Not every full duality is strong!

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Abstract. Is every full duality strong? We show that the answer is 'no', thereby answering a question that goes back to the earliest foundations of the theory of natural dualities.

Let $\underline{\mathbf{M}}$ be a finite algebra and let $\underset{\sim}{\mathbf{M}}$ be a structure with (possibly partial) operations and relations that has the same underlying set as $\underline{\mathbf{M}}$ and carries the discrete topology. Then the algebra $\underline{\mathbf{M}}$ and its 'alter ego' $\underset{\sim}{\mathbf{M}}$ yield a **duality** if, in a canonical sense, every member $\mathbf{A}$ of the quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}\,\underline{\mathbf{M}}$ generated by $\underline{\mathbf{M}}$ is isomorphic to the algebra of all continuous homomorphisms into $\underset{\sim}{\mathbf{M}}$ from some member $\mathbf{X}$ of the topological quasi-variety $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ generated by $\underset{\sim}{\mathbf{M}}$. For example, let $\underline{\mathbf{2}} = \langle 2; \vee, \wedge, ', 0, 1\rangle$ and $\underset{\sim}{\mathbf{2}} = \langle 2; \mathcal{T}\rangle$ be the two-element Boolean algebra and two-element discrete space respectively. The quasi-variety generated by $\underline{\mathbf{2}}$ is the class $\boldsymbol{\mathcal{B}}$ of all Boolean algebras, while the topological quasi-variety generated by $\underset{\sim}{\mathbf{2}}$ is the class $\boldsymbol{\mathfrak{Z}}$ of all Boolean spaces. Since every $\mathbf{A} \in \boldsymbol{\mathcal{B}}$ is, by Stone duality, canonically isomorphic to the algebra of all clopen subsets of some $\mathbf{X} \in \boldsymbol{\mathfrak{Z}}$ and hence to the algebra of all continuous maps from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$, it follows that $\underline{\mathbf{2}}$ and $\underset{\sim}{\mathbf{2}}$ yield a duality in the above sense.

Better yet, $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a **full duality** if they yield a duality and, in a canonical sense, the association between $\mathbf{A}$ and $\mathbf{X}$ is a dual equivalence between the quasi-variety $\boldsymbol{\mathcal{A}}$ and the dual category $\boldsymbol{\mathfrak{X}}$. For example, the variety of Boolean algebras is dually equivalent by Stone duality to the category of Boolean spaces, and so $\underline{\mathbf{2}}$ and $\underset{\sim}{\mathbf{2}}$ yield a full duality. (See Clark and Davey [2], 1998, for a detailed account.)

These notions generalizing Stone duality for Boolean algebras were introduced by Davey and Werner [10], 1983. They developed the foundations of a general theory of natural dualities, and observed that both duality and full duality often require the injectivity of the alter ego $\underset{\sim}{\mathbf{M}}$ in special subclasses of $\boldsymbol{\mathfrak{X}}$. Examining the literature they found that, in every known full duality, the necessary instances of injectivity had been established by verifying a seemingly stronger condition, namely, that $\underset{\sim}{\mathbf{M}}$ is injective in (all of) $\boldsymbol{\mathfrak{X}}$.

In a parallel development, Clark and Krauss [4], 1984, isolated a condition which guarantees that a duality is full. For a non-empty set S, they defined a subset $X \subseteq M^S$ to be **hom-closed** if, for every set I and every subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}^I$, the set X is closed under every homomorphism from $\mathbf{A}$ to $\underline{\mathbf{M}}$ applied pointwise on X. This led them to the following characterization of full duality [2, 3.1.7].

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Full Duality Theorem. *Assume that* $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yield a duality. Then* $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yield a full duality if and only if every topologically closed substructure of a non-zero power of* $\underset{\sim}{\mathbf{M}}$ *is isomorphic to a hom-closed substructure of a non-zero power of* $\underset{\sim}{\mathbf{M}}$*.*

Examining the literature, Clark and Krauss discovered that every known full duality satisfied a seemingly stronger condition, namely, that every topologically closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$ is (as it stands) hom-closed.

A decade later, Clark and Davey [1], 1995, defined a **strong duality** to be a duality in which every topologically closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$ is hom-closed. Reviewing the intervening literature, they confirmed that every subsequently discovered full duality was strong and had an alter ego that was injective in the dual category. They then connected these conditions by proving that a duality is strong precisely when it is full and the alter ego is injective in the dual category [2, 3.2.4]. This development focused attention on the question we address in this note.

Full-vs-Strong Duality Question. *Is every full duality strong?*

Clark and Davey reiterated this question in their 1998 monograph [2, 3.2.7], where they describe it as "one of the oldest and most tantalising open questions in the foundations of duality theory." But it took yet another full decade before real progress on this question began to emerge.

An important tool for recognizing many natural dualities was discovered by Zadori [16], 1995, and independently by Willard, though published four years later [15], 1999. (See [2, 2.2.11].) We say that $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ yield a duality **at the finite level** if the canonical isomorphism exists for every finite member $\mathbf{A}$ of $\boldsymbol{\mathcal{A}}$.

Duality Compactness Theorem. *Assume that* $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yield a duality at the finite level and that* $\underset{\sim}{\mathbf{M}}$ *has finite type. Then* $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yield a duality.*

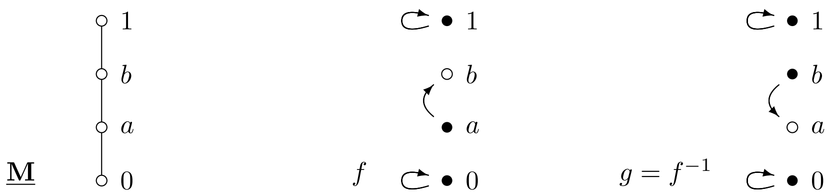
Progress toward a solution to the full-vs-strong question began when Davey, Haviar and Willard [9], 2005, found an example of an alter ego $\underset{\sim}{\mathbf{3}}$ for the three-element lattice $\underline{\mathbf{3}}$ that yields a duality that is full but not strong *at the finite level*, although it fails to be full beyond the finite level. More recently, Davey, Haviar, Niven and Perkal [8], 2005, showed that a similar example can be produced with $\underline{\mathbf{3}}$ replaced by any finite non-Boolean distributive lattice. These examples have the interesting consequence that the Duality Compactness Theorem does not extend to full duality. Nevertheless, Davey [6], 2006, found that a slightly restricted version does extend.

Full Duality Compactness Theorem. *Assume that* $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yield a full duality at the finite level and that* $\underset{\sim}{\mathbf{M}}$ *has finite type. If* $\underset{\sim}{\mathbf{M}}$ *has no partial operations in its type, then* $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yield a full duality.*

This offers us a strategy for proving that a duality is full without proving that it is strong. For example, we can apply the finite-level version of the Full Duality Theorem [2, 3.1.7] to upgrade a duality at the finite level to a full duality at the finite level by checking only finite powers. Then use the previous theorem to lift to a full duality.

While these results seem to suggest a negative answer to the full-vs-strong question, a recent study by Davey, Haviar and Niven [7], 2006, suggests just the opposite. These authors address the question by restricting the choice of $\underline{\mathbf{M}}$. They prove that every full duality is strong if $\underline{\mathbf{M}}$ is either an abelian group, a semilattice, a relative Stone Heyting algebra or a bounded distributive lattice. After all of this uncertainty, we can at last resolve this nagging question.

Let $\underline{\mathbf{M}} = \langle M; \vee, \wedge, 0, 1, t\rangle$, where $\langle M; \vee, \wedge, 0, 1\rangle$ is the four-element bounded chain and t is the ternary discriminator operation. The quasi-primal algebra $\underline{\mathbf{M}}$ has exactly two non-identity partial automorphisms, f and g, as shown in the following diagram.



So the Quasi-primal Strong Duality Theorem [2, 3.3.13] tells us that $\underline{\mathbf{M}}$ and its alter ego $\underset{\sim}{\mathbf{M}}_1 := \langle M; f, g, \mathcal{T}\rangle$ yield a strong duality. Now define $r := \operatorname{graph}(f)$ and define the alter ego

$$\underset{\sim}{\mathbf{M}}_2 := \langle M; r, \mathcal{T}\rangle.$$

The quite contrary action of replacing an operation by its graph creates our long-sought-after example.

The Example. $\underline{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}_2$ *yield a duality that is full but not strong.*

Proof. As remarked above, $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}_1$ yield a strong duality. The relation r entails f and g; use the constructs (4) Permutation and (10) Graph from Theorem 2.4.6 [2]. Therefore $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}_2$ yield a duality. But this duality is not strong, since any subset of a power of M that is not closed under f and g forms a substructure that is not hom-closed.

It remains to show that the duality induced by $\underset{\sim}{\mathbf{M}}_2$ is full. Since this is the first time that a duality has been shown to be full without proving that it is in fact strong, we present two different versions of the proof. The first uses the Full Duality Theorem and, optionally, avoids mention of topology altogether by appealing to the Full Duality Compactness Theorem. It gives the first application of the Full Duality Theorem with an isomorphism other than an identity map. The second is a direct proof from the definition of full duality.

Hom-Closure Proof. Let $\mathbf{X}$ be a substructure of $\underset{\sim}{\mathbf{M}}_2^S$, for some finite non-empty set S. Define $\psi : X \to M^S \times M^2$ by

$$\psi(x) := \begin{cases} (x, a, b) & \text{if } x \notin \pi_1(r^{\mathbf{X}}) \cup \pi_2(r^{\mathbf{X}}), \\ (x, 0, 0) & \text{if } x \in \pi_1(r^{\mathbf{X}}) \cup \pi_2(r^{\mathbf{X}}). \end{cases}$$

Clearly $Z := \psi(X)$ forms a substructure of $\underset{\sim}{\mathbf{M}}_2^S \times \underset{\sim}{\mathbf{M}}_2^2$ and $\psi : \mathbf{X} \to \mathbf{Z}$ is an isomorphism. Moreover, $\mathbf{Z}$ is closed under f and g, and is therefore hom-closed, as $\underset{\sim}{\mathbf{M}}_1$ strongly dualises $\underline{\mathbf{M}}$. By the finite-level version of the Full Duality Theorem, $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}_2$ yield a full duality at the finite level. As $\underset{\sim}{\mathbf{M}}_2$ is purely relational, the Full Duality Compactness Theorem implies that $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}_2$ yield a full duality.

Remark. At the expense of making the index set more complicated, we can extend this proof to closed substructures of arbitrary powers of $\underset{\sim}{\mathbf{M}}_2$. Then the Full Duality Theorem applies and we don't need to appeal to the Full Duality Compactness Theorem. Indeed, let $\mathbf{X}$ be a closed substructure of $\underset{\sim}{\mathbf{M}}_2^S$, for some non-empty set S. Let T be any family of non-empty clopen subsets of $\mathbf{X}$ that are disjoint from both $\pi_1(r^{\mathbf{X}})$ and $\pi_2(r^{\mathbf{X}})$ and cover $X \backslash (\pi_1(r^{\mathbf{X}}) \cup \pi_2(r^{\mathbf{X}}))$. Define $\psi : X \to M^S \times (M^2)^T$ by $\psi(x) = (x, \gamma_x)$, where $\gamma_x : T \to M^2$ is given by $\gamma_x(U) := (a, b)$, if $x \in U$, and $\gamma_x(U) := (0, 0)$, if $x \notin U$. The reader can readily verify that ψ is an isomorphism onto a topologically closed substructure of $\underset{\sim}{\mathbf{M}}_2^S \times (\underset{\sim}{\mathbf{M}}_2^2)^T$ that is closed under f and g.

Direct Proof. Let $\mathbf{X}$ be a topologically closed substructure of $\underset{\sim}{\mathbf{M}}_2^S$, for some non-empty set S. We must show that $\varepsilon_{\mathbf{X}} : \mathbf{X} \to DE(\mathbf{X})$ is surjective. Clearly $\varepsilon_{\mathbf{X}}$ is the empty isomorphism when $X = \varnothing$, so assume $X \neq \varnothing$. Now let $h : E(\mathbf{X}) \to \underline{\mathbf{M}}$. We must find $y \in X$ such that $h = \varepsilon_{\mathbf{X}}(y)$. By Jónsson's Lemma, there is an ultrafilter U on X such that $\vartheta_U \subseteq \ker(h)$, where $\vartheta_U := \theta_U{\restriction}_{E(\mathbf{X})}$ and θ_U is the congruence on $\underline{\mathbf{M}}^X$ determined by the ultrafilter U. Since $h(E(\mathbf{X})) \leqslant \underline{\mathbf{M}}$, we have $\ker(h) \neq \mathbf{1}$. Then $E(\mathbf{X})/\vartheta_U \leqslant \underline{\mathbf{M}}^X/\theta_U \cong \underline{\mathbf{M}}$, since $\underline{\mathbf{M}}$ is finite, and therefore $E(\mathbf{X})/\vartheta_U$ is simple so $\vartheta_U = \ker(h)$.

Let $\widehat{U} := \{\, Y \in U \mid Y \text{ is clopen in } \mathbf{X} \,\}$. Then $\widehat{U}$ is an ultrafilter of the Boolean algebra of clopen subsets of $\mathbf{X}$. By Stone duality, there exists $z \in X$ such that $\widehat{U}$ consists of the clopen subsets of X that contain z. For α, β in $E(\mathbf{X})$, the set $\operatorname{eq}(\alpha, \beta) := \{\, x \in X \mid \alpha(x) = \beta(x) \,\}$ is clopen and therefore

$$\begin{aligned} h(\alpha) = h(\beta) &\iff (\alpha, \beta) \in \ker(h) = \vartheta_U \iff \operatorname{eq}(\alpha, \beta) \in U \\ &\iff \operatorname{eq}(\alpha, \beta) \in \widehat{U} \iff \alpha(z) = \beta(z) \\ &\iff \varepsilon_{\mathbf{X}}(z)(\alpha) = \varepsilon_{\mathbf{X}}(z)(\beta). \end{aligned}$$

Thus $\ker(h) = \ker(\varepsilon_{\mathbf{X}}(z))$. Let $\mathbf{M}_h := h(E(\mathbf{X}))$ and $\mathbf{M}_z := \varepsilon_{\mathbf{X}}(z)(E(\mathbf{X}))$. Then there is an isomorphism $k : \mathbf{M}_z \to \mathbf{M}_h$ such that $h = k \circ \varepsilon_{\mathbf{X}}(z)$. As $\mathbf{M}_h, \mathbf{M}_z \leqslant \underline{\mathbf{M}}$, the map k can only be f, g or an identity map. If $k \subseteq \mathrm{id}_M$, then $h = \varepsilon_{\mathbf{X}}(z)$. Thus, by symmetry, we may assume that $k = f$. Then $M_z = \{0, a, 1\}$ and $M_h = \{0, b, 1\}$.

Before we complete the proof, we study the structure of $E(\mathbf{X})$ more closely. As $\mathbf{X}$ satisfies the universal Horn theory of $\underset{\sim}{\mathbf{M}}_2$, the following sentences are true in $\mathbf{X}$:

$$\begin{aligned} &\forall uvw([r(u, v) \ \&\ r(u, w)] \to v = w), \\ &\forall uvw([r(u, w) \ \&\ r(v, w)] \to u = v), \\ &\forall uvw([r(u, v) \ \&\ r(v, w)] \to u = v = w). \end{aligned}$$

Hence $r^{\mathbf{X}}$ is the graph of an injective unary operation F on X (possibly partial, possibly empty). Moreover, if we let

$$U_{01} := \{x \in \mathrm{dom}(F) \mid F(x) = x\}, \quad U_a := \mathrm{dom}(F)\backslash U_{01} \text{ and } U_b := F(U_a),$$

then U_{01}, U_a and U_b are pairwise disjoint. Let $V := X\backslash(U_{01} \cup U_a \cup U_b)$. Since $r^{\mathbf{X}}$ is topologically closed in $\mathbf{X}$, the set $V = X\backslash(\pi_1(r^{\mathbf{X}}) \cup \pi_2(r^{\mathbf{X}}))$ is open in $\mathbf{X}$.

We can then describe $E(\mathbf{X})$ as the set of all continuous maps $\alpha : X \to M$ such that $\alpha(U_{01}) \subseteq \{0, 1\}$, $\alpha(U_a) \subseteq \{0, a, 1\}$ and $\alpha(F(x)) = f(\alpha(x))$, for all $x \in U_a$. Since V is open, it is easy to see that $\varepsilon_{\mathbf{X}}(u)(E(\mathbf{X})) = M$, for each $u \in V$. Since $M_z = \{0, a, 1\}$, we conclude that $z \in U_a$. Now let $\alpha \in E(\mathbf{X})$. Since $z \in U_a$, we have $f(\alpha(z)) = \alpha(F(z))$, so

$$h(\alpha) = f(\varepsilon_{\mathbf{X}}(z)(\alpha)) = f(\alpha(z)) = \alpha(F(z)) = \varepsilon_{\mathbf{X}}(F(z))(\alpha).$$

Thus $h = \varepsilon_{\mathbf{X}}(F(z))$. □

The preceding proof of the existence of $z \in X$ such that h factors through $\varepsilon_{\mathbf{X}}(z)$ goes back to Davey [5, Prop. 3.2], 1976.

Note that, as with all previous finite-level full but not strong dualities [9, 8, 7], our example draws on the existence of a relatively straightforward strong duality to create a less straightforward full but not strong one. Our example does not settle the following problem, which at the moment appears to be much more difficult.

Full-vs-Strong Dualisability Question. *Is every fully dualisable algebra strongly dualisable? That is, if a finite algebra has an alter ego with which it yields a full duality, must it also have an alter ego with which it yields a strong duality?*

We can only say that the evidence accumulated so far is consistent with a positive answer to this question. In many classes of algebras, we know that every dualisable algebra is strongly dualisable; for example, the class of finite algebras that generate a congruence-distributive variety [1], the variety of commutative rings with identity [3], and the classes of graph algebras and flat graph algebras [14]. In such classes the answer to this question is trivially 'yes'. But the answer is also 'yes' in the class consisting of three-element unary algebras [11, 12, 13], a class in which not every duality can be upgraded to a strong duality.

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