let $\psi : X \to C$ preserve q. We will define an n-ary term function τ of $\mathbf{C}$ by first defining functions f_1 and f_2 as in condition (iii) of Lemma 4.5.

Part 1. Defining the function $f_2 : \{0,2\}^n \to \{0,2\}$.

We can define the subset S_2 of $\{0,2\}^n$ by

$$S_2 := \{\, \vec{c} - \vec{d} \mid \vec{c}, \vec{d} \in X \text{ with } \vec{c} \equiv_\mu \vec{d} \,\}.$$

We shall next prove that S_2 is closed under $+$ and that $S_2 + X \subseteq X$.

Let $\vec{a}, \vec{b} \in S_2$ and $\vec{c} \in X$. As X is closed under q, we have $\vec{a} + \vec{c} \in X$. Thus $S_2 + X \subseteq X$, and it follows that $\vec{a} + \vec{b} + \vec{c} \in X$. Since $\vec{a} + \vec{b} \in \{0,2\}^n$, we have $\vec{a} + \vec{b} + \vec{c} \equiv_\mu \vec{c}$, which implies that $\vec{a} + \vec{b} \in S_2$. Thus S_2 is closed under $+$.

Now let $\vec{a} \in S_2$ and $\vec{c}, \vec{d} \in X$. Since $S_2 + X \subseteq X$, we know that $\vec{a} + \vec{c} \in X$ and $\vec{a} + \vec{d} \in X$. We also have $\vec{a} + \vec{d} \equiv_\mu \vec{d}$. As ψ preserves q, this implies that

$$\psi(\vec{a} + \vec{d}) \equiv_\mu \psi(\vec{d}) \quad \text{and} \quad \psi(\vec{a} + \vec{c}) = \psi(\vec{a} + \vec{d}) - \psi(\vec{d}) + \psi(\vec{c}).$$

So $\psi(\vec{a} + \vec{c}) - \psi(\vec{c}) = \psi(\vec{a} + \vec{d}) - \psi(\vec{d}) \in \{0,2\}$. Thus we can unambiguously define the function $g_2 : S_2 \to \{0,2\}$ by

$$g_2(\vec{a}\,) := \psi(\vec{a} + \vec{c}) - \psi(\vec{c}),$$

where $\vec{c}$ is any element of X.

To see that g_2 preserves $+$, let $\vec{a}, \vec{b} \in S_2$ and choose some $\vec{c} \in X$. Then

$$\begin{aligned} g_2(\vec{a} + \vec{b}\,) &= \psi(\vec{a} + \vec{b} + \vec{c}) - \psi(\vec{c}) \\ &= \psi(\vec{a} + \vec{b} + \vec{c}) - \psi(\vec{b} + \vec{c}) + \psi(\vec{b} + \vec{c}) - \psi(\vec{c}) \\ &= g_2(\vec{a}\,) + g_2(\vec{b}\,). \end{aligned}$$

Hence g_2 preserves $+$, and it is easy to extend g_2 to a function $f_2 : \{0,2\}^n \to \{0,2\}$ that also preserves $+$.

Part 2. Defining the function $f_1 : \{0,1\}^n \to C$.

For each $\vec{c} \in C^n$, we can define $\vec{c}_1$ to be the unique element of $\{0,1\}^n$ such that $\vec{c} \equiv_\mu \vec{c}_1$. Now define the subset S_1 of $\{0,1\}^n$ by

$$S_1 := \{\, \vec{c}_1 \mid \vec{c} \in X \,\}.$$

Consider $\vec{c}, \vec{d} \in X$ with $\vec{c}_1 = \vec{d}_1$. We must have $\vec{c} \equiv_\mu \vec{d}$, and so $\vec{c} - \vec{d} \in S_2$. Thus

$$\begin{aligned} \psi(\vec{c}\,) - \psi(\vec{d}\,) &= \psi(\vec{c} - \vec{d} + \vec{d}\,) - \psi(\vec{d}\,) \\ &= g_2(\vec{c} - \vec{d}\,) \\ &= f_2\big((\vec{c} - \vec{c}_1) - (\vec{d} - \vec{d}_1)\big) \\ &= f_2(\vec{c} - \vec{c}_1) - f_2(\vec{d} - \vec{d}_1). \end{aligned}$$

Therefore $\psi(\vec{c}\,) - f_2(\vec{c} - \vec{c}_1) = \psi(\vec{d}\,) - f_2(\vec{d} - \vec{d}_1)$. This proves that we can unambiguously define the function $g_1 : S_1 \to C$ by

$$g_1(\vec{c}_1) := \psi(\vec{c}\,) - f_2(\vec{c} - \vec{c}_1).$$

Extend g_1 arbitrarily to a function $f_1 : \{0,1\}^n \to C$.

Part 3. Defining a term function that extends ψ.

Now Lemma 4.5 guarantees the existence of an n-ary term function τ of $\mathbf{C}$ such that $\tau(\vec{a}+\vec{b}) = f_1(\vec{a}) + f_2(\vec{b})$, for all $\vec{a} \in \{0,1\}^n$ and $\vec{b} \in \{0,2\}^n$. To check that τ extends ψ, let $\vec{c} \in X$. Then we have

$$\begin{aligned}\tau(\vec{c}) &= \tau\big(\vec{c}_1 + (\vec{c} - \vec{c}_1)\big)\\ &= f_1(\vec{c}_1) + f_2(\vec{c} - \vec{c}_1)\\ &= g_1(\vec{c}_1) + f_2(\vec{c} - \vec{c}_1)\\ &= \psi(\vec{c}) - f_2(\vec{c} - \vec{c}_1) + f_2(\vec{c} - \vec{c}_1)\\ &= \psi(\vec{c}).\end{aligned}$$

So ψ extends to a term function of $\mathbf{C}$, whence (IC) holds. Thus q dualises $\mathbf{C}$, by the IC Duality Theorem, 4.1. ☐

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