# Congruence meet-semidistributive locally finite varieties and a finite basis theorem

George F. McNulty and Ross Willard

*In memory of Bjarni Jónsson, the mathematician and the man*

**Abstract.** We provide several conditions that, among locally finite varieties, characterize congruence meet-semidistributivity and we use these conditions to give a new proof of a finite basis theorem published by Baker, McNulty, and Wang in 2004. This finite basis theorem extends Willard's Finite Basis Theorem.

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**Keywords.** Mal'cev condition, congruence meet-semidistributive, finitely based algebras.

## 1. Introduction

In 2000, Ross Willard [10] published the following theorem.

**Willard's Finite Basis Theorem.**
*Every variety with a finite signature and a finite residual bound that is congruence meet-semidistributive is finitely based.*

Willard's argument depended on a list of conditions on a variety $\mathcal{V}$, each characterizing when $\mathcal{V}$ is congruence meet-semidistributive. Among these conditions one is a Mal'cev condition and another, which we call the $\Diamond$-condition, played key roles in Willard's proof.

In 2016, Jovanović, Marković, McKenzie, and Moore [5] provided four different strong Mal'cev conditions that characterize when a locally finite variety is congruence meet-semidistributive. They note that the first of these strong Mal'cev conditions was devised by Kozik, Krokhin, Valeriote and Willard [6].

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Here we develop, for the locally finite case, conditions equivalent to congruence meet-semidistributivity that resemble some of those on Willard's list. In particular, we have the half $\diamondsuit$-condition, explained below.

Finally, we use the half $\diamondsuit$-condition to give another proof of an extension of Willard's Finite Basis Theorem published in 2004 by Baker, McNulty, and Wang [1]. We note that in the same year Maróti and McKenzie [9] extended Willard's Finite Basis Theorem to quasivarieties, by a method like the one used by Baker, McNulty, and Wang.

## 2. Some notation at the start

Let $\mathbf{A}$ be an algebra. A *basic translation* is a function $\lambda\colon A \to A$ so that there is an operation symbol $Q$ of rank $r > 0$ and elements $a_0, \ldots, a_{r-1} \in A$ so that for some $i < r$

$$\lambda(x) = Q^{\mathbf{A}}(a_0, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{r-1}) \text{ for all } x \in A.$$

Basic translations have *complexity* 1. A *translation* is just a composition of some finite sequence of basic translations. For a natural number $\ell$, we say that a translation *has complexity* $\ell$ if it is the composition of a sequence of length no more than $\ell$ of basic translations. Since a translation may arise as a composition of basic translations in several ways, we have framed the notion of complexity of a translation so that the set of complexities of a translation form an infinite interval of natural numbers. The identity map on $A$ is a translation of complexity 0. Each translation is associated with a term that is built from the terms associated to basic translations by means of repeated substitution.

It is sometimes convenient to expand the set of basic translations (and hence the set of translations) by allowing some finite set $T$ of terms to also play the role of basic operation symbols in the definition above. We call the expanded set of basic translations obtained in this way $T$-*enhanced* basic translations, and we call their compositions $T$-*enhanced* translations. In this context, we understand the complexities of a $T$-enhanced translation to be measured against $T$-enhanced basic translations.

Recall the following theorem of A. I. Mal'cev [7, 8].

**Mal'cev's Principal Congruence Generation Theorem.**
*Suppose that* $\mathbf{A}$ *is an algebra and that* $a, b, p,$ *and* $q$ *be elements of* $A$. *Then* $\langle p, q\rangle \in \mathrm{Cg}^{A}(a, b)$ *if and only if there is a natural number* $n$ *and elements* $r_0, r_1, \ldots, r_n \in A$ *and translations* $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ *of* $\mathbf{A}$ *so that* $p = r_0$ *and* $r_n = q$ *and*

$$\{r_i, r_{i+1}\} = \{\lambda_i(a), \lambda_i(b)\} \quad \textit{for all } i < n. \qquad \square$$

Figure 1 illustrates the content of this theorem. Arrows in the left diagram are labelled with translations while on the right they are labelled with complexities.

We use $\{a, b\} \looparrowright^{n}_{\ell} \{p, q\}$ to denote that there is a sequence $r_0, \ldots, r_n$ of no more than $n{+}1$ elements of $A$ and a sequence $\lambda_0, \ldots, \lambda_{n-1}$ of $n$ translations

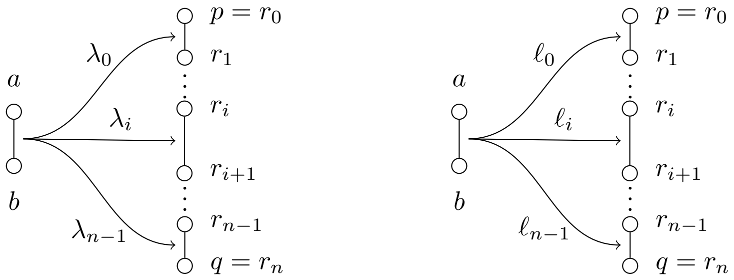



FIGURE 1. Principal congruence generation

of $\mathbf{A}$ (or $n$ $T$-enhanced translations of $\mathbf{A}$, depending on the context), each of complexity no more than $\ell$, that witness $(p,q) \in \mathrm{Cg}^A(a,b)$. The sequence $r_0, r_1, \dots, r_n$ is called a *principal congruence sequence.*

In case the signature is finite, for variables $x, y, z$, and $w$, we can regard $\{x,y\} \looparrowright^n_\ell \{z,w\}$ as an elementary formula (a formula of first order logic) with free variables $x, y, z$, and $w$ that has a string of existential quantifiers (here asserting the existence of $n+1$ elements, as well as the existence of lots of coefficients for the translations) followed by a disjunction of a conjunction of a lot of equations.

## 3. Conditions equivalent to SD$_\wedge$ in the locally finite case

Ross Willard [10] presents, in Theorem 2.1, a number of conditions that, for a variety $\mathcal{V}$, are equivalent to $\mathcal{V}$ being congruence meet-semidistributive. In [5] Jovanović, Marković, McKenzie, and Moore provide four strong Mal'cev conditions that characterize congruence meet-semidistributivity for locally finite varieties. Here we will present several more conditions, some resembling Willard's conditions, that characterize, among locally finite varieties, those that are congruence meet-semidistributive. We put these into play in the next section to give another proof of an extension of Willard's Finite Basis Theorem.

Here are three of the conditions Willard provides that characterize when a variety $\mathcal{V}$ is congruence meet-semidistributive (using the numbering in Theorem 2.1 in Willard's paper [10]):

(5) For some finite ordered tree $T$ the finite set $\mathcal{C}_2(T)$ of equations holds in $\mathcal{V}$.

(6) There exists a finite family $\{\langle s_p(x,y,z), t_p(x,y,z)\rangle \mid p \in T\}$ of pairs of ternary terms such that

$$\mathcal{V} \models s_p(x,y,x) \approx t_p(x,y,x) \text{ for all } p \in T$$

$$\mathcal{V} \models \forall x,y\Big(x \approx y \Leftrightarrow \bigwedge_{p\in T}[s_p(x,x,y) \approx t_p(x,x,y) \Leftrightarrow s_p(x,y,y) \approx t_p(x,y,y)]\Big).$$

(7) There exists a finite set $T$ of terms such that, for all $\mathbf{A} \in \mathcal{V}$ and all finite sequences $p = r_0, \ldots, r_n = q \in A$, if $p \neq q$, then there are $i < n$ and $e, f \in A$ with $e \neq f$ so that

$$\{r_i, r_{i+1}\} \looparrowright^2_1 \{e, f\} \text{ and } \{p, q\} \looparrowright^2_1 \{e, f\}$$

with respect to $T$-enhanced translations.

Regarding the Mal'cev condition (5), the sets $\mathcal{C}_2(T)$ are described in Willard's paper [10]. We have altered the statement of (7) above from Willard's formulation; however this is what he proves. Figure 2 illustrates condition (7), which we call the $\diamondsuit$-*condition.*

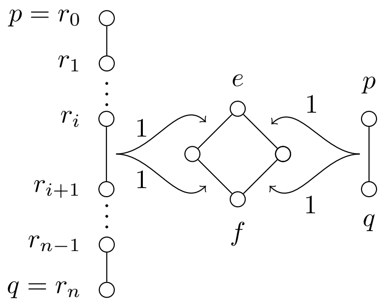



FIGURE 2. The $\diamondsuit$-condition

We will say a variety $\mathcal{V}$ satisfies the *half* $\diamondsuit$-*condition* provided it satisfies the following condition (illustrated in Figure 3):

There exists a finite set $T$ of terms such that, for all $\mathbf{A} \in \mathcal{V}$ and all finite sequences $p = r_0, \ldots, r_n = q \in A$, if $p \neq q$, then there exists $i < n$ and $e, f \in A$ with $e \neq f$ so that $\{r_i, r_{i+1}\} \looparrowright^2_1 \{e, f\}$ and $\{p, q\} \looparrowright^1_1 \{e, f\}$ with respect to $T$-enhanced translations.

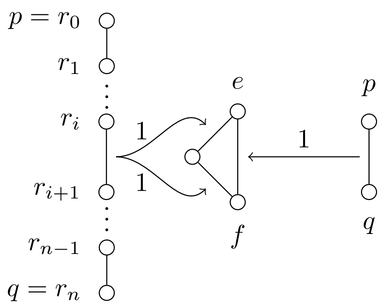



FIGURE 3. The half $\diamondsuit$-condition

We also say $\mathcal{V}$ satisfies the *half $\diamondsuit$-condition for principal congruence sequences* provided $\mathcal{V}$ satisfies the restriction of the above condition to principal congruence sequences.

Among locally finite varieties it turns out that the half $\diamondsuit$-condition characterizes congruence meet-semidistributive varieties. As regards Willard's condition (6), we are able to replace it, in locally finite varieties, with the following condition which we call (6*):

There exists an idempotent 4-ary term $t(x, y, z, w)$ such that

$$\begin{aligned}
&\mathcal{V} \models t(y, x, x, y) \approx t(y, x, y, y)\\
&\mathcal{V} \models t(y, y, x, x) \approx t(x, y, x, x), t(y, x, y, x) \approx t(x, x, y, x)\\
&\mathcal{V} \models \forall x, y\big(\theta(x, y) \to x \approx y\big) \text{ where } \theta(x, y) \text{ is the formula}\\
&\quad t(y, y, x, y) \approx y \mathbin{⩕} t(y, x, x, x) \approx x \mathbin{⩕} t(y, y, y, x) \approx t(x, y, y, x).
\end{aligned}$$

We also give below other conditions that resemble special cases of Willard's condition (6).

In the locally finite setting, we consider several strong Mal'cev conditions as replacements for Willard's (5). The first replacement is the following theorem proven in the paper [5] of Jovanović, Marković, McKenzie, and Moore as Theorem 3.2.

**Theorem.** *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ is locally finite and congruence meet-semidistributive, then $\mathcal{V}$ realizes the strong Mal'cev condition* (SM3) *given by*

$$\begin{aligned}
&t(x, x, x, x) \approx x,\\
&t(y, x, x, x) \approx t(x, y, x, x) \approx t(x, x, y, x) \approx t(x, x, x, y)\\
&\qquad\qquad \approx t(y, y, x, x) \approx t(y, x, y, x) \approx t(x, y, y, x).
\end{aligned} \tag{SM3}$$

*On the other hand, if $\mathcal{V}$ realizes the strong Mal'cev condition* (SM3)*, then $\mathcal{V}$ is congruence meet-semidistributive.* □

We have followed Jovanović, Marković, McKenzie, and Moore in naming this condition.

**The Half $\diamondsuit$ Theorem.**
*Let $\mathcal{V}$ be a variety.*

(a) *If* (SM3) *holds for $\mathcal{V}$, then* (6*) *holds for $\mathcal{V}$;*
(b) *If* (6*) *holds for $\mathcal{V}$, then $\mathcal{V}$ satisfies the half $\diamondsuit$-condition;*
(c) *If $\mathcal{V}$ satisfies the half $\diamondsuit$-condition, then the variety $\mathcal{V}$ is congruence meet-semidistributive.*

*Proof.* For (a), in (6*) take $t(x, y, z, w)$ to be the term specified in (SM3). Then the first line of (6*) holds. To confirm the second line, let $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$. Suppose $t^{\mathbf{A}}(b, a, a, b) = b$, $t^{\mathbf{A}}(b, a, a, a) = a$, and that $t^{\mathbf{A}}(b, b, b, a) =$

$t^{\mathbf{A}}(a,b,b,a)$. Then observe

$$t^{\mathbf{A}}(b,a,a,a) = t^{\mathbf{A}}(a,b,b,a) \qquad t^{\mathbf{A}}(b,b,b,a) = t^{\mathbf{A}}(b,b,a,b)$$
$$a = t^{\mathbf{A}}(b,a,a,a) \qquad t^{\mathbf{A}}(a,b,b,a) = t^{\mathbf{A}}(b,b,b,a) \qquad t^{\mathbf{A}}(b,b,a,b) = b.$$

The equations in the top line of this display are instances of equations in (SM3) while the equations in the bottom line are the instances of hypotheses of (6*) that we are assuming.

For (b), we assume that (6*) holds for $\mathcal{V}$ witnessed by the term $t$. We will show that $\mathcal{V}$ satisfies the half $\diamondsuit$-condition witnessed by the set $T = \{t\}$. Let $\mathbf{A} \in \mathcal{V}$ and also let $p = r_0, \ldots, r_n = q \in A$ with $p \neq q$. Invoking (6*) in its contrapositive form, we see that there are three alternatives:

$$t^{\mathbf{A}}(q,q,p,q) \neq q \quad \text{or} \quad t^{\mathbf{A}}(q,p,p,p) \neq p \quad \text{or} \quad t^{\mathbf{A}}(q,q,q,p) \neq t^{\mathbf{A}}(p,q,q,p).$$

**Alternative I:** $t^{\mathbf{A}}(q,q,p,q) \neq q = t^{\mathbf{A}}(q,q,q,q)$
Observe that $t^{\mathbf{A}}(q,p,p,q) = t^{\mathbf{A}}(q,p,q,q)$ by the first line of equations in (6*). This means that the equation $t^{\mathbf{A}}(q,x,p,q) = t^{\mathbf{A}}(q,x,q,q)$ holds at the left end of the sequence $p = r_0, r_1, \ldots, r_{n-1}, r_n = q$ but fails at the right end. There must be a leftmost place in this sequence where the equation fails. Let this element at this position be $r_{i+1}$. Then

$$t^{\mathbf{A}}(q,r_i,p,q) = t^{\mathbf{A}}(q,r_i,q,q) \quad \text{and} \quad t^{\mathbf{A}}(q,r_{i+1},p,q) \neq t^{\mathbf{A}}(q,r_{i+1},q,q).$$

Let $e = t^{\mathbf{A}}(q,r_{i+1},p,q)$ and $f = t^{\mathbf{A}}(q,r_{i+1},q,q)$. This completes the proof of the half $\diamondsuit$-condition under Alternative I.

**Alternative II:** $t^{\mathbf{A}}(q,p,p,p) \neq p = t^{\mathbf{A}}(p,p,p,p)$
Observe that $t^{\mathbf{A}}(q,q,p,p) = t^{\mathbf{A}}(p,q,p,p)$ by the first line of equations in (6*). This means that the equation $t^{\mathbf{A}}(q,x,p,p) = t^{\mathbf{A}}(p,x,p,p)$ holds at the right end of the sequence $p = r_0, r_1, \ldots, r_{n-1}, r_n = q$ but fails at the left end. There must be a rightmost place in this sequence where the equation fails. Let this element at this position be $r_i$. Then

$$t^{\mathbf{A}}(q,r_{i+1},p,p) = t^{\mathbf{A}}(p,r_{i+1},p,p) \quad \text{and} \quad t^{\mathbf{A}}(q,r_i,p,p) \neq t^{\mathbf{A}}(p,r_i,p,p).$$

Let $e = t^{\mathbf{A}}(q,r_i,p,p)$ and $f = t^{\mathbf{A}}(p,r_i,p,p)$. This completes the proof of the half $\diamondsuit$-condition under Alternative II.

**Alternative III:** $t^{\mathbf{A}}(q,q,q,p) \neq t^{\mathbf{A}}(p,q,q,p)$
The equation $t^{\mathbf{A}}(q,x,q,p) = t^{\mathbf{A}}(p,x,q,p)$ fails at the right end of the sequence $p = r_0, r_1, \ldots, r_{n-1}, r_n = q$ but holds at the left end, since $t(y,x,y,x) \approx t(x,x,y,x)$ is one of the equations in this first line of (6*). There must be a leftmost place in this sequence where the equation fails. Let this element at this position be $r_{i+1}$. Then

$$t^{\mathbf{A}}(q,r_i,q,p) = t^{\mathbf{A}}(p,r_i,q,p) \quad \text{and} \quad t^{\mathbf{A}}(q,r_{i+1},q,p) \neq t^{\mathbf{A}}(p,r_{i+1},q,p).$$

Let $e = t^{\mathbf{A}}(q,r_{i+1},q,p)$ and $f = t^{\mathbf{A}}(p,r_{i+1},q,p)$.

This completes the proof of the half $\diamondsuit$-condition under Alternative III.

For (c), we can use the same proof that Ross Willard [10] used to prove that the $\diamondsuit$-condition implies congruence meet-semidistributivity. We include it here for the convenience of the reader. Assume that $\mathcal{V}$ satisfies the half

$\diamondsuit$-condition. To prove that $\mathcal{V}$ is congruence meet-semidistributive, it suffices to show that if $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$ are congruences such that $\alpha \cap \beta = \alpha \cap \gamma = 0_A$, then $\alpha \cap (\beta \vee \gamma) = 0_A$. Suppose instead that $\alpha \cap (\beta \vee \gamma) \neq 0_A$. Thus we may choose $p = r_0, \ldots, r_n = q$ in $A$ such that $p \neq q$ and $(p, q) \in \alpha$ and $(r_i, r_{i+1}) \in \beta \cup \gamma$ for each $i < n$. By the half $\diamondsuit$-condition, there must exist $i < n$ and $e, f \in A$ with $e \neq f$ such that $(e, f) \in \mathrm{Cg}^{\mathbf{A}}(p, q) \subseteq \alpha$ and either $(e, f) \in \mathrm{Cg}^{\mathbf{A}}(r_i, r_{i+1}) \subseteq \beta$ or $(e, f) \in \mathrm{Cg}^{\mathbf{A}}(r_i, r_{i+1}) \subseteq \gamma$. But then either $\alpha \cap \beta \neq 0_A$ or $\alpha \cap \gamma \neq 0_A$, as desired. (See Figure 4.) □

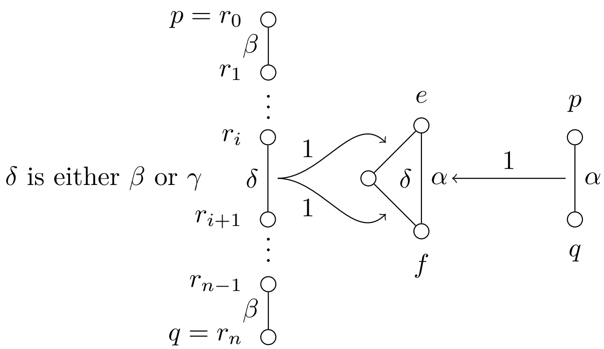



FIGURE 4. Proof of $\mathrm{SD}_\wedge$

Our next condition resembles (SM2) from the paper [5] by Jovanović, Marković, McKenzie and Moore. They note that this strong Mal'cev condition was established by Janko and Maróti in unpublished work. We say a variety $\mathcal{V}$ *satisfies* (SM2′) provided there are terms $r_0(x, y, z), r_1(x, y, z)$, and $r_2(x, y, z)$ so that all the equations below are true in $\mathcal{V}$.

$$
(\mathrm{SM2}') \quad \begin{cases}
r_0(x, y, y) \approx r_2(x, y, y) \\
r_1(x, x, y) \approx r_2(x, y, x) \\
r_0(x, x, y) \approx r_0(y, x, y) \\
r_1(x, y, y) \approx r_1(x, y, x) \\
r_2(x, x, y) \approx r_1(x, x, y) \\
r_0(x, x, x) \approx r_1(x, x, x) \approx r_2(x, x, x) \approx x
\end{cases}
$$

Like (SM2), the condition (SM2′) uses three ternary terms but needs fewer equations.

There is a companion condition that we will call (6′): there are idempotent terms $r_0(x, y, z)$, $r_1(x, y, z)$, and $r_2(x, y, z)$ so that

$$
\begin{aligned}
&\mathcal{V} \models r_0(x, x, y) \approx r_0(y, x, y), \; r_1(x, y, y) \approx r_1(x, y, x), \\
&\mathcal{V} \models r_2(x, x, y) \approx r_1(x, x, y), \\
&\mathcal{V} \models \forall x, y\, [\theta(x, y) \implies x \approx y] \text{ where } \theta(x, y) \text{ is the formula} \\
&\quad r_0(x, y, y) \approx y \;\&\; r_1(x, x, y) \approx x \;\&\; r_2(x, y, y) \approx r_2(x, y, x).
\end{aligned}
$$

**The SM2′ Theorem.**
*Let $\mathcal{V}$ be a variety.*

(a) *If $\mathcal{V}$ satisfies* (SM3)*, then $\mathcal{V}$ satisfies* (SM2′)*.*
(b) *If $\mathcal{V}$ satisfies* (SM2′)*, then $\mathcal{V}$ satisfies* (6′)*.*
(c) *If $\mathcal{V}$ satisfies* (6′)*, then it has the half $\diamondsuit$-condition.*

*Proof.* For the first part of the theorem just let $r_0(x,y,z)$ be $t(z,y,x,z)$, let $r_1(x,y,z)$ be $t(z,y,x,x)$, and let $r_2(x,y,z)$ be $t(z,y,y,x)$.

To confirm the second part, let $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$. Suppose $r_0^{\mathbf{A}}(a,b,b) = b$, $r_1^{\mathbf{A}}(a,a,b) = a$, and $r_2^{\mathbf{A}}(a,b,b) = r_2^{\mathbf{A}}(a,b,a)$. Then observe

$$\begin{array}{ccc} r_1^{\mathbf{A}}(a,a,b) = r_2^{\mathbf{A}}(a,b,a) & r_2^{\mathbf{A}}(a,b,b) = r_0^{\mathbf{A}}(a,b,b) & \\ a = r_1^{\mathbf{A}}(a,a,b) & r_2^{\mathbf{A}}(a,b,a) = r_2^{\mathbf{A}}(a,b,b) & r_0^{\mathbf{A}}(a,b,b) = b. \end{array}$$

The equations in the top line of this display are instances of equations in (SM2′), while the equations in the bottom line are the instances of hypotheses of (6′) that we are assuming.

To see the last part, assume that $\mathcal{V}$ satisfies (6′) witnessed by the terms $r_0, r_1, r_2$. We will show that $\mathcal{V}$ satisfies the half $\diamondsuit$-condition witnessed by the set $T = \{r_0, r_1, r_2\}$. Let $\mathbf{A} \in \mathcal{V}$ and also let $p = r_0, \dots, r_n = q \in A$ with $p \neq q$. Invoking the last condition of (6′) in its contrapositive form, we see there are three alternatives:

$$r_0^{\mathbf{A}}(p,q,q) \neq q \text{ or } r_1^{\mathbf{A}}(p,p,q) \neq p \text{ or } r_2^{\mathbf{A}}(p,q,q) \neq r_2^{\mathbf{A}}(p,q,p).$$

**Alternative I:** $r_0^{\mathbf{A}}(p,q,q) \neq q = r_0^{\mathbf{A}}(q,q,q)$
Observe that $r_0^{\mathbf{A}}(p,p,q) = r_0^{\mathbf{A}}(q,p,q)$ by the first equation in (6′). This means that the equation $r_0^{\mathbf{A}}(p,x,q) = r_0^{\mathbf{A}}(q,x,q)$ holds at the left end of the sequence $p = r_0, r_1, \dots, r_{n-1}, r_n = q$ but fails at the right end. There must be a leftmost place in this sequence where the equation fails. Let the element at this position be $r_{i+1}$. Then

$$r_0^{\mathbf{A}}(p,r_i,q) = r_0^{\mathbf{A}}(q,r_i,q) \quad \text{and} \quad r_0^{\mathbf{A}}(p,r_{i+1},q) \neq r_0^{\mathbf{A}}(q,r_{i+1},q).$$

Let $e = r_0^{\mathbf{A}}(p,r_{i+1},q)$ and $f = r_0^{\mathbf{A}}(q,r_{i+1},q)$. This completes the proof of the half $\diamondsuit$-condition under Alternative I.

**Alternative II:** $r_1^{\mathbf{A}}(p,p,q) \neq p = r_1^{\mathbf{A}}(p,p,p)$
Observe that $r_1^{\mathbf{A}}(p,q,q) = r_1^{\mathbf{A}}(p,q,p)$ by the second equation in (6′). This means that the equation $r_1^{\mathbf{A}}(p,x,q) = r_1^{\mathbf{A}}(p,x,p)$ holds at the right end of the sequence $p = r_0, r_1, \dots, r_{n-1}, r_n = q$ but fails at the left end. There must be a rightmost place in this sequence where the equation fails. Let the element at this position be $r_i$. Then

$$r_1^{\mathbf{A}}(p,r_{i+1},q) = r_1^{\mathbf{A}}(p,r_{i+1},p) \quad \text{and} \quad r_1^{\mathbf{A}}(p,r_i,q) \neq r_1^{\mathbf{A}}(p,r_i,p).$$

Let $e = r_1^{\mathbf{A}}(p,r_i,q)$ and $f = r_1^{\mathbf{A}}(p,r_i,p)$. This completes the proof of the half $\diamondsuit$-condition under Alternative II.

**Alternative III:** $r_2^{\mathbf{A}}(p,q,q) \neq r_2^{\mathbf{A}}(p,q,p)$, but we also assume that Alternative II does not hold.
Observe that $r_2^{\mathbf{A}}(p,p,q) = r_2^{\mathbf{A}}(p,p,p) = p$ by the failure of Alternative II

and the third equation in (6′). This means that the equation $r_2^{\mathbf{A}}(p,x,q) = r_2^{\mathbf{A}}(p,x,p)$ holds at the left end of the sequence $p = r_0, r_1, \dots, r_{n-1}, r_n = q$ but fails at the right end. There must be a leftmost place in this sequence where the equation fails. Let the element at this position be $r_{i+1}$. Then

$$r_2^{\mathbf{A}}(p,r_i,q) = r_2^{\mathbf{A}}(p,r_i,p) \quad \text{and} \quad r_2^{\mathbf{A}}(p,r_{i+1},q) \neq r_2^{\mathbf{A}}(p,r_{i+1},p).$$

Let $e = r_2^{\mathbf{A}}(p,r_{i+1},q)$ and $f = r_2^{\mathbf{A}}(p,r_{i+1},p)$. This completes the proof of the half $\Diamond$-condition under Alternative III. □

We give a second pair of conditions that use three ternary terms. We say a variety $\mathcal{V}$ *satisfies* (SM2″) provided there are terms $t_1(x,y,z), t_2(x,y,z)$ and $s_2(x,y,z)$ so that all the equations below are true in $\mathcal{V}$.

$$(\text{SM2}'') \qquad \begin{cases} t_1(x,x,x) \approx x \\ t_1(x,y,x) \approx t_1(x,y,y) \\ t_1(x,x,y) \approx s_2(x,x,y) \\ s_2(x,y,y) \approx t_2(x,y,y) \\ t_2(x,x,y) \approx t_1(y,x,y) \\ s_2(x,y,x) \approx t_2(x,y,x) \\ t_1(y,x,y) \approx t_2(x,y,y) \\ t_2(y,x,y) \approx t_2(x,x,y) \end{cases}$$

Notice that $t_1, t_2$, and $s_2$ are idempotent, by the equations above. The equation $t_2(x,x,y) \approx t_2(x,y,y)$ follows from the equations above as well.

Here is a companion condition to (SM2″) that we will call (6″): there are idempotent terms $t_1(x,y,z), t_2(x,y,z)$, and $s_2(x,y,z)$ so that

$$\begin{aligned} &\mathcal{V} \models t_1(x,y,x) \approx t_1(x,y,y), t_2(y,x,y) \approx t_2(x,x,y), \\ &\mathcal{V} \models t_2(x,x,y) \approx s_2(y,x,y), t_2(x,y,y) \approx s_2(x,y,y), \\ &\mathcal{V} \models \forall x,y\, [\theta(x,y) \implies x \approx y] \text{ where } \theta(x,y) \text{ is the formula} \\ &\quad x \approx t_1(x,x,y) \mathbin{\&} t_2(x,y,y) \approx y \mathbin{\&} s_2(x,x,y) \approx t_2(x,x,y). \end{aligned}$$

**The SM2″ Theorem.**
*Let $\mathcal{V}$ be a variety.*

(a) *If $\mathcal{V}$ satisfies* (SM3), *then $\mathcal{V}$ satisfies* (SM2″).
(b) *If $\mathcal{V}$* (SM2″), *then $\mathcal{V}$ satisfies* (6″).
(c) *If $\mathcal{V}$ satisfies* (6″), *then it has the half $\Diamond$-condition.*

*Proof.* For part (a), let $t(x,y,z,w)$ be the term specified in (SM3). Then let $t_1(x,y,z)$ be $t(x,y,z,x)$, let $t_2(x,y,z)$ be $t(x,y,z,z)$, and let $s_2(x,y,z)$ be $t(x,y,y,z)$. Then the equations given in (SM2″) follow from those given in (SM3).

For part (b), observe that the equations specified on the first line of (6″) either belong to those listed in (SM2″) or follow from them. For the second line, suppose that $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$. Then note

$$t_1^{\mathbf{A}}(a,a,b) = s_2^{\mathbf{A}}(a,a,b) \quad t_2^{\mathbf{A}}(a,a,b) = t_2^{\mathbf{A}}(a,b,b)$$
$$a = t_1^{\mathbf{A}}(a,a,b) \quad s_2^{\mathbf{A}}(a,a,b) = t_2^{\mathbf{A}}(a,a,b) \quad t_2^{\mathbf{A}}(a,b,b) = b.$$

The equations in the top row above either belong to (SM2$''$) or are consequences of it, while the equations in the bottom row are the hypotheses of the second line of (6$''$).

For part (c), we assume that $\mathcal{V}$ satisfies (6$''$) witnessed by the terms $t_1, t_2, s_2$. We will show that $\mathcal{V}$ satisfies the half $\diamondsuit$-condition witnessed by the set $T = \{t_1, t_2, s_2\}$. Let $\mathbf{A} \in \mathcal{V}$ and also let $p = r_0, \ldots, r_n = q \in A$ with $p \neq q$. Invoking (6$''$) in its contrapositive form, we see there are three alternatives:

$$t_1^{\mathbf{A}}(p,p,q) \neq p \text{ or } t_2^{\mathbf{A}}(p,q,q) \neq q \text{ or } s_2^{\mathbf{A}}(p,p,q) \neq t_2^{\mathbf{A}}(p,p,q).$$

**Alternative I:** $t_1^{\mathbf{A}}(p,p,q) \neq p = t_1^{\mathbf{A}}(p,p,p)$
Observe that $t_1^{\mathbf{A}}(p,q,q) = t_1^{\mathbf{A}}(p,q,p)$ by the first equation in (6$''$). This means that the equation $t_1^{\mathbf{A}}(p,x,q) = t_1^{\mathbf{A}}(p,x,p)$ holds at the right end of the sequence $p = r_0, r_1, \ldots, r_{n-1}, r_n = q$ but fails at the left end. There must be a rightmost place in this sequence where the equation fails. Let the element at this position be $r_i$. Then

$$t_1^{\mathbf{A}}(p,r_i,q) \neq t_1^{\mathbf{A}}(p,r_i,p) \quad \text{and} \quad t_1^{\mathbf{A}}(p,r_{i+1},q) = t_1^{\mathbf{A}}(p,r_{i+1},p).$$

Let $e = t_1^{\mathbf{A}}(p,r_i,q)$ and $f = t_1^{\mathbf{A}}(p,r_i,p)$. This completes the proof of the half $\diamondsuit$-condition under Alternative I.

**Alternative II:** $t_2^{\mathbf{A}}(p,q,q) \neq q = t_2^{\mathbf{A}}(q,q,q)$
Observe that $t_2^{\mathbf{A}}(p,p,q) = t_2^{\mathbf{A}}(q,p,q)$ by the second equation in (6$''$). This means that the equation $t_2^{\mathbf{A}}(p,x,q) = t_2^{\mathbf{A}}(q,x,q)$ holds at the left end of the sequence $p = r_0, r_1, \ldots, r_{n-1}, r_n = q$ but fails at the right end. There must be a leftmost place in this sequence where the equation fails. Let the element at this position be $r_{i+1}$. Then

$$t_2^{\mathbf{A}}(p,r_i,q) = t_2^{\mathbf{A}}(q,r_i,q) \quad \text{and} \quad t_2^{\mathbf{A}}(p,r_{i+1},q) \neq t_2^{\mathbf{A}}(q,r_{i+1},q).$$

Let $e = t_2^{\mathbf{A}}(p,r_{i+1},q)$ and $f = t_2^{\mathbf{A}}(q,r_{i+1},q)$. This completes the proof of the half $\diamondsuit$-condition under Alternative II.

**Alternative III:** $s_2^{\mathbf{A}}(p,p,q) \neq t_2^{\mathbf{A}}(p,p,q)$ and we assume Alternative II fails. Observe that $t_2^{\mathbf{A}}(p,p,q) = s_2^{\mathbf{A}}(q,p,q)$ by the third equation in (6$''$). This means that the equation $s_2^{\mathbf{A}}(p,x,q) = s_2^{\mathbf{A}}(q,x,q)$ fails at the left end of the sequence $p = r_0, r_1, \ldots, r_{n-1}, r_n = q$. On the other hand, at the right end we have

$$s_2(p,q,q) = t_2(p,q,q) = q = s_2(q,q,q)$$

where the first equality comes from the fourth equation in (6$''$), the second equality comes from the failure of Alternative II. So our equation $s_2(p,x,q) = s_2(q,x,q)$ holds at the right end of the sequence. There must be a rightmost place in this sequence where the equation fails. Let the element at this position be $r_i$. Then

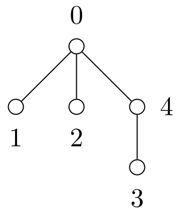
$$s_2^{\mathbf{A}}(p,r_i,q) \neq s_2^{\mathbf{A}}(q,r_i,q) \quad \text{and} \quad s_2^{\mathbf{A}}(p,r_{i+1},q) = s_2^{\mathbf{A}}(q,r_{i+1},q).$$

Let $e = s_2^{\mathbf{A}}(p, r_{i+1}, q)$ and $f = s_2^{\mathbf{A}}(q, r_{i+1}, q)$. This completes the proof of the half $\diamondsuit$-condition under Alternative III. $\square$

Here is a condition that is a single instance of Willard's Mal'cev condition (5). We say a variety $\mathcal{V}$ *satisfies* $\mathcal{C}_2(T^\bullet)$ provided there are four pairs of terms $(s_i(x,y,z), t_i(x,y,z))$ with $1 \le i < 5$ so that all the equations below are true in $\mathcal{V}$.

$$\mathcal{C}_2(T^\bullet) \qquad \begin{cases} x \approx s_1(x,x,y) \\ s_1(x,y,y) \approx t_1(x,y,y) \\ t_1(x,x,y) \approx s_2(x,x,y) \\ s_2(x,y,y) \approx t_2(x,y,y) \\ t_2(x,x,y) \approx s_4(x,x,y) \\ s_4(x,y,y) \approx s_3(x,y,y) \\ s_3(x,x,y) \approx t_3(x,x,y) \\ t_3(x,y,y) \approx t_4(x,y,y) \\ t_4(x,x,y) \approx y \\ s_1(x,y,x) \approx t_1(x,y,x) \\ s_2(x,y,x) \approx t_2(x,y,x) \\ s_3(x,y,x) \approx t_3(x,y,x) \\ s_4(x,y,x) \approx t_4(x,y,x) \end{cases}$$

In fact, $\mathcal{C}_2(T^\bullet)$ is the condition Willard associated with the ordered tree $T^\bullet$ depicted below.




See Willard's paper [10] for the details of how to get the equations from the tree. A slightly different rendition of $\mathcal{C}_2(T^\bullet)$, following the method of bracket expressions devised in [1], yields the following equations:

$$\Sigma_\beta \begin{cases} \qquad\qquad x \approx d_0(x,y,z) & \\ d_9(x,y,z) \approx z & \\ d_i(x,x,y) \approx d_{i+1}(x,x,y) & \text{if } i \text{ is even} \\ d_i(x,y,y) \approx d_{i+1}(x,y,y) & \text{if } i \text{ is odd} \\ d_i(x,y,x) \approx d_j(x,y,x) & \text{if } i \text{ and } j \text{ are matched in } \beta \end{cases}$$

where $\beta$ is the bracket expression $\langle\langle\rangle\langle\rangle\langle\langle\rangle\rangle\rangle$ and the brackets are labelled from left to right by $0, 1, \ldots, 9$. So the pairs of matched indices are

$$(0,9), (1,2), (3,4), (5,8), \text{ and } (6,7).$$

This form of Willard's $\mathcal{C}_2(T^\bullet)$ for congruence meet-semidistributivity reveals its close syntactical connection to Jónsson's characterization [3] of congruence distributivity.

The connection between $\mathcal{C}_2(T^\bullet)$ and $\Sigma_\beta$ is given by the following definitions of the terms $d_i(x,y,z)$ with the help of the terms appearing in $\mathcal{C}_2(T^\bullet)$:

$$\begin{aligned}
d_0(x,y,z) &:= x & d_5(x,y,z) &:= s_4(x,y,z)\\
d_1(x,y,z) &:= s_1(x,y,z) & d_6(x,y,z) &:= s_3(x,y,z)\\
d_2(x,y,z) &:= t_1(x,y,z) & d_7(x,y,z) &:= t_3(x,y,z)\\
d_3(x,y,z) &:= s_2(x,y,z) & d_8(x,y,z) &:= t_4(x,y,z)\\
d_4(x,y,z) &:= t_2(x,y,z) & d_9(x,y,z) &:= z
\end{aligned}$$

As a companion to the condition $\mathcal{C}_2(T^\bullet)$ we say a variety $\mathcal{V}$ *satisfies condition* $(6^\bullet)$ provided there are three pairs $(s_i(x,y,z), t_i(x,y,z))$, for $1 \le i < 4$, of terms such that

$$(6^\bullet)\quad \begin{cases}
\mathcal{V} \models t_i(x,y,x) \approx s_i(x,y,x) \text{ for } i = 1,2,3,\\
\mathcal{V} \models x \approx s_1(x,x,y),\ t_3(x,y,y) \approx y,\\
\mathcal{V} \models s_1(x,y,y) \approx t_1(x,y,y),\ s_2(x,y,y) \approx t_2(x,y,y),\\
\mathcal{V} \models s_3(x,x,y) \approx t_3(x,x,y),\\
\mathcal{V} \models \forall x,y\,[\,\theta(x,y) \implies x \approx y] \text{ where } \theta(x,y) \text{ is the formula}\\
\qquad x \approx t_1(x,x,y) \curlywedge s_2(x,x,y) \approx t_2(x,x,y) \curlywedge s_3(x,y,y) \approx y.
\end{cases}$$

Consider a single case of Willard's condition (6)—namely the case when $T = \{1,2,3\}$. We say a variety $\mathcal{V}$ satisfies condition $(6_3)$ provided there are three pairs $(s_i(x,y,z), t_i(x,y,z))$, for $1 \le i < 4$, of terms so that

$$(6_3)\quad \begin{cases}
\mathcal{V} \models t_i(x,y,x) \approx s_i(x,y,x) \text{ for all } i \text{ with } 1 \le i < 4\\
\mathcal{V} \models \forall x,y\,[\,\theta(x,y) \implies x \approx y] \text{ where } \theta(x,y) \text{ is the formula}\\
\qquad \bigcurlywedge_{1\le i<4} \Big( s_i(x,x,y) \approx t_i(x,x,y) \Leftrightarrow s_i(x,y,y) \approx t_i(x,y,y) \Big).
\end{cases}$$

The condition $(6_4)$ is like $(6_3)$, but for four pairs of terms.

**A Willard Style Theorem Characterizing $\mathbf{SD}_\wedge$ in the Locally Finite Case.** *Let $\mathcal{V}$ be a variety.*

(a) *If $\mathcal{V}$ satisfies* (SM2$''$)*, then $\mathcal{V}$ satisfies $\mathcal{C}_2(T^\bullet)$.*
(b) *If $\mathcal{V}$ satisfies $\mathcal{C}_2(T^\bullet)$, then $\mathcal{V}$ satisfies* $(6_4)$.
(c) *If $\mathcal{V}$ satisfies* $(6_4)$*, then $\mathcal{V}$ satisfies the $\diamondsuit$-condition.*
(d) *If $\mathcal{V}$ satisfies* (SM2$''$)*, then $\mathcal{V}$ satisfies* $(6^\bullet)$.
(e) *If $\mathcal{V}$ satisfies* $(6^\bullet)$*, then $\mathcal{V}$ satisfies* $(6_3)$.
(f) *if $\mathcal{V}$ satisfies* $(6_3)$*, then $\mathcal{V}$ satisfies the $\diamondsuit$-condition.*

*Proof.* For part (a) let $t_1(x,y,z), t_2(x,y,z)$, and $s_2(x,y,z)$ be the terms specified in (SM2$''$) and take $s_1(x,y,z)$ to be $t_1(x,y,x)$, take $s_3(x,y,z)$ to be $t_2(x,y,z)$, take $s_4(x,y,z)$ to be $t_2(x,x,z)$, take $t_3(x,y,z)$ to be $t_2(z,y,z)$, and take $t_4(x,y,z)$ to be $z$. With these definitions, the equations in $\mathcal{C}_2(T^\bullet)$ emerge from the equations of (SM2$''$).

Parts (b) and (c) were established as part of Theorem 2.1 in Ross Willard's paper [10].

Part (d) follows directly from part (b) of the SM2$''$ Theorem.

For part (e), assume ($6^\bullet$) and let $\mathbf{A} \in \mathcal{V}$ with $a, b \in A$ so that, for each $i \in \{1,2,3\}$ we have $s_i^{\mathbf{A}}(a,a,b) = t_i^{\mathbf{A}}(a,a,b)$ if and only if $s_i^{\mathbf{A}}(a,b,b) = t_1^{\mathbf{A}}(a,b,b)$. But according to ($6^\bullet$) we also have $s_1^{\mathbf{A}}(a,b,b) = t_1^{\mathbf{A}}(a,b,b)$, as well as $s_2^{\mathbf{A}}(a,b,b) = t_2^{\mathbf{A}}(a,b,b)$ and $s_3^{\mathbf{A}}(a,a,b) = t_3^{\mathbf{A}}(a,a,b)$. We obtain

$$\begin{aligned} a = s_1^{\mathbf{A}}(a,a,b) &= t_1^{\mathbf{A}}(a,a,b) \\ s_2^{\mathbf{A}}(a,a,b) &= t_2^{\mathbf{A}}(a,a,b) \\ s_3^{\mathbf{A}}(a,b,b) &= t_3^{\mathbf{A}}(a,b,b) = b. \end{aligned}$$

An appeal to the last line of the definition of ($6^\bullet$) yields $a = b$, as required.

Part (f) is proved in [10]. □

We gather the conclusions above together.

**Corollary.**

(a) *Every locally finite congruence meet-semidistributive variety satisfies each of the conditions* (SM2$'$), (SM2$''$), $\mathcal{C}_2(T^\bullet)$, ($6'$), ($6''$), ($6^\bullet$),($6_3$), ($6_4$) *and the half $\diamondsuit$-condition.*

(b) *Every variety that satisfies any one of the conditions listed in part (a), is congruence meet-semidistributive.* □

## 4. The proof of an extension of Willard's Finite Basis Theorem

Willard in [10] proved that every variety of finite signature that is congruence meet-semidistributive and has a finite residual bound is finitely based. Extending this, Baker, McNulty, and Wang [1] proved:

**The Half $\diamondsuit$ Finite Basis Theorem.**
*Let $\mathcal{V}$ be a variety of finite signature such that*

(a) *$\mathcal{V}$ has bounded critical depth,*

(b) *$\mathcal{V}$ is congruence meet-semidistributive,*

(c) *$\mathcal{V}$ is locally finite, and*

(d) *$\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable.*

*Then $\mathcal{V}$ is finitely based.*

A class $\mathcal{K}$ of algebras, all of the same signature, is said to have *bounded critical depth* provided there is a number $\ell$ such that for every $\mathbf{A} \in \mathcal{K}_{\mathrm{si}}$ and for all $a, b, p, q \in A$ with $a \neq b$ and $(p, q)$ a critical pair (that is, a non-identical pair belonging to the monolith), we have $\{a, b\} \looparrowright_\ell^n \{p, q\}$ for some natural number $n$.

The article [1] of Baker, McNulty, and Wang includes an example devised by Ralph McKenzie where the theorem just above applies but which does not have a finite residual bound.

Rather than giving a new proof of Willard's Finite Basis Theorem here, we provide another proof of the extension of it given by Baker, McNulty, and Wang.

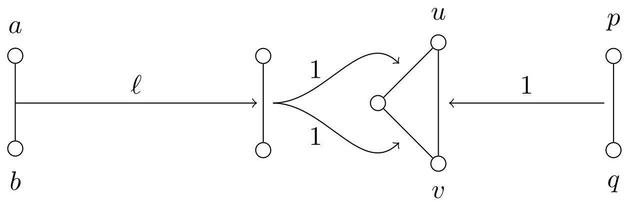
Let $\mathcal{K}$ be a class of algebras of the same signature. We say $\mathcal{K}$ has the *definite atoms property* provided there is a finite set $T$ of terms so that

- $\mathcal{K}$ has the half $\Diamond$-condition for principal congruence sequences witnessed by the set $T$, and
- there is a natural number $\ell$ such that for all $\mathbf{A} \in \mathcal{K}$ and for all $a, b \in A$ and all atoms $\alpha \subseteq \mathrm{Cg}^{\mathbf{A}}(a,b)$ and for all $(p,q) \in \alpha$ such that $p \neq q$, we have

  $$\mathbf{A} \models \exists u, v[\neg u \approx v \Wedge \{a,b\} \looparrowright^1_\ell \circ \looparrowright^2_1 \{u,v\} \Wedge \{p,q\} \looparrowright^1_1 \{u,v\}]$$

  with respect to $T$-enhanced translations.

See the diagram below for an illustration of the second stipulation in the definite atoms property.




We need two lemmas. The first traces its ancestry to Bjarni Jónsson's 1979 paper [4] giving a proof of Kirby Baker's Finite Basis Theorem and it has a close connection to the proof of Stan Burris [2] of Baker's Theorem, also from 1979.

**The Definite Atoms Finite Basis Lemma.**
*Let $\mathcal{V}$ be a variety in a finite signature. If*

- *the variety $\mathcal{V}$ has the definite atoms property,*
- *$\mathcal{V}$ is locally finite, and*
- *$\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable,*

*then $\mathcal{V}$ is finitely based.*

*Proof.* Let $T$ be a finite set of terms witnessing the definite atoms property. Let $\ell$ be a bound guaranteed by the third stipulation of the definite atoms property. For any natural number $m$ let $\pi_m(x_0, y_0, x_1, y_1)$ denote the formula

$$\exists u, v\left[\neg u \approx v \Wedge \{x_0, y_0\} \looparrowright^1_m \circ \looparrowright^2_2 \{u,v\} \Wedge \{x_1, y_1\} \looparrowright^1_m \circ \looparrowright^2_2 \{u,v\}\right]$$

where $\looparrowright^n_\ell$ is interpreted in the context of $T$-enhanced translations.

**Contention.** *If $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$ so that $\mathrm{Cg}^{\mathbf{A}}(a,b) \cap \mathrm{Cg}^{\mathbf{A}}(c,d)$ is not trivial, then $\mathbf{A} \models \pi_\ell(a,b,c,d)$.*

*Proof of contention.* Suppose that $\mathbf{A} \in \mathcal{V}$ and that $a, b, c, d \in A$ so that $\mathrm{Cg}^{A}(a,b) \cap \mathrm{Cg}^{A}(c,d)$ is not trivial. Now pick $p', q' \in A$ with $p' \neq q'$ so that $(p', q') \in \mathrm{Cg}^{A}(a,b) \cap \mathrm{Cg}^{A}(c,d)$. The computations witnessing this can be carried out in a finitely generated subalgebra $\mathbf{B}$ of $\mathbf{A}$. But $\mathcal{V}$ is locally finite, so $\mathbf{B}$ is finite. Observe that $\mathbf{B} \in \mathcal{V}$. Since $\mathbf{B}$ is finite every nontrivial congruence of $\mathbf{B}$ lies above an atom. Now pick $p, q$ with $p \neq q$ and $(p, q) \in \mathrm{Cg}^{\mathbf{B}}(p', q')$ so that $\mathrm{Cg}^{\mathbf{B}}(p, q)$ is an atom. Using the definite atoms property, we pick $p_0, p_1 \in B$ with $p_0 \neq p_1$ and

$$\mathbf{B} \models \{a, b\} \looparrowright^1_\ell \circ \looparrowright^2_1 \{p_0, p_1\} \wedge\!\!\!\wedge \{p, q\} \looparrowright^1_1 \{p_0, p_1\}.$$

Notice that $(p_0, p_1) \in \mathrm{Cg}^{\mathbf{B}}(c, d)$. Invoking the definite atoms property again, we obtain $r_0, r_1 \in B$ with $r_0 \neq r_1$ so that

$$\mathbf{B} \models \{c, d\} \looparrowright^1_\ell \circ \looparrowright^2_1 \{r_0, r_1\} \wedge\!\!\!\wedge \{p_0, p_1\} \looparrowright^1_1 \{r_0, r_1\}.$$

Now $\{a, b\} \looparrowright^1_\ell \circ \looparrowright^2_1 \{p_0, p_1\} \looparrowright^1_1 \{r_0, r_1\}$ yields $\{a, b\} \looparrowright^1_\ell \circ \looparrowright^2_2 \{r_0, r_1\}$. Hence

$$\mathbf{B} \models \{a, b\} \looparrowright^1_\ell \circ \looparrowright^2_2 \{r_0, r_1\} \wedge\!\!\!\wedge \{c, d\} \looparrowright^1_\ell \circ \looparrowright^2_2 \{r_0, r_1\}.$$

That is $\mathbf{B} \models \pi_\ell(a, b, c, d)$. But this is an existential formula. So $\mathbf{A}$ is a model as well, as desired. •

Let $\psi$ be the following sentence

$$\forall x_0, y_0, x_1, y_1 \left[\pi_{\ell+1}(x_0, y_0, x_1, y_1) \implies \pi_\ell(x_0, y_0, x_1, y_1)\right].$$

It follows from the contention above that $\mathcal{V} \models \psi$. Let $\mathcal{W} = \mathrm{Mod}\{\sigma, \psi\}$. We see $\mathcal{W}$ is a finitely axiomatizable class that includes $\mathcal{V}$. Let $\phi$ be the sentence

$$\forall x_0, y_0, x_1, y_1 \left[(\neg x_0 \approx y_0 \wedge\!\!\!\wedge \neg x_1 \approx y_1) \Leftrightarrow \pi_\ell(x_0, y_0, x_1 . y_1)\right].$$

It is our contention that $\{\sigma, \psi, \phi\}$ axiomatizes $\mathcal{W}_{\mathrm{fsi}}$. Certainly every model of those three sentences belongs to $\mathcal{W}_{\mathrm{fsi}}$. We will concern ourselves with the reverse inclusion. Let $\mathbf{A} \in \mathcal{W}_{\mathrm{fsi}}$. We know $\mathbf{A} \models \{\sigma, \psi\}$. We have to establish $\mathbf{A} \models \phi$. To this end, let $a, b, c, d \in A$. First suppose that $a \neq b$ and $c \neq d$. Since $\mathbf{A}$ is finitely subdirectly irreducible, we can pick $r, s \in A$ with $r \neq s$ and $(r, s) \in \mathrm{Cg}^{\mathbf{A}}(a, b) \cap \mathrm{Cg}^{\mathbf{A}}(c, d)$. By Mal'cev's description of principal congruences, there are numbers $n$ and $m$ so that

$$\{a, b\} \looparrowright^n_m \{r, s\} \text{ and } \{c, d\} \looparrowright^n_m \{r, s\}.$$

Since $\mathbf{A} \models \sigma$, we can apply the definite atom property (twice as we did above, but this time using the half $\diamondsuit$-condition on principal congruence sequences) to obtain

$$\mathbf{A} \models \pi_m(a, b, c, d).$$

This doesn't quite get us $\mathbf{A} \models \phi$ since $m$ might be larger than $\ell$. But now we can invoke $\psi$ to decrease $m$ step by step to $\ell$. So we have established the left-to-right direction within $\phi$. To obtain the other implication, let us suppose $a = b$. $\pi_\ell(a, b, c, d)$ cannot hold since we would have to have some $r_0, r_1 \in A$ with $r_0 \neq r_1$ and $(r_0, r_1) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$. The case when $c \neq d$ is similar. This means that $\mathbf{A} \models \phi$ as desired.

Let us gather together what we know. There is a finitely axiomatizable class $\mathcal{W}$ such that $\mathcal{V} \subseteq \mathcal{W}$ and $\mathcal{W}_{\mathrm{fsi}}$ is finitely axiomatizable. We also know that $\mathcal{V}_{\mathrm{fsi}}$ is finitely axiomatizable (it is one of our hypotheses).

Let $\tau$ be a sentence that axiomatizes $\mathcal{V}_{\rm fsi}$. Then we have

$$\mathcal{V} \models \sigma \mathbin{⩓\!\!\!⩓} \psi \mathbin{⩓\!\!\!⩓} (\phi \implies \tau).$$

Let $\Gamma$ be a finite set of equations true in $\mathcal{V}$ so that $\Gamma \models \sigma \mathbin{⩓\!\!\!⩓} \psi \mathbin{⩓\!\!\!⩓} (\phi \implies \tau)$ and put $\mathcal{V}' = \operatorname{Mod}\Gamma$. Then $\mathcal{V} \subseteq \mathcal{V}' \subseteq \mathcal{W}$ and $\mathcal{V}_{\rm fsi} = \mathcal{V}'_{\rm fsi}$, so $\mathcal{V} = \mathcal{V}'$. Since $\mathcal{V}'$ is finitely based, we conclude that $\mathcal{V}$ is also. □

We still need to obtain the definite atoms property. The second lemma does that. It traces its heritage ultimately back to Kirby Baker's original proof of Baker's Finite Basis Theorem.

**The Half ♢ Single Sequence Lemma.**
*Every locally finite variety of finite signature with the half ♢-condition and that has bounded critical depth, has the definite atoms property.*

*Proof.* Let $\mathcal{V}$ be a variety satisfying the hypotheses of the lemma. Let $T$ be a finite set of terms witnessing the half ♢-condition for $\mathcal{V}$, and let $\ell$ witness the bounded critical depth property for $\mathcal{V}$. Our aim is to show that $T$ and $\ell$ witness the second stipulation in the definite atoms property for $\mathcal{V}$. Let $\mathbf{A} \in \mathcal{V}$, $a, b \in A$, $\alpha \subseteq \mathrm{Cg}^{\mathbf{A}}(a, b)$, and $(p, q) \in \alpha$ be as in the second stipulation of the definite atoms property.

Pick a congruence $\theta \in \operatorname{Con} \mathbf{A}$ that is maximal with respect to separating $p$ and $q$. Then $\mathbf{A}/\theta$ is subdirectly irreducible and $(p/\theta, q/\theta)$ is a critical pair of $\mathbf{A}/\theta$. Notice that $\theta$ must also separate $a$ and $b$.

Now $(p/\theta, q/\theta) \in \mathrm{Cg}^{\mathbf{A}/\theta}(a/\theta, b/\theta)$. According to Mal'cev there is finite sequence $r'_0, \ldots, r'_k$ in $\mathbf{A}/\theta$ and translations $\lambda'_0, \ldots, \lambda'_k$ such that

$$\begin{gathered} p/\theta = r'_0 \\ \{\lambda'_i(a/\theta), \lambda'_i(b/\theta)\} = \{r'_i, r'_{i+1}\} \text{ for all } i < k \\ r'_k = q/\theta \end{gathered}$$

and, moreover, the complexity of the translations $\lambda'_i$ never exceeds $\ell$. We pick representatives of the congruence classes and denote this by removing the primes. We get

$$\begin{gathered} p \mathrel{\theta} r_0 \\ \{\lambda_i(a), \lambda_i(b)\} \mathrel{\theta} \{r_i, r_{i+1}\} \text{ for all } i < k \\ r_k \mathrel{\theta} q. \end{gathered}$$

To make the middle line of the above display more specific, for each $i < k$ pick $a_i$ and $b_i$ so that $\{a_i, b_i\} = \{a, b\}$ with $\lambda_i(a_i) \mathrel{\theta} r_i$ and $\lambda_i(b_i) \mathrel{\theta} r_{i+1}$. We obtain

$$\begin{gathered} p \mathrel{\theta} r_0 \mathrel{\theta} \lambda_0(a_0) \\ \lambda_1(a_1) \mathrel{\theta} r_1 \mathrel{\theta} \lambda_0(b_0) \\ \lambda_1(b_1) \mathrel{\theta} r_2 \mathrel{\theta} \lambda_2(a_2) \\ \vdots \\ \lambda_{k-1}(b_{k-1}) \mathrel{\theta} r_k \mathrel{\theta} q \end{gathered}$$

where we have made the harmless assumption that $k$ is even. This leads to

$$\begin{gathered} p\ \theta\ \lambda_0(a_0) \\ \lambda_1(a_1)\ \theta\ \lambda_0(b_0) \\ \lambda_1(b_1)\ \theta\ \lambda_2(a_2) \\ \vdots \\ \lambda_{k-1}(b_{k-1})\ \theta\ q. \end{gathered}$$

Consider the *single sequence* made by traversing the display above in a back-and-forth manner:

$$p,\ \lambda_0(a_0),\ \lambda_0(b_0),\ \lambda_1(a_1),\ \lambda_1(b_1),\ \ldots,\ \lambda_{k-1}(a_{k-1}),\ \lambda_{k-1}(b_{k-1}),\ q.$$

According to the half $\diamondsuit$-condition, there is a pair $(c,d)$ of adjacent members of this sequence so that $\{c,d\} \looparrowright^2_1 \{e,f\}$ and $\{p,q\} \looparrowright^1_1 \{e,f\}$. Consider the diagram in Figure 5. There the single sequence is displayed vertically on the left. Notice that every other link indicates a pair in $\theta$. The other links join $\lambda_i(a_i)$ and $\lambda_i(b_i)$, for some $i$.

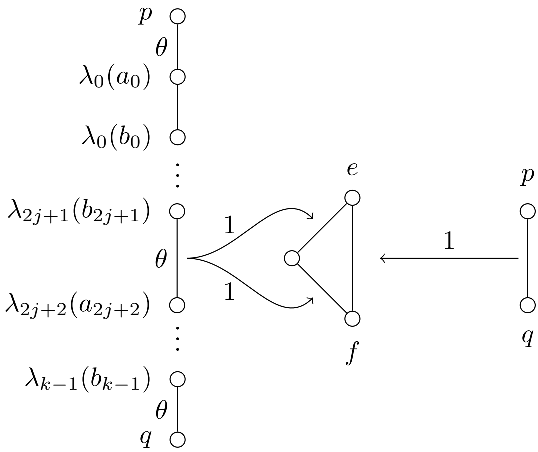



FIGURE 5. Proof of the Definite Atoms Property

First, suppose that the half $\diamondsuit$-condition is witnessed on an edge labelled $\theta$, as shown in the diagram. Then we see that $(e,f) \in \mathrm{Cg}^{\mathbf{A}}(p,q)$. But $\mathrm{Cg}^{\mathbf{A}}(p,q)$ is an atom and $e \neq f$. Hence $(e,f)$ generates this atom. But we also see that $(e,f) \in \theta$. Now $\theta$ lies above the atom. This is impossible, since $\theta$ was chosen to separate $p$ and $q$. So the half $\diamondsuit$-condition must be witnessed by an edge linking $\lambda_i(a_i)$ and $\lambda_i(b_i)$ for some $i$. Then $\lambda_i$ witnesses that $\{a,b\} \looparrowright^1_\ell \{c,d\}$, as desired. This completes the proof of the Half $\diamondsuit$ Single Sequence Lemma. □

The Half $\diamondsuit$ Finite Basis Theorem follows immediately from the Half $\diamondsuit$ Single Sequence Lemma and the Definite Atoms Finite Basis Lemma.

## 5. The variety of vector spaces over a finite field

The proof above of the extension of Willard's Finite Basis Theorem is broken into two parts. The first part is the Definite Atoms Finite Basis Lemma and the second is the Half $\diamondsuit$ Single Sequence Lemma. The Definite Atoms Finite Basis Lemma houses that part of the argument that uses neither the bounded critical depth nor the hypothesis of congruence meet-semidistributivity. In the Half $\diamondsuit$ Single Sequence Lemma, the bound on critical depth is used to bound the parameter $\ell$ in Mal'cev's description of principal congruences, while the congruence meet-semidistributivity is used to reduce the parameter $n$ to 2. But the definite atoms property seems very close to the half $\diamondsuit$-condition. At least in the locally finite case, it might seem equivalent to congruence meet-semidistributivity. But this is not the case.

Given a finite field $\mathbf{K}$, the class of vector spaces over $\mathbf{K}$ is a finitely based variety. In this variety, the subdirectly irreducible algebras coincide with the simple algebras, which are the one dimensional vector spaces. This variety has a finite residual bound and the finitely subdirectly irreducible algebras and the subdirectly irreducible algebras are the same. Here we observe that this variety of vector spaces also has the definite atoms property. The usefulness of this observation lies not in the conclusion that the Definite Atoms Finite Basis Lemma reveals that the variety is finitely based (which we knew already), but that a variety with the definite atoms property need not be congruence meet-semidistributive.

To see why the variety of vector spaces over $\mathbf{K}$ has the definite atoms property, first consider what the translations on a vector space must be. The basic translations either have the form $x + v$ or $\beta x$ where $v$ is an arbitrary vector and $\beta$ is an arbitrary nonzero scalar. A small amount of work shows that any translation is of the form $\beta x + v$. This means that the translations have complexity at most 2. We can reduce this to at most 1 by enhancing the basic operation symbols by treating each term $\beta x + y$ as a basic operation symbol, for each nonzero $\beta \in K$. Considering subspaces in place of congruences, we see that the atoms in the congruence lattice correspond to subspaces of dimension 1 and these are exactly the subspaces corresponding to principal congruences.

Now let $\sigma$ be any elementary sentence axiomatizing the variety of vector spaces over $\mathbf{K}$. Let $\mathbf{A}$ be a vector space over $\mathbf{K}$ and let $a, b, p = r_0, \ldots, r_n = q \in A$ with $p \neq q$ and the sequence $p = r_0, \ldots, r_n = q$ witnessing $(p, q) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$. We have an arbitrary principal congruence sequence. Reason as follows. Pick $i < n$ so that $r_i \neq r_{i+1}$. Next select nonzero scalars $\alpha$ and $\gamma$ so that

$$p - q = \alpha(a - b) \text{ and}$$
$$r_i - r_{i+1} = \gamma(a - b).$$

This leads to $p = \frac{\alpha}{\gamma}(r_i - r_{i+1}) + q$. Let the translation $\mu(x) := \frac{\alpha}{\gamma}(x - r_{i+1}) + q$. Then $\mu(r_i) = p$ and $\mu(r_{i+1}) = q$. We conclude that

$$\{r_i, r_{i+1}\} \looparrowright^1_1 \{p, q\} \;{}^1_1\!\looparrowleft \{p, q\}.$$

Therefore the Half $\diamondsuit$-condition holds for principal congruence sequences. But we also have $\{a,b\} \looparrowright_1^1 \{r_i, r_{i+1}\}$. So the last stipulation of the definition of the Definite Atoms Property is fulfilled with $\ell = 1$, since atoms and principal congruences coincide in this case.

## References

[1] Baker, K.A., McNulty, G.F., Wang, J.: An extension of Willard's finite basis theorem: congruence meet-semidistributive varieties of finite critical depth. Algebra Universalis **52**, 289–302 (2004).

[2] Burris, S.: On Baker's finite basis theorem for congruence distributive varieties. Proc. Amer. Math. Soc. **73**, 141–148 (1979).

[3] Jónsson, B.: Algebras whose congruence lattices are distributive. Mathematica Scandinavica, **21**, 100–121 (1967)

[4] Jónsson, B.: On finitely based varieties of algebras. Colloq. Math. **42**, 255–261 (1979).

[5] Jovanović, J., Marković, P., McKenzie, R., Moore, M.: Optimal strong Mal'cev conditions for congruence meet-semidistributivity in locally finite varieties. Algebra Universalis **76**, 305–325 (2016).

[6] Kozik, M., Krokhin, A., Valeriote, M., Willard, R.: Characterizations of several Maltsev conditions. Algebra Universalis **73**, 205–224 (2015).

[7] Mal'cev, A.I.: On the general theory of algebraic systems. Mat. Sb. N.S. **35(77)**, 3–20 (1954).

[8] Mal'cev, A.I.: On the general theory of algebraic systems. Amer. Math. Soc. Transl. (2) **27**, 125–142 (1963).

[9] Maróti, M., McKenzie, R.: Finite basis problems and results for quasivarieties. Studia Logica **78**(1-2), 293–320 (2004).

[10] Willard, R.: A finite basis theorem for residually finite, congruence meet-semidistributive varieties. J. Symbolic Logic **65**, 187–200 (2000).

George F. McNulty
Department of Mathematics
University of South Carolina
Columbia, SC 29208
USA
URL: `http://people.math.sc.edu/mcnulty/`
e-mail: `mcnulty@math.sc.edu`

Ross Willard
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
Canada
URL: `http://www.math.uwaterloo.ca/~rdwillar/`
e-mail: `ross.willard@uwaterloo.ca`