The lattice of alter egos

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Abstract

We introduce a new Galois connection for partial operations on a finite set, which induces a natural quasi-order on the collection of all partial algebras on this set. The quasi-order is compatible with the basic concepts of natural duality theory, and we use it to turn the set of all alter egos of a given finite algebra into a doubly algebraic lattice. The Galois connection provides a framework for us to develop further the theory of natural dualities for partial algebras. The development unifies several fundamental concepts from duality theory and reveals a new understanding of full dualities, particularly at the finite level.

The Galois connection of clone theory (between operations and relations) provides a natural framework for comparing and studying the algebras on a fixed finite set. We start this paper by introducing a new Galois connection that we will use as a framework for comparing and studying the partial algebras on a fixed finite set.

Our motivation for studying partial algebras comes from the theory of natural dualities. In natural duality theory, one typically starts with a fixed finite algebra $\mathbf{M}$ and considers *alter egos* of $\mathbf{M}$, which are discretely topologised structures $\underset{\sim}{\mathbf{N}}$ on the same set as $\mathbf{M}$ but whose type may include finitary relations and partial operations. The structure of an alter ego $\underset{\sim}{\mathbf{N}}$ is required to be 'compatible' with the operations of $\mathbf{M}$. The aim is to find an alter ego of the algebra $\mathbf{M}$ that can be used to build a category that is dually equivalent to the quasi-variety generated by $\mathbf{M}$. A classic example is Priestley duality [**23**], which is based on the two-element bounded lattice $\mathbf{M} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ and the two-element chain with the discrete topology $\underset{\sim}{\mathbf{N}} = \langle \{0,1\}; \leqslant, \mathcal{T}\rangle$. This pair yields a dual equivalence between bounded distributive lattices and Priestley spaces and, at the finite level, a dual equivalence between finite bounded distributive lattices and finite ordered sets.

The new Galois connection induces a quasi-order on the collection of all alter egos of a finite algebra $\mathbf{M}$, under which they form a doubly algebraic lattice $\mathcal{A}_{\mathbf{M}}$. Our aim is to understand how the basic concepts of duality theory sit within this lattice. This aim leads us on a new path through the foundations of natural duality theory. In particular, we further develop the basic theory of natural dualities for partial algebras. Through this development, we hope to bring out the similarities between duality theory and clone theory. We highlight the key role of entailment in duality theory, and the link with entailment in clone theory.

In order to describe our duality-theoretic results more precisely, we require some specialised terminology. These terms will be defined in Section 3, before they are needed in the main body of the paper.

The way that duality and strong duality sit within the lattice of alter egos $\mathcal{A}_{\mathbf{M}}$ is essentially already known. For any finite algebra $\mathbf{M}$, it is easy to use the basic theory to show that:

– the alter egos that yield a finite-level duality form a principal filter of the lattice $\mathcal{A}_{\mathbf{M}}$ and, if any of these alter egos yield a duality, then they all do (4.1);

2000 *Mathematics Subject Classification* 08A05 (primary), 06A15, 08A55, 08C15 (secondary).

The second author was supported by ARC Discovery Project Grant DP0556248 and the third author by a Discovery Grant from NSERC, Canada.

– the alter egos that yield a finite-level strong duality form the top element of the lattice $\mathcal{A}_{\mathbf{M}}$ (4.6), and so there is essentially only one candidate alter ego for strong duality.

In contrast, our understanding of full dualities has previously been fairly sketchy. From our new perspective, we will find that full dualities become less mysterious, especially at the finite level. We can give a simple and natural intrinsic description of when an alter ego yields a finite-level full duality (4.3).

However, our new perspective also highlights the differences between full duality and the better behaved concepts of duality and strong duality. In particular, the alter egos that yield a full duality do not necessarily form an increasing subset of $\mathcal{A}_{\mathbf{M}}$. In other words, by enriching the structure of an alter ego of $\mathbf{M}$, we can destroy a full duality. We can nevertheless completely characterise when enrichment will not destroy full duality (5.3), and thus show that:

– the alter egos that yield a finite-level full duality form a complete sublattice $\mathcal{F}_{\mathbf{M}}$ of the lattice of alter egos $\mathcal{A}_{\mathbf{M}}$, and those that yield a full duality form an increasing subset of $\mathcal{F}_{\mathbf{M}}$ (5.5).

We do not know in general whether the alter egos that yield a full duality must form a filter of $\mathcal{F}_{\mathbf{M}}$.

We finish with two applications of our results to specific examples. First we consider the four-element quasi-primal algebra $\mathbf{R}$ of Clark, Davey and Willard [**4**], which provided the first known example of a full but not strong duality. We draw the lattice of all 17 full dualities based on $\mathbf{R}$ (5.7). We then give a short proof of a result due to Davey, Haviar, Niven and Perkal [**8**]: every finite non-boolean distributive lattice has a duality that is full but not strong at the finite level (5.10).

Our development of duality theory for *partial algebras* comes from our desire to understand full dualities for *algebras*. We will find that the symmetry inherent in this approach provides additional insights. For example, we shall see the sense in which the conditions (IC) and (FTC) are dual to each other (4.5), and the sense in which the Structural Entailment and the Closure Theorems are the same (3.8, 3.9).

NOTE. We do not want the development of our general theory to be bogged down by 'trivial' technicalities. So we work within a somewhat restricted setting. For example, when Priestley duality is recast in our setting, the one-element lattice is removed from the usual algebraic category and the empty Priestley space is removed from the usual topological category. While this is a very minor change, still we might prefer not to make it in practice. The setting that we use in this paper is chosen to smooth the development of the general theory, the interesting part of which is certainly at the non-trivial level. In an appendix we explain how any particular duality formulated in our setting can easily be reformulated in various alternative (but essentially equivalent) settings.

1. *A Galois connection for partial operations*

Fix a finite non-empty set M. We shall define a *partial operation* on M to be a map $f : r \to M$ with an associated arity $k \geqslant 1$, where $\varnothing \neq r \subseteq M^k$. (In other words, we consider finitary, non-nullary partial operations with non-empty domains.) Similarly, we define a *relation* on M to be a set r with an associated arity $k \geqslant 1$, where $\varnothing \neq r \subseteq M^k$. Now let $\mathcal{P}_M$ denote the set of all partial operations on M, and let $\mathcal{R}_M$ denote the set of all relations on M.

NOTE 1.1. We choose to exclude partial operations that are nullary or empty in order to smooth the development of the duality theory in later sections. See the appendix for a discussion of three alternative (but essentially equivalent) settings.

$$\begin{array}{ccccccc}
\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1\ell} \\ a_{21} & a_{22} & \cdots & a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{k\ell} \end{array}\right] & & & & \begin{array}{c} \to \\ \to \\ \\ \to \end{array} & \begin{array}{c} g(r_1) \\ g(r_2) \\ \vdots \\ g(r_k) \end{array} \\
\downarrow & \downarrow & & \downarrow & & \Downarrow \\
f(c_1) & f(c_2) & \cdots & f(c_\ell) & \Rightarrow & f\big(g(r_1),\ldots,g(r_k)\big) = g\big(f(c_1),\ldots,f(c_\ell)\big)
\end{array}$$

FIGURE 1.1. *Compatible partial operations f and g*

DEFINITION 1.2. Let f and g be partial operations on M, with arities $k, \ell \geqslant 1$. We say that f and g are *compatible* provided the following condition holds, for each $k \times \ell$ matrix A of elements of M:

– if each row r_i of A is in $\mathrm{dom}(g)$ and each column c_j of A is in $\mathrm{dom}(f)$, then
  * $\big(g(r_1),\ldots,g(r_k)\big) \in \mathrm{dom}(f)$,
  * $\big(f(c_1),\ldots,f(c_\ell)\big) \in \mathrm{dom}(g)$, and
  * $f\big(g(r_1),\ldots,g(r_k)\big) = g\big(f(c_1),\ldots,f(c_\ell)\big)$;

see Figure 1.1. Compatibility of partial operations is symmetric.

The following basic lemma gives a few alternative formulations of compatibility.

LEMMA 1.3. *Let f and g be partial operations on a finite non-empty set M. Then the following are equivalent*:

(i) *f and g are compatible*;
(ii) *f preserves* $\mathrm{dom}(g)$ *and* $\mathrm{graph}(g)$, *and g preserves* $\mathrm{dom}(f)$ *and* $\mathrm{graph}(f)$;
(iii) *f preserves* $\mathrm{dom}(g)$, *and g preserves* $\mathrm{graph}(f)$;
(iv) *g is a homomorphism with respect to the partial algebra $\langle M; f\rangle$.*

When interpreted in familiar settings, this is the familiar notion of compatibility. For example, take a total operation $f : M^k \to M$ and a relation $r \subseteq M^\ell$, with $k, \ell \geqslant 1$. Then f preserves the relation r if and only if f is compatible with the restricted projection operation $\pi_1{\restriction}_r$.

DEFINITION 1.4. For each set F of partial operations on M, define

$$F^\Diamond := \big\{\, g \in \mathcal{P}_M \;\big|\; (\forall f \in F)\ f \text{ and } g \text{ are compatible} \,\big\}.$$

Then the map $F \mapsto F^\Diamond$ is an order-reversing Galois connection from the power-set lattice $\wp(\mathcal{P}_M)$ to itself.

We seek a general description of the closure $F^{\Diamond\Diamond}$ of a set of partial operations $F \subseteq \mathcal{P}_M$. This task breaks up into two parts. By Lemma 1.3, a partial operation $g \in \mathcal{P}_M$ belongs to $F^{\Diamond\Diamond}$ if and only if

(†) g preserves the graph of every partial operation in $F^\Diamond$, and
(‡) $\mathrm{dom}(g)$ is closed under every partial operation in $F^\Diamond$.

We will see that these two conditions lead to three constructs for building the closure $F^{\Diamond\Diamond}$. The first two constructs are adding the projections and composing operations.

NOTATION 1.5. Let Π denote the set of all total projection operations on M. For $F \subseteq \mathcal{P}_M$, we let $\mathrm{Clo_p}(F)$ denote the *partial clone* on M generated by F: it is the smallest subset of $\mathcal{P}_M$ containing $F \cup \Pi$ that is closed under composition (with maximum, non-empty domain). This corresponds to the usual definition of a partial clone, except that we exclude empty operations.

The following lemma describes the partial operations satisfying condition (†). Within clone theory, this lemma essentially dates back to Geiger [**16**]. Within duality theory, it was used by Davey, Haviar and Priestley [**10**] to describe hom-entailment (see [**3**, 9.4.1]).

For $F \subseteq \mathcal{P}_M$ and $R \subseteq \mathcal{R}_M$, we introduce the notation

$$F{\restriction}R := \big\{\, f{\restriction}_r \;\big|\; f \in F \text{ and } r \in R \text{ with } r \subseteq \mathrm{dom}(f) \,\big\}.$$

For a single relation r on M, we can view the subset $\mathrm{Clo_p}(F){\restriction}\{r\}$ of M^r as a $|r|$-ary relation on M, and this relation is closed under every partial operation in F.

LEMMA 1.6. *Let $F \cup \{g\}$ be a set of partial operations on a finite non-empty set M. Then the following are equivalent*:

(i) *every relation on M that is closed under all $f \in F$ is also closed under g*;
(ii) *the relation $\mathrm{Clo_p}(F){\restriction}\{r\}$ on M is closed under g, where $r := \mathrm{dom}(g)$*;
(iii) *g has an extension in $\mathrm{Clo_p}(F)$*;
(iv) *g preserves the graph of every partial operation in $F^\Diamond$.*

Proof. The three implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (i) are straightforward. Now let $g : r \to M$ have arity $k \geqslant 1$ and define $r^* := \mathrm{Clo_p}(F){\restriction}\{r\}$. We assume (ii) to prove (iii). The projections $\rho_1, \ldots, \rho_k : r \to M$ are elements of the relation $r^* \subseteq M^r$, with $(\rho_1, \ldots, \rho_k) \in r^{M^r}$. Since we are assuming that r^* is closed under g, we get $g = g^{M^r}(\rho_1, \ldots, \rho_k) \in r^*$. Thus g has an extension in $\mathrm{Clo_p}(F)$.

Now assume (iv) to prove (ii). Choose $a \in r$. Then the projection $\rho_a : r^* \to M$ is in $F^\Diamond$. So g preserves $\mathrm{graph}(\rho_a)$, and it follows that g preserves $\mathrm{dom}(\rho_a) = r^*$. □

The third construct for building the closure $F^{\Diamond\Diamond}$ is restriction of domains to 'definable' relations.

NOTATION 1.7. Let $F \subseteq \mathcal{P}_M$. We will view F as a set of concrete partial operations on M and also as a set of abstract partial operation symbols with associated arities. So we can consider the partial algebra $\mathbf{M} = \langle M; F\rangle$ of type F. For a short discussion of first-order logic for partial algebras, see [**3**, pp. 24–25].

We shall call a conjunction of atomic formulæ $\Psi(\vec{v}) = [\psi_1(\vec{v}) \;\&\; \cdots \;\&\; \psi_n(\vec{v})]$ a *conjunct-atomic formula*. We say that a k-ary relation r on M is *conjunct-atomic definable* from F if it is described in $\mathbf{M}$ by a k-variable conjunct-atomic formula $\Psi(\vec{v})$ of type F, that is, if

$$r = \big\{\, (a_1, \ldots, a_k) \in M^k \;\big|\; \Psi(a_1, \ldots, a_k) \text{ is true in } \mathbf{M} \,\big\}.$$

We define $\mathrm{Def_{ca}}(F)$ to be the set of all relations on M that are conjunct-atomic definable from F. Note that $\mathrm{Def_{ca}}(F)$ consists of all relations on M that can be obtained as a finite intersection of equalisers of partial operations in $\mathrm{Clo_p}(F)$.

DEFINITION 1.8. A *primitive positive* formula is of the form $\exists\vec{w}\,\Psi(\vec{v}, \vec{w})$, where $\Psi(\vec{v}, \vec{w})$ is a conjunct-atomic formula. For sets of partial operations F_1 and F_2 on M, we say that a k-ary relation r on M is *primitive-positive definable* from F_1 with *existence witnessed* by F_2 if

– there is a $(k+\ell)$-variable conjunct-atomic formula $\Psi(\vec{v},\vec{w})$ of type F_1, for some $\ell \geqslant 0$, such that $\exists \vec{w}\, \Psi(\vec{v},\vec{w})$ describes the relation r on M, and
– there is an ℓ-tuple $\vec{\tau}$ of k-ary terms of type F_2 such that $\Psi(\vec{v},\vec{\tau}(\vec{v}))$ describes the relation r on M.

The proof of the following lemma is a straightforward modification of the original proof of the Dual Entailment Theorem 3.6 [**10**]. We will use the lemma in this section to describe the partial operations satisfying condition (‡). In Section 5, the lemma will be used to obtain a common generalisation of two entailment theorems from duality theory.

Lemma 1.9. *Let F_1 and F_2 be sets of partial operations on a finite non-empty set M such that $F_1 \subseteq F_2$, and let r be a relation on M. Then the following are equivalent*:
(i) *r is closed under every partial operation in $F_1^{\Diamond}$ with domain closed under all partial operations in F_2*;
(ii) *r is closed under every partial operation in $F_1^{\Diamond}$ with domain $\mathrm{Clo_p}(F_2)\restriction\{r\}$*;
(iii) *r is primitive-positive definable from F_1 with existence witnessed by F_2.*

Proof of (ii)$\Rightarrow$(iii). Assume that (ii) holds. The relation $r^* := \mathrm{Clo_p}(F_2)\restriction\{r\} \subseteq M^r$ contains the projections $\rho_1,\dots,\rho_k : r \to M$, where r has arity $k \geqslant 1$. Enumerate the elements of $r^*\backslash\{\rho_1,\dots,\rho_k\}$ as $h_1,\dots,h_\ell$, where $\ell \geqslant 0$. Each $h_j : r \to M$ has an extension in $\mathrm{Clo_p}(F_2)$, and so there is a k-ary term τ_j of type F_2 with $h_j = \tau_j^M\restriction_r$. This gives us our required ℓ-tuple of terms $\vec{\tau} := (\tau_1,\dots,\tau_\ell)$.

We next choose a finite subset F_0 of F_1. For each partial operation $g : r^* \to M$ that is not in $F_1^{\Diamond}$, choose some $f_g \in F_1$ such that f_g and g are not compatible. Define F_0 to be the set of all these partial operations f_g. Then F_0 is finite, as M is finite.

The relation $r^* = \mathrm{Clo_p}(F_2)\restriction\{r\} = \{\rho_1,\dots,\rho_k,h_1,\dots,h_\ell\}$ is closed under F_0. So we can construct a $(k+\ell)$-variable conjunct-atomic formula $\Psi(\vec{v},\vec{w})$ of type F_1 describing, via the correspondences $v_i \leftrightarrow \rho_i$ and $w_j \leftrightarrow h_j$,
– which of the projections $\rho_1,\dots,\rho_k$ in r^* are equal, and
– how each partial operation $f \in F_0$ acts pointwise on $r^* \subseteq M^r$.

Now define two k-ary relations on M:
– let $s_\exists$ be described by the formula $\exists\vec{w}\ \Psi(\vec{v},\vec{w})$, and
– let s_τ be described by the formula $\Psi(\vec{v},\vec{\tau}(\vec{v}))$.

To see that (iii) holds, it remains only to show that $r = s_\exists = s_\tau$. Since $s_\tau \subseteq s_\exists$, it is enough to show that $r \subseteq s_\tau$ and $s_\exists \subseteq r$.

Let $\vec{a} = (a_1,\dots,a_k) \in r$. To check that $\vec{a} \in s_\tau$, we need to show that $\Psi(\vec{a},\vec{\tau}(\vec{a}))$ is true in M. By the construction of Ψ, we have

$$\begin{aligned} & \Psi(\rho_1,\dots,\rho_k,h_1,\dots,h_\ell) \text{ in } M^r, \\ \Longrightarrow\ & \Psi\big(\rho_1(\vec{a}),\dots,\rho_k(\vec{a}),h_1(\vec{a}),\dots,h_\ell(\vec{a})\big) \text{ in } M, \\ \Longrightarrow\ & \Psi\big(a_1,\dots,a_k,\tau_1^M(\vec{a}),\dots,\tau_\ell^M(\vec{a})\big) \text{ in } M. \end{aligned}$$

So $\vec{a} \in s_\tau$, whence $r \subseteq s_\tau$.

Finally, to see that $s_\exists \subseteq r$, let $\vec{a} \in s_\exists$. There exists $\vec{c} \in M^\ell$ such that $\Psi(\vec{a},\vec{c})$ is true in M. Since any equalities amongst the projections $\rho_1,\dots,\rho_k$ are expressed in the formula Ψ, we can define the partial operation $g : r^* \to M$ by $g(\rho_i) := a_i$ and $g(h_j) := c_j$.

As $\Psi(\vec{a},\vec{c})$ holds in M, the construction of Ψ ensures that g preserves $\mathrm{graph}(f)$, for all $f \in F_0$. Since $\mathrm{dom}(g) = r^*$ is closed under F_0, it follows by Lemma 1.3 that $g \in F_0^{\Diamond}$. The choice of F_0 now guarantees that $g \in F_1^{\Diamond}$. By (ii), the partial operation g preserves r. Since $(\rho_1,\dots,\rho_k) \in r^{M^r}$, this implies that $\vec{a} = \big(g(\rho_1),\dots,g(\rho_k)\big) \in r$. Hence $s_\exists \subseteq r$. $\square$

Proof of (iii) $\Rightarrow$ (i). Assume that (iii) holds via the $(k+\ell)$-variable formula $\Psi(\vec{v},\vec{w})$ and the ℓ-tuple $\vec{\tau}$ of k-ary terms. Let g be an n-ary partial operation in $F_1^\Diamond$ with domain closed under F_2. We want to show that g preserves r. So let $\vec{a}_1,\dots,\vec{a}_k \in \mathrm{dom}(g) \subseteq M^n$ with $(\vec{a}_1,\dots,\vec{a}_k) \in r^{M^n}$.

Fix $j \in \{1,\dots,\ell\}$. We can assume that the variable w_j actually occurs in the formula $\Psi(\vec{v},\vec{w})$. Since $\Psi(\vec{v},\vec{\tau}(\vec{v}))$ describes r on M, this implies that $r \subseteq \mathrm{dom}(\tau_j^M)$. As $\mathrm{dom}(g)$ is closed under F_2, we can define $\vec{b}_j := \tau_j^{M^n}(\vec{a}_1,\dots,\vec{a}_k) \in \mathrm{dom}(g)$. Again since $\Psi(\vec{v},\vec{\tau}(\vec{v}))$ describes r on M, it now follows that

$$\begin{aligned}
&\Psi\big(\vec{a}_1,\dots,\vec{a}_k,\vec{b}_1,\dots,\vec{b}_\ell\big) \text{ in } M^n, \\
\Longrightarrow\ &\Psi\big(g(\vec{a}_1),\dots,g(\vec{a}_k),g(\vec{b}_1),\dots,g(\vec{b}_\ell)\big) \text{ in } M, && \text{as } g \in F_1^\Diamond \text{ (see Lemma 1.3)}, \\
\Longrightarrow\ &\big(g(\vec{a}_1),\dots,g(\vec{a}_k)\big) \in r, && \text{as } \exists\vec{w}\,\Psi(\vec{v},\vec{w}) \text{ describes } r \text{ on } M.
\end{aligned}$$

Thus g preserves r. □

The statement of this lemma simplifies in the case that $F_1 = F_2$.

Lemma 1.10. *Let F be a set of partial operations on a finite non-empty set M, and let r be a relation on M. Then the following are equivalent*:

(i) *r is closed under every partial operation in $F^\Diamond$*;
(ii) *r is closed under every partial operation in $F^\Diamond$ with domain $\mathrm{Clo_p}(F)\restriction\{r\}$*;
(iii) *r is conjunct-atomic definable from F.*

Finally, we can describe the closed sets of our Galois connection.

Theorem 1.11. *Let F be a set of partial operations on a finite non-empty set M.*

(i) *A partial operation g on M belongs to the closure $F^{\Diamond\Diamond}$ if and only if*
  (a) *g has an extension in $\mathrm{Clo_p}(F)$, and*
  (b) *the domain of g belongs to $\mathrm{Def_{ca}}(F)$.*
(ii) *The set F is closed (that is, $F = F^{\Diamond\Diamond}$) if and only if F is a partial clone on M closed under restriction of domain to relations in $\mathrm{Def_{ca}}(F)$.*

Proof. Recall conditions (†) and (‡) given after Definition 1.4. We now see that (†) is equivalent to condition (i)(a) above, by Lemma 1.6, and that (‡) is equivalent to (i)(b), by Lemma 1.10. So part (i) follows from Lemma 1.3, and part (ii) follows easily from part (i). □

Part (i) of the previous theorem tells us that the closure operator $F \mapsto F^{\Diamond\Diamond}$ on the power-set $\wp(\mathcal{P}_M)$ is algebraic. So the lattice $\mathcal{G}_M$, of all closed subsets of $\mathcal{P}_M$ ordered by inclusion, is algebraic. But, since it comes from a Galois connection, the lattice $\mathcal{G}_M$ is self-dual and therefore doubly algebraic.

Corollary 1.12. *Let M be a finite non-empty set. Define the Galois connection $F \mapsto F^\Diamond$ on $\wp(\mathcal{P}_M)$ as in 1.4. Then the lattice of closed sets $\mathcal{G}_M$ is doubly algebraic.*

This Galois connection is the basis for our development of duality theory: we use it as a framework for studying finite structures in the next section, and then alter egos in the sections following. For the remainder of this section, we briefly explore some links to clone theory.

1.1. *Clones*

The results of this section can be used to deduce the standard descriptions of the closure operators for the familiar Galois connection between total operations and relations on the finite set M [**16**, **2**] (see also [**22**, **25**]). First let $F \cup \{g\}$ be a set of total operations on M. Because $\mathrm{Clo_p}(F) = \mathrm{Clo}(F)$ in this case, it follows straight from Lemma 1.6 that the following are equivalent:

- every relation on M that is closed under all $f \in F$ is also closed under g;
- $g \in \mathrm{Clo}(F)$.

Now let $S \cup \{r\}$ be a set of relations on M. Using Lemma 1.9 with $F_1 = \Pi{\restriction}S$ and $F_2 = \mathcal{P}_M$, we see that the following are equivalent:

- every total operation on M that preserves all $s \in S$ also preserves r;
- r is primitive-positive definable from S.

Note that it follows straight from Theorem 1.11 that the map $C \mapsto C^{\Diamond\Diamond}$ is an order-embedding from the lattice $\mathcal{C}_M$ of all clones on M into the lattice $\mathcal{G}_M$ of all closed sets under our Galois connection.

1.2. *Strong partial clones*

We first describe a natural way to restrict any Galois connection between a pair of lattices $\mathbf{L}$ and $\mathbf{K}$, given by maps $^{\blacktriangleright} : L \to K$ and $^{\blacktriangleleft} : K \to L$. Choose some $k \in K$ with $k = k^{\blacktriangleleft\blacktriangleright}$. Then the new maps

$$^{\triangleright} : L \to {\downarrow}k \quad \text{and} \quad {}^{\triangleleft} : {\downarrow}k \to L, \quad \text{given by} \quad a^{\triangleright} := a^{\blacktriangleright} \wedge k \quad \text{and} \quad b^{\triangleleft} := b^{\blacktriangleleft},$$

form a Galois connection between $\mathbf{L}$ and the principal ideal ${\downarrow}k$ in $\mathbf{K}$. It is easy to check that

- the lattice of closed elements $L^{\triangleright\triangleleft}$ is precisely the principal filter generated by $k^{\blacktriangleleft}$ in the lattice of closed elements $L^{\blacktriangleright\blacktriangleleft}$, and
- the lattice of closed elements $({\downarrow}k)^{\triangleleft\triangleright}$ is precisely the principal ideal generated by k in the lattice of closed elements $K^{\blacktriangleleft\blacktriangleright}$.

Our Galois connection is essentially an extension of the known Galois connection between partial operations and relations on M [**16**, **24**] (see also [**25**]). To see this, first note that the set $\Pi{\restriction}\mathcal{R}_M$ of all restricted projections on M satisfies $(\Pi{\restriction}\mathcal{R}_M)^{\Diamond} = \Pi{\restriction}\mathcal{R}_M$. So, as in the discussion above, we can use the closed element $\Pi{\restriction}\mathcal{R}_M$ to restrict our Galois connection to one between partial operations and restricted projections on M.

In order to obtain the known Galois connection between partial operations and relations, we need to associate restricted projections with their domains. A partial operation f preserves a relation r if and only if it is compatible with any one (equivalently, all) of the restricted projections $\pi_i{\restriction}_r$. By tweaking our restricted Galois connection, we obtain the one we want:

$$F^{\succ} := \big\{\, \mathrm{dom}(g) \;\big|\; g \in F^{\Diamond} \cap (\Pi{\restriction}\mathcal{R}_M) \,\big\} = \big\{\, r \in \mathcal{R}_M \;\big|\; (\forall f \in F)\ f \text{ preserves } r \,\big\},$$
$$R^{\prec} := (\Pi{\restriction}R)^{\Diamond} = \big\{\, f \in \mathcal{P}_M \;\big|\; (\forall r \in R)\ f \text{ preserves } r \,\big\},$$

for $F \subseteq \mathcal{P}_M$ and $R \subseteq \mathcal{R}_M$.

It follows that the lattice of sets of partial operations of the form $F = F^{\succ\prec}$ is precisely the principal filter in $\mathcal{G}_M$ generated by $\Pi{\restriction}\mathcal{R}_M$, while the lattice of sets of relations of the form $R = R^{\prec\succ}$ is isomorphic via $R \mapsto \Pi{\restriction}R$ to the principal ideal in $\mathcal{G}_M$ generated by $\Pi{\restriction}\mathcal{R}_M$.

The two results we used to prove Theorem 1.11 can be used directly to obtain the closure operators for the Galois connection between partial operations and relations on M [**16**, **24**]. The closed sets of partial operations are described by Lemma 1.6: they are the partial clones on M closed under arbitrary restriction of domain and are called *strong partial clones*. The closed sets of relations are described by Lemma 1.10 (applied with $F = \Pi{\restriction}S$, for some set S of relations).

2. *The lattice of structures*

We now use the Galois connection from the previous section as a framework for comparing the different structures on a fixed finite set. In this section, we start by setting up the natural notions of reduct, equivalence and compatibility between finite structures induced by the Galois connection. While these will simply correspond to the standard notions from duality theory [**3**, **12**, **6**], we believe that the Galois connection provides the most natural and transparent approach.

First define a *base structure* to be a finite non-empty structure $\mathbf{M} = \langle M; F, R\rangle$, where F and R are sets of partial operations and relations on M, respectively.

Note 2.1. As in the previous section, all partial operations and relations on the fixed set M are assumed to be non-nullary and non-empty (unless explicitly stated otherwise).

In general, we are interested not simply in the base structure $\mathbf{M} = \langle M; F, R\rangle$ itself, but in a class of structures that it generates. When we consider an arbitrary structure $\mathbf{A} = \langle A; F^{\mathbf{A}}, R^{\mathbf{A}}\rangle$ of type $\langle F, R\rangle$, we must allow $F^{\mathbf{A}}$ and $R^{\mathbf{A}}$ to be sets of possibly empty partial operations and relations on A. (For example, a substructure of $\mathbf{M}$ may have empty partial operations or relations in its type.)

A base structure $\mathbf{M}$ gives rise to several different sets of (non-empty) partial operations and relations on the set M. A partial operation on M is said to be a *term function* of $\mathbf{M}$ if it belongs to $\mathrm{Clo_p}(F)$. A relation on M is *conjunct-atomic definable* from $\mathbf{M}$ if it is described in $\mathbf{M}$ by a conjunct-atomic formula of type $\langle F, R\rangle$. We use $\mathrm{Def_{ca}}(\mathbf{M})$ to denote the set of all such relations.

Definition 2.2. Let $\mathbf{M} = \langle M; F, R\rangle$ be a base structure, and let $f : r \to M$ be a k-ary partial operation on M, for some $k \geqslant 1$.

- We call f a *structural function* of $\mathbf{M}$ if
  - f extends to a term function of $\mathbf{M}$, and
  - the domain of f is conjunct-atomic definable from $\mathbf{M}$.
- We call f a *compatible partial operation* on $\mathbf{M}$ if it forms a homomorphism $f : \mathbf{r} \to \mathbf{M}$, where $\mathbf{r}$ is a substructure of $\mathbf{M}^k$.

Remark 2.3. In terms of the Galois connection in the previous section, we are associating the base structure $\mathbf{M} = \langle M; F, R\rangle$ with the set $G := F \cup (\Pi{\restriction}R)$ of partial operations on M, where $\Pi{\restriction}R$ consists of all projections with domain in R. The structural functions of $\mathbf{M}$ form the closed set $G^{\Diamond\Diamond} = \mathrm{Clo_p}(G){\restriction}\,\mathrm{Def_{ca}}(G)$; see Theorem 1.11. The compatible partial operations on $\mathbf{M}$ form the closed set $G^{\Diamond}$; see Lemma 1.3.

Using structural functions, we now define a natural quasi-order on the set of all structures on M (analogously to using term functions to define the term-reduct quasi-order on the set of all algebras on M).

Definition 2.4. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be base structures on a finite set M.

- We say that $\mathbf{M}_1$ is a *structural reduct* of $\mathbf{M}_2$ (and write $\mathbf{M}_1 \sqsubseteq \mathbf{M}_2$, for short) if every structural function of $\mathbf{M}_1$ is also a structural function of $\mathbf{M}_2$.
- We say that $\mathbf{M}_1$ and $\mathbf{M}_2$ are *structurally equivalent* (and write $\mathbf{M}_1 \equiv \mathbf{M}_2$) if they have the same structural functions.

If $\mathbf{M}_1$ and $\mathbf{M}_2$ are algebras, then structural reduct coincides with term reduct.

LEMMA 2.5. *Let* $\mathbf{M}_1 = \langle M; F_1, R_1\rangle$ *and* $\mathbf{M}_2 = \langle M; F_2, R_2\rangle$ *be base structures. Then the following are equivalent*:

(i) $\mathbf{M}_1$ *is a structural reduct of* $\mathbf{M}_2$;
(ii) (a) *each partial operation in* F_1 *extends to a term function of* $\mathbf{M}_2$, *and*
(b) *each relation in* $R_1 \cup \mathrm{dom}(F_1)$ *is conjunct-atomic definable from* $\mathbf{M}_2$;
(iii) *all compatible partial operations on* $\mathbf{M}_2$ *are compatible partial operations on* $\mathbf{M}_1$.

Proof. For each $i \in \{1, 2\}$, define the set of partial operations $G_i := F_i \cup (\Pi{\restriction}R_i)$. Using basic properties of a Galois connection, we have

$$G_1^{\Diamond\Diamond} \subseteq G_2^{\Diamond\Diamond} \iff G_2^{\Diamond} \subseteq G_1^{\Diamond} \iff G_1 \subseteq G_2^{\Diamond\Diamond}.$$

By Remark 2.3, it follows that conditions (i), (iii) and (ii) are equivalent. □

We now turn the collection of structures on the set M into a lattice. First, equip the collection of all base structures on M with the structural-reduct quasi-order. Then factor in the natural way to obtain the ordered set $\mathcal{S}_M$. Using Remark 2.3, we can easily set up an order-isomorphism from $\mathcal{S}_M$ to the lattice of closed sets $\mathcal{G}_M$ from Section 1. So, by applying Corollary 1.12, we obtain the following.

THEOREM 2.6. *The collection of all base structures on a finite non-empty set* M *forms a doubly algebraic lattice* $\mathcal{S}_M$, *under the structural-reduct quasi-order.*

The lattice $\mathcal{S}_M$ can be viewed as an analogue for structures to the lattice of clones $\mathcal{C}_M$ for algebras. We work with representatives of the elements of $\mathcal{S}_M$ (which are individual structures on M), rather than the actual elements of $\mathcal{S}_M$ (which are equivalence classes of structures on M).

REMARK 2.7. We can easily describe join and meet in $\mathcal{S}_M$, as it essentially comes from the closure operator $F \mapsto F^{\Diamond\Diamond}$ on $\wp(\mathcal{P}_M)$. Consider a set of base structures $\mathbf{M}_i = \langle M; F_i, R_i\rangle$ indexed by a non-empty set I. The *join* of the corresponding subset of $\mathcal{S}_M$ is represented by the union of the structures:

$$\textstyle\bigcup_{i\in I} \mathbf{M}_i := \langle M; \bigcup_{i\in I} F_i, \bigcup_{i\in I} R_i\rangle.$$

The *meet* is represented by the structure

$$\textstyle\bigcap_{i\in I} \mathbf{M}_i := \langle M; \bigcap_{i\in I} G_i^{\Diamond\Diamond}\rangle,$$

where $G_i := F_i \cup (\Pi{\restriction}R_i)$ and so $G_i^{\Diamond\Diamond}$ is the set of structural functions of $\mathbf{M}_i$.

Our Galois connection also induces the natural notion of compatibility between structures. We have already defined a compatible partial operation on a base structure $\mathbf{M}$; see 2.2. We define a *compatible relation* on $\mathbf{M}$ to be the underlying set of a substructure $\mathbf{r}$ of $\mathbf{M}^k$, for some $k \geqslant 1$.

We say that base structures $\mathbf{M}_1$ and $\mathbf{M}_2$ on the same set are *compatible* if they satisfy the equivalent conditions of the following lemma.

LEMMA 2.8. *Let* $\mathbf{M}_1 = \langle M; F_1, R_1\rangle$ *and* $\mathbf{M}_2 = \langle M; F_2, R_2\rangle$ *be base structures. Then the following are equivalent*:

(i) *every structural function of* $\mathbf{M}_1$ *is a compatible partial operation on* $\mathbf{M}_2$;
(ii) *every structural function of* $\mathbf{M}_2$ *is a compatible partial operation on* $\mathbf{M}_1$;
(iii) (a) *each* $f \in F_1$ *is a compatible partial operation on* $\mathbf{M}_2$, *and*
(b) *each* $r \in R_1$ *is a compatible relation on* $\mathbf{M}_2$.

Proof. For $i \in \{1, 2\}$, define $G_i := F_i \cup (\Pi{\restriction}R_i)$. Then

$$G_1^{\Diamond\Diamond} \subseteq G_2^{\Diamond} \iff G_2^{\Diamond\Diamond} \subseteq G_1^{\Diamond} \iff G_1 \subseteq G_2^{\Diamond}.$$

So the three conditions are equivalent, by Remark 2.3. □

Fixing a base structure $\mathbf{M} = \langle M; F, R\rangle$, we can now consider the set of all structures compatible with $\mathbf{M}$. This set forms a principal ideal of the lattice of structures $\mathcal{S}_M$: the top element of the ideal is represented by the structure $\langle M; G^{\Diamond}\rangle$, where $G := F \cup (\Pi{\restriction}R)$ and so $G^{\Diamond}$ is the set of all compatible partial operations on $\mathbf{M}$. We denote the resulting complete sublattice by $\mathcal{A}_{\mathbf{M}}$; it will be referred to as the *lattice of alter egos* of $\mathbf{M}$. Since the lattice of structures $\mathcal{S}_M$ is doubly algebraic, by Theorem 2.6, so is the lattice of alter egos $\mathcal{A}_{\mathbf{M}}$.

Starting in Section 4, the lattice of alter egos will provide us with a language for relating the different natural dualities admitted by a finite structure. We use three particular algebras as our main source of examples. The first is the two-element bounded lattice $\mathbf{2}$.

EXAMPLE 2.9. *The lattice of alter egos* $\mathcal{A}_{\mathbf{2}}$ *has size* 2.

Proof. Let $\mathbf{2} = \langle \{0, 1\}; \vee, \wedge, \underline{0}, \underline{1}\rangle$ with $0 \leqslant 1$. We claim that each base structure compatible with $\mathbf{2}$ is equivalent to either the set $\mathbf{2}_{\mathrm{s}} := \langle \{0, 1\}; \rangle$ or the ordered set $\mathbf{2}_{\mathrm{o}} := \langle \{0, 1\}; \leqslant\rangle$. It is known (for example, from Priestley duality) that every compatible partial operation on $\mathbf{2}$ is a restricted projection. So each base structure compatible with $\mathbf{2}$ is essentially relational.

Now consider a k-ary compatible relation r on $\mathbf{2}$, for some $k \geqslant 1$. First assume that the associated sublattice $\mathbf{r}$ of $\mathbf{2}^k$ is boolean. Then r is conjunct-atomic definable in the empty language, by Baker–Pixley [**1**]. Thus r can be removed from the type of any structure on $\{0, 1\}$.

We can now assume that $\mathbf{r}$ is non-boolean. There is an element a of r that does not have a complement in the lattice $\mathbf{r}$. So $\{a(1), \ldots, a(k)\} = \{0, 1\}$, and it is easy to check that the order relation $\leqslant$ is conjunct-atomic definable from r as follows:

$$\leqslant = \big\{ (b_0, b_1) \in \{0, 1\}^2 \;\big|\; \big(b_{a(1)}, b_{a(2)}, \ldots, b_{a(k)}\big) \in r \big\}.$$

The relation r is conjunct-atomic definable from $\leqslant$, by Baker–Pixley [**1**]. Thus the relation r can be replaced by $\leqslant$ in the type of a structure on $\{0, 1\}$. It now follows that $\mathbf{2}_{\mathrm{s}}$ and $\mathbf{2}_{\mathrm{o}}$ are essentially the only base structures compatible with $\mathbf{2}$. □

The second algebra we use in examples is the three-element bounded lattice $\mathbf{3}$. In a follow-up paper [**9**], we will see that the lattice of alter egos $\mathcal{A}_{\mathbf{3}}$ is uncountable. This is somewhat counter-intuitive, given that

– all compatible partial operations on $\mathbf{3}$ are essentially binary, and
– all compatible relations on $\mathbf{3}$ are conjunct-atomic definable from binary compatible relations on $\mathbf{3}$.

(The first of these claims uses congruence distributivity [**21**, A.7.6]; the second uses Baker–Pixley [**1**].)

Third, we shall use the four-element quasi-primal algebra that provided the first example of a full but not strong duality (see Example 5.1).

EXAMPLE 2.10. *For every quasi-primal algebra* $\mathbf{Q}$, *the lattice of alter egos* $\mathcal{A}_{\mathbf{Q}}$ *is finite.*

Proof. Let $k \geqslant 1$. We say that a k-ary relation r on Q is *indecomposable* if it is not essentially a product of two relations; more formally, if there is no two-block partition $\{I, J\}$ of the index set $\{1, \ldots, k\}$ such that

$$r = \big\{\, \vec{a} \in Q^k \;\big|\; \pi_I(\vec{a}) \in \pi_I(r) \text{ and } \pi_J(\vec{a}) \in \pi_J(r) \,\big\}.$$

P. Krauss [**19**] has characterised the k-ary indecomposable compatible relations on the quasi-primal algebra $\mathbf{Q}$: each such relation r is of the form

$$r = \big\{\, (a_1, \ldots, a_k) \in Q^k \;\big|\; a_1 \in A_1 \text{ and } a_i = f_i(a_1) \text{ for all } 2 \leqslant i \leqslant k \,\big\},$$

for some subalgebra $\mathbf{A}_1$ of $\mathbf{Q}$ and partial automorphisms $f_i : \mathbf{A}_1 \to \mathbf{A}_i$ of $\mathbf{Q}$.

We can choose a finite set R of compatible relations on $\mathbf{Q}$ such that every indecomposable compatible relation on $\mathbf{Q}$ is interdefinable with a relation from R via coordinate manipulation. Then we can construct a finite set R^+ of compatible relations on $\mathbf{Q}$ so that it contains one product of each non-empty subset of R.

Let $h : s \to Q$ be a k-ary compatible partial operation on $\mathbf{Q}$. We wish to find a compatible partial operation $g : r \to Q$ on $\mathbf{Q}$ such that

- $r \in R^+$,
- r and s are conjunct-atomic interdefinable, and
- there is an extension of g in $\mathrm{Clo_p}(h)$ and an extension of h in $\mathrm{Clo_p}(g)$.

It will then follow that g can replace h in the type of any structure, and thence that $\mathcal{A}_{\mathbf{Q}}$ is finite.

We can assume that s is a product of relations from R. (It is interdefinable with such a relation via coordinate manipulation.) The relations in R^+ have no repeated factors. So we show how to get rid of a repeated factor from s. Say that $s = p \times q \times q$ and let $r := p \times q$. Then r and s are conjunct-atomic interdefinable:

$$r = \big\{\, (\vec{a}, \vec{b}) \;\big|\; (\vec{a}, \vec{b}, \vec{b}) \in s \,\big\} \quad \text{and} \quad s = \big\{\, (\vec{a}, \vec{b}, \vec{c}) \;\big|\; (\vec{a}, \vec{b}) \in r \;\&\; (\vec{a}, \vec{c}) \in r \,\big\}$$

Since $\mathbf{Q}$ is quasi-primal, the compatible partial operation $h : s \to Q$ is essentially unary (again, use congruence distributivity [**21**, A.7.6]). So, by symmetry, we can assume that h does not depend on the third factor of s. Now define the compatible partial operation $g : r \to Q$ by $g(\vec{x}, \vec{y}) := h(\vec{x}, \vec{y}, \vec{y})$. Then g is in $\mathrm{Clo_p}(h)$, and the partial operation $h'(\vec{x}, \vec{y}, \vec{z}) := g(\vec{x}, \vec{y})$ is an extension of h in $\mathrm{Clo_p}(g)$. □

3. *Entailment in duality theory*

Our Galois connection leads to a symmetric setting for developing duality theory, starting from a pair of compatible base structures $\mathbf{M}_0$ and $\mathbf{M}_1$ on a finite set M. The complete symmetry at the finite level will give us 'two theorems for the price of one' in the next section.

In this section, we quickly run through the basic definitions in our new setting. Then we look in more detail at entailment, which will be our key to understanding natural dualities at the finite level. In particular, we extend the Dual Entailment and Structural Entailment Theorems [**10**, **14**, **12**]. The early chapters of the Clark–Davey text [**3**] give a detailed introduction to the theory of natural dualities for algebras—we call this the *usual setting*. Some basic duality theory for structures has been set down by Davey [**6**]. (More generally, Hofmann [**17**] has looked at natural duality theory based on sketches.)

We are starting from two compatible base structures $\mathbf{M}_0$ and $\mathbf{M}_1$. The topological structure $\underset{\sim}{\mathbf{M}}_1$, obtained by adding the discrete topology to $\mathbf{M}_1$, is called an *alter ego* of $\mathbf{M}_0$. We aim to set up a correspondence between

- the 'algebraic' class $\mathcal{A} := \mathbb{ISP}^+(\mathbf{M}_0)$, which consists of all isomorphic copies of substructures of powers of $\mathbf{M}_0$, and
- the topological class $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$, which consists of all isomorphic copies of topologically closed substructures of powers of $\underset{\sim}{\mathbf{M}}_1$.

Note 3.1. Unlike in the usual setting, we choose to exclude the empty structure and the empty-indexed power from both classes $\mathcal{A}$ and $\mathcal{X}$. See the appendix for a discussion of three alternative (but essentially equivalent) settings.

As usual, we can set up functors $\mathrm{D} : \mathcal{A} \to \mathcal{X}$ and $\mathrm{E} : \mathcal{X} \to \mathcal{A}$, where

- for each $\mathbf{A} \in \mathcal{A}$, the dual $\mathrm{D}(\mathbf{A})$ is the topologically closed substructure of $(\underset{\sim}{\mathbf{M}}_1)^A$ whose universe is the set $\hom(\mathbf{A}, \mathbf{M}_0)$ of all homomorphisms from $\mathbf{A}$ to $\mathbf{M}_0$, and
- for each $\mathbf{X} \in \mathcal{X}$, the dual $\mathrm{E}(\mathbf{X})$ is the substructure of $(\mathbf{M}_0)^X$ whose universe is the set $\hom(\mathbf{X}, \underset{\sim}{\mathbf{M}}_1)$ of all continuous homomorphisms from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}_1$.

For every $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$, there are natural evaluation embeddings $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$, where

- each continuous homomorphism $e_{\mathbf{A}}(a) : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}_1$ is given by $x \mapsto x(a)$,
- each homomorphism $\varepsilon_{\mathbf{X}}(x) : \mathrm{E}(\mathbf{X}) \to \mathbf{M}_0$ is given by $\alpha \mapsto \alpha(x)$.

We can now give the main definitions of natural duality theory within our setting.

Definition 3.2. Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be compatible base structures. We say that

- $\underset{\sim}{\mathbf{M}}_1$ *dualises* $\mathbf{M}_0$ [*at the finite level*] if the natural embedding $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism, for each [finite] structure $\mathbf{A} \in \mathcal{A} := \mathbb{ISP}^+(\mathbf{M}_0)$,
- $\underset{\sim}{\mathbf{M}}_1$ *co-dualises* $\mathbf{M}_0$ [*at the finite level*] if the natural embedding $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, for each [finite] topological structure $\mathbf{X} \in \mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$, and
- $\underset{\sim}{\mathbf{M}}_1$ *fully dualises* $\mathbf{M}_0$ [*at the finite level*] if $\underset{\sim}{\mathbf{M}}_1$ both dualises and co-dualises $\mathbf{M}_0$ [at the finite level].

Note that, if $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ [at the finite level], then $(\mathrm{D}, \mathrm{E}, e, \varepsilon)$ gives a dual equivalence between the categories [of finite members of] $\mathcal{A}$ and $\mathcal{X}$.

We will use the following lemma to show that our notion of structural equivalence is consistent with the above definitions. The lemma has two versions: either include or exclude the phrases in square brackets. The topological version gives the usual definition of structural reduct in natural duality theory [**12**].

Lemma 3.3. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be base structures on the same set. Then $\mathbf{M}_1$ is a structural reduct of $\mathbf{M}_2$ if and only if, for all non-empty sets S and T,*

(i) *every* [*topologically closed*] *substructure $\mathbf{X}$ of $(\underset{\sim}{\mathbf{M}}_2)^S$ has the same underlying set as a* [*topologically closed*] *substructure $\mathbf{X}^\flat$ of $(\underset{\sim}{\mathbf{M}}_1)^S$, and*

(ii) *every* [*continuous*] *homomorphism $\varphi : \mathbf{X} \to \mathbf{Y}$ has the same underlying set-map as a* [*continuous*] *homomorphism $\varphi^\flat : \mathbf{X}^\flat \to \mathbf{Y}^\flat$, for* [*topologically closed*] *substructures $\mathbf{X}$ of $(\underset{\sim}{\mathbf{M}}_2)^S$ and $\mathbf{Y}$ of $(\underset{\sim}{\mathbf{M}}_2)^T$.*

Proof. We use Lemma 2.5. Condition 3.3(ii) above implies condition 2.5(iii), and so we obtain the 'if' direction. It is straightforward to check that both 3.3(i) and 3.3(ii) follow from 2.5(ii). This gives the 'only if' direction. □

Remark 3.4. Assume that $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ are structurally equivalent alter egos of $\mathbf{M}_0$. By Lemma 3.3, the two concrete categories $\mathbb{S}_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$ and $\mathbb{S}_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ have essentially the same objects and morphisms. So $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ are equivalent from the point of view of the above definitions. For instance, if we know that $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ at the finite level, then it follows easily that $\underset{\sim}{\mathbf{M}}_2$ fully dualises $\mathbf{M}_0$ at the finite level as well. We do not consider the duality provided by $\underset{\sim}{\mathbf{M}}_2$ to be 'new'; it is essentially the same as that provided by $\underset{\sim}{\mathbf{M}}_1$.

Example 3.5. We found the two essentially different alter egos of the two-element bounded lattice $\mathbf{2}$ in Example 2.9. The alter ego $\underset{\sim}{\mathbf{2}}_{\mathrm{s}}$ doesn't even dualise $\mathbf{2}$ at the finite level: for every non-trivial bounded distributive lattice $\mathbf{A}$, the double dual $\mathrm{E_s D_s}(\mathbf{A})$ is boolean. The alter ego $\underset{\sim}{\mathbf{2}}_{\mathrm{o}}$ fully dualises $\mathbf{2}$: the associated dual equivalence between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{X}}_{\mathrm{o}}$ is Priestley duality [**23**].

Entailment will be our main tool, in the next section, for obtaining intrinsic descriptions of natural duality at the finite level. Let r be a compatible relation on $\mathbf{M}_0$ and let $\mathbf{X}$ be a *concrete structure* in $\boldsymbol{\mathcal{X}}$, that is, let $\mathbf{X} \leqslant (\underset{\sim}{\mathbf{M}}_1)^S$, for some non-empty set S. We can interpret the relation r pointwise on X, even if r is not in the type of $\mathbf{M}_1$. We say that $\underset{\sim}{\mathbf{M}}_1$ *entails* r *on* $\mathbf{X}$ if every continuous homomorphism $\varphi : \mathbf{X} \to \underset{\sim}{\mathbf{M}}_1$ preserves r. We now use Lemma 1.9 to extend a fundamental theorem of duality theory to our setting.

Dual Entailment Theorem 3.6 [**10**, **14**]. *Let* $\mathbf{M}_0$ *and* $\mathbf{M}_1$ *be compatible base structures, and let* r *be a compatible relation on* $\mathbf{M}_0$. *Then the following are equivalent*:

(i) $\underset{\sim}{\mathbf{M}}_1$ *entails* r *on each concrete structure in* $\mathbb{S}_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$ *that is closed under all partial operations in* $\hom(\mathbf{r}, \mathbf{M}_0)$;
(ii) $\underset{\sim}{\mathbf{M}}_1$ *entails* r *on the dual* $\mathrm{D}(\mathbf{A})$, *for all* $\mathbf{A} \in \mathbb{ISP}^+(\mathbf{M}_0)$;
(iii) $\underset{\sim}{\mathbf{M}}_1$ *entails* r *on the dual* $\mathrm{D}(\mathbf{r})$;
(iv) r *is primitive-positive definable from* $\mathbf{M}_1$ *with existence witnessed by* $\hom(\mathbf{r}, \mathbf{M}_0)$.

Proof. The implications (i) ⇒ (ii) ⇒ (iii) and (iv) ⇒ (i) are straightforward. Now say that $\mathbf{M}_1 = \langle M; H_1, R_1\rangle$. To establish (iii) ⇒ (iv), we apply Lemma 1.9 with $F_1 := H_1 \cup (\Pi{\restriction}R_1)$ and $F_2 := F_1 \cup \hom(\mathbf{r}, \mathbf{M}_0)$. This ensures that $\mathrm{Clo_p}(F_2){\restriction}\{r\} = \hom(\mathbf{r}, \mathbf{M}_0)$, which is the underlying set of $\mathrm{D}(\mathbf{r})$. □

We shall use Lemma 1.10 to extend another fundamental theorem of duality theory to our setting, based on the following observation [**12**, 4.1].

Remark 3.7. Let $\underset{\sim}{\mathbf{M}}_1 = \langle M; F_1, R_1, \mathcal{T}\rangle$ be an alter ego of a base structure $\mathbf{M}_0$ and define $G_1 := F_1 \cup (\Pi{\restriction}R_1)$. Now let r be a k-ary compatible relation on $\mathbf{M}_0$, for some $k \geqslant 1$. It is easy to see that the following are equivalent, for each compatible partial operation $h : r \to M$ on $\mathbf{M}_0$:

– h has an extension in $\mathrm{Clo_p}(G_1)$;
– h extends to a term function of $\mathbf{M}_1$;
– h belongs to the substructure of the dual $\mathrm{D}(\mathbf{r})$ generated by the projections $\rho_1, \ldots, \rho_k : \mathbf{r} \to \mathbf{M}_0$.

So $\mathrm{Clo}_{\mathrm{p}}(G_1)\restriction\{r\}$ is the underlying set of the substructure of $\mathrm{D}(\mathbf{r})$ generated by the projections. Structure is imposed pointwise on this set in both of its guises: as an r-ary relation on M and as a substructure of $\mathrm{D}(\mathbf{r})$.

Structural Entailment Theorem 3.8 [**12**, 2.6]. *Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be compatible base structures, and let r be a compatible relation on $\mathbf{M}_0$. Then the following are equivalent*:

(i) $\underset{\sim}{\mathbf{M}}_1$ *entails r on every concrete structure in $\mathbb{S}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}}_1)$*;
(ii) $\underset{\sim}{\mathbf{M}}_1$ *entails r on the substructure of $\mathrm{D}(\mathbf{r})$ generated by the projections*;
(iii) *r is conjunct-atomic definable from $\mathbf{M}_1$.*

Proof. Say that $\mathbf{M}_1 = \langle M; F_1, R_1\rangle$. To prove that (ii) $\Rightarrow$ (iii), use Lemma 1.10 with $F := F_1 \cup (\Pi\restriction R_1)$. □

We obtained the Structural Entailment Theorem by interpreting Lemma 1.10 for an alter ego $\underset{\sim}{\mathbf{M}}_1$. In our symmetric setting, we can also interpret Lemma 1.10 for the structure $\mathbf{M}_0$, explaining another fundamental result in duality theory.

Closure Lemma 3.9. *Let $\mathbf{M}$ be a base structure and let r be a relation on M. Then the following are equivalent*:

(i) *r is closed under all compatible partial operations on $\mathbf{M}$*;
(ii) *r is conjunct-atomic definable from $\mathbf{M}$.*

Notes. In the usual setting, this lemma is the finite-level version of Clark and Krauss's Closure Theorem [**5**] (see also [**3**, 3.1.3]). Condition (i) above corresponds to '*r is hom-closed*', and condition (ii) to '*r is term-closed*'. In fact, the general Closure Theorem can also be extended to our setting. For non-empty sets S and X with $X \subseteq M^S$, let '*X is term-closed*' correspond to X being definable via a possibly infinite conjunction of atomic formulæ of the same type as $\mathbf{M}$ in the variables $\{\, v_s \mid s \in S \,\}$. Then the proof of the Closure Theorem [**3**, 3.1.3] can be modified for our setting.

4. *Natural duality at the finite level*

In this section, we give simple and natural intrinsic descriptions of duality, full duality and strong duality at the finite level. The descriptions will tell us how these concepts sit within the lattice of alter egos of a base structure $\mathbf{M}$.

First consider a k-ary compatible relation r on $\mathbf{M}$, for some $k \geqslant 1$, and let $\mathbf{r}$ denote the corresponding substructure of $\mathbf{M}^k$. Then r is a *hom-minimal relation* on $\mathbf{M}$ if the only homomorphisms from $\mathbf{r}$ to $\mathbf{M}$ are projections. The following simple but important result shows that the hom-minimal relations are 'unavoidable' for obtaining a duality at the finite level.

Duality Lemma 4.1. *Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be compatible base structures. Then the following are equivalent*:

(i) $\underset{\sim}{\mathbf{M}}_1$ *dualises $\mathbf{M}_0$ at the finite level*;
(ii) *every compatible relation on $\mathbf{M}_0$ is entailed by $\underset{\sim}{\mathbf{M}}_1$ on duals*;
(iii) *every hom-minimal relation on $\mathbf{M}_0$ is conjunct-atomic definable from $\mathbf{M}_1$*;
(iv) *for each $r \in \mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_0)$, every compatible partial operation on $\mathbf{M}_1$ with domain r extends to a term function of $\mathbf{M}_0$.*

Notes. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are essentially known in the usual setting [**3**, **18**, **12**, **20**]. The equivalence (i) $\Leftrightarrow$ (iv) is a finite-level version of the Third Duality Theorem [**5**] (see also [**3**, 3.1.6]) and can be extended to the infinite level; refer to the notes for the Closure Lemma 3.9.

Proof. The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows straight from the Dual and Structural Entailment Theorems, 3.6 and 3.8. We next show that (iii) $\Rightarrow$ (i), by adapting the proof of the Brute Force Duality Theorem [**3**, 2.3.1].

Let $\mathbf{A} \leqslant (\mathbf{M}_0)^n$, for some $n \geqslant 1$, with $\hom(\mathbf{A}, \mathbf{M}_0) = \{x_1, \ldots, x_k\}$. Define the embedding $\xi : \mathbf{A} \to (\mathbf{M}_0)^k$ by $\xi(a) := (x_1(a), \ldots, x_k(a))$. Then $r := \xi(A)$ is a hom-minimal relation on $\mathbf{M}_0$. As $(x_1, \ldots, x_k) \in r^{\mathrm{D}(\mathbf{A})}$, every morphism $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}_1$ that preserves r is an evaluation. It follows that (iii) $\Rightarrow$ (i).

Finally, we show how (ii) translates into (iv). The Dual Entailment Theorem 3.6 tells us that entailment on duals is the same as entailment on finite concrete structures closed under all the compatible partial operations on $\mathbf{M}_0$ (as the latter condition lies between 3.6(i) and 3.6(iii)). Thus condition (ii) can be rewritten as:

(ii)′ for each relation r on M that is closed under all compatible partial operations on $\mathbf{M}_0$, every compatible partial operation on $\mathbf{M}_1$ with domain r preserves all compatible relations on $\mathbf{M}_0$.

But '*closed under all compatible partial operations on* $\mathbf{M}_0$' is the same as '*conjunct-atomic definable from* $\mathbf{M}_0$', by Lemma 3.9, and '*preserves all compatible relations on* $\mathbf{M}_0$' is the same as '*extends to a term function of* $\mathbf{M}_0$', by Lemma 1.6. Therefore (ii)′ is equivalent to (iv). □

The equivalence (i) $\Leftrightarrow$ (iii) in the previous lemma provides an intrinsic description of duality at the finite level. It tells us that the alter egos of a base structure $\mathbf{M}$ that yield a finite-level duality form a principal filter of the lattice $\mathcal{A}_{\mathbf{M}}$, and thus form a complete sublattice $\mathcal{D}_{\mathbf{M}}$. If the structure $\mathbf{M}$ is dualisable, then every alter ego that yields a finite-level duality in fact yields a duality. (Use (i) $\Rightarrow$ (ii) in the previous lemma.) So then $\mathcal{D}_{\mathbf{M}}$ is the lattice of all dualities based on $\mathbf{M}$. By the end of this section, we shall form a general picture of the lattice $\mathcal{D}_{\mathbf{M}}$.

We also want an intrinsic description of finite-level full duality. At the finite level, the topological side is discrete and so our setting is completely symmetric. Thus we immediately obtain the following.

Co-duality Lemma 4.2. *Let* $\mathbf{M}_0$ *and* $\mathbf{M}_1$ *be compatible base structures. Then* $\underset{\sim}{\mathbf{M}}_1$ *co-dualises* $\mathbf{M}_0$ *at the finite level if and only if* $\underset{\sim}{\mathbf{M}}_0$ *dualises* $\mathbf{M}_1$ *at the finite level.*

The previous two lemmas yield a new intrinsic description of full duality at the finite level.

Full Duality Lemma 4.3. *Let* $\mathbf{M}_0$ *and* $\mathbf{M}_1$ *be compatible base structures. Then the following are equivalent*:

(i) $\underset{\sim}{\mathbf{M}}_1$ *fully dualises* $\mathbf{M}_0$ *at the finite level*;
(ii) $\underset{\sim}{\mathbf{M}}_0$ *fully dualises* $\mathbf{M}_1$ *at the finite level*;
(iii) $\underset{\sim}{\mathbf{M}}_1$ *dualises* $\mathbf{M}_0$ *at the finite level and* $\underset{\sim}{\mathbf{M}}_0$ *dualises* $\mathbf{M}_1$ *at the finite level*;
(iv) (a) *every hom-minimal relation on* $\mathbf{M}_0$ *is conjunct-atomic definable from* $\mathbf{M}_1$, *and*
(b) *for each* $r \in \mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_1)$, *every compatible partial operation on* $\mathbf{M}_0$ *with domain* r *extends to a term function of* $\mathbf{M}_1$.

Notes. The implication (i) $\Rightarrow$ (iv) was already known in the usual setting [**18**, **13**]. The reverse implication (iv) $\Rightarrow$ (i) also holds in this setting, provided $\mathbf{M}_1$ has *enough nullaries* (that

is, for all $a \in M$, the nullary $\widehat{a}$ should be a term function of $\mathbf{M}_1$ whenever it is compatible with $\mathbf{M}_0$); see the appendix for more details.

Using the previous lemma, we can show that the finite-level full dualities based on $\mathbf{M}$ form a complete lattice $\mathcal{F}_{\mathbf{M}}$, and so there is always a 'smallest' one.

Corollary 4.4. *Let $\mathbf{M}$ be a base structure. The collection of all alter egos that fully dualise $\mathbf{M}$ at the finite level forms an algebraic lattice $\mathcal{F}_{\mathbf{M}}$, under the structural-reduct quasi-order. Moreover, meets in the lattice $\mathcal{F}_{\mathbf{M}}$ coincide with meets in the lattice of alter egos $\mathcal{A}_{\mathbf{M}}$.*

Proof. First let $\mathcal{F}_{\mathbf{M}}$ denote the sub-ordered set of the lattice of alter egos $\mathcal{A}_{\mathbf{M}}$ determined by those that fully dualise $\mathbf{M}$ at the finite level. Using the Full Duality Lemma 4.3 (i) $\Leftrightarrow$ (iv) and Remark 2.7, it is easy to check that $\mathcal{F}_{\mathbf{M}}$ includes the top element of $\mathcal{A}_{\mathbf{M}}$ and that $\mathcal{F}_{\mathbf{M}}$ is closed under meets and directed joins in $\mathcal{A}_{\mathbf{M}}$. We know that $\mathcal{A}_{\mathbf{M}}$ is an algebraic lattice, and so the result follows. □

However, it is not clear from the Full Duality Lemma whether joins in $\mathcal{F}_{\mathbf{M}}$ agree with joins in $\mathcal{A}_{\mathbf{M}}$. We shall study the lattice $\mathcal{F}_{\mathbf{M}}$ further in the next section.

We turn now to finite-level strong duality, though we delay the definition until after two more lemmas. The first follows from Lemmas 1.6, 3.9 and 4.1.

IC Lemma 4.5. *Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be compatible base structures. The following conditions are equivalent and imply that $\underset{\sim}{\mathbf{M}}_1$ dualises $\mathbf{M}_0$ at the finite level:*

(IC) *every compatible partial operation on $\mathbf{M}_1$ extends to a term function of $\mathbf{M}_0$;*

$(\text{FTC})^{\partial}$ *every compatible relation on $\mathbf{M}_0$ is conjunct-atomic definable from $\mathbf{M}_1$.*

Notes. The first condition above corresponds in the standard theory to '$\mathbf{M}_0$ and $\underset{\sim}{\mathbf{M}}_1$ satisfy (IC)' and the second to '$\mathbf{M}_1$ and $\underset{\sim}{\mathbf{M}}_0$ satisfy (FTC)'. So, within our symmetric setting, we see that the conditions (IC) and (FTC) are dual to each other. In the case that $\mathbf{M}_0$ is an algebra, the first condition implies that $\mathbf{M}_1$ is injective in $\mathbb{ISP}^+_{\text{fin}}(\mathbf{M}_1)$; see Subsection 4.2.

Strong Duality Lemma 4.6. *Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be compatible base structures. Then the following are equivalent:*

(i) *$\underset{\sim}{\mathbf{M}}_1$ is structurally equivalent to the top alter ego of $\mathbf{M}_0$;*

(ii) *$\underset{\sim}{\mathbf{M}}_0$ is structurally equivalent to the top alter ego of $\mathbf{M}_1$;*

(iii) (a) *$\underset{\sim}{\mathbf{M}}_1$ dualises $\mathbf{M}_0$ at the finite level, and*
  (b) *every compatible relation on $\mathbf{M}_1$ is conjunct-atomic definable from $\mathbf{M}_0$;*

(iv) (a) *$\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ at the finite level, and*
  (b) *every compatible partial operation on $\mathbf{M}_1$ extends to a term function of $\mathbf{M}_0$;*

(v) *$\mathbf{M}_0$ and $\underset{\sim}{\mathbf{M}}_1$ satisfy (IC) and (FTC);*

(vi) *$\mathbf{M}_0$ and $\underset{\sim}{\mathbf{M}}_1$ satisfy (IC), and $\mathbf{M}_1$ and $\underset{\sim}{\mathbf{M}}_0$ satisfy (IC);*

(vii) *$\mathbf{M}_0$ and $\underset{\sim}{\mathbf{M}}_1$ satisfy (FTC), and $\mathbf{M}_1$ and $\underset{\sim}{\mathbf{M}}_0$ satisfy (FTC).*

Notes. Condition (iii) corresponds to the standard definition of finite-level strong duality. In the case that $\mathbf{M}_0$ is an algebra, condition (iv) is a familiar description of finite-level strong duality [**3**, 3.2.4], as then condition (iv)(b) implies that $\mathbf{M}_1$ is injective in $\mathbb{ISP}^+_{\text{fin}}(\mathbf{M}_1)$. Condition (v) comes from the Third Strong Duality Theorem [**3**, 3.2.11], and the equivalence (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) is a finite-level version of the Two-for-One Strong Duality Theorem [**3**, 3.3.2].

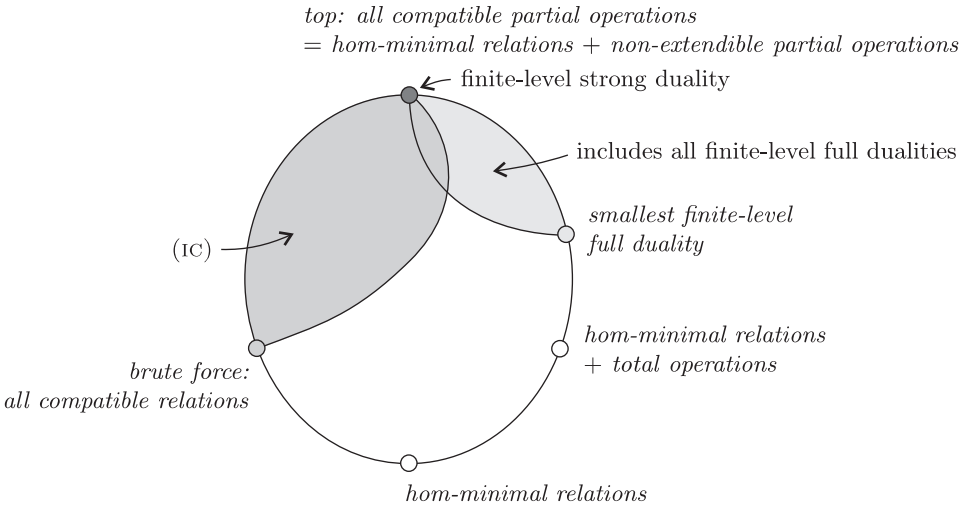

FIGURE 4.2. *The lattice $\mathcal{D}_{\mathbf{M}}$ of alter egos that dualise $\mathbf{M}$ at the finite level*

Proof. Based on Remark 2.3, define the set of partial operations $G_i := F_i \cup (\Pi{\restriction}R_i)$ for the structure $\mathbf{M}_i = \langle M; F_i, R_i\rangle$. As $\mathbf{M}_0$ and $\mathbf{M}_1$ are compatible, we have $G_0^{\Diamond\Diamond} \subseteq G_1^{\Diamond}$. So our Galois connection gives the equivalences

$$G_1^{\Diamond\Diamond} = G_0^{\Diamond} \iff G_0^{\Diamond\Diamond} = G_1^{\Diamond} \iff G_1^{\Diamond} \subseteq G_0^{\Diamond\Diamond},$$

whence conditions (i), (ii) and (v) are all equivalent. The IC Lemma 4.5 immediately gives (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). Use the IC Lemma 4.5 with the Duality Lemma 4.1 for (iii) $\Leftrightarrow$ (v), and with the Full Duality Lemma 4.3 for (iv) $\Leftrightarrow$ (v). □

If the equivalent conditions of the previous lemma hold, then we will say that $\underset{\sim}{\mathbf{M}}_1$ *strongly dualises* $\mathbf{M}_0$ *at the finite level*. The symmetric conditions (i) and (ii) tell us that finite-level strong duality is symmetric.

By compiling the results in this section, we can depict the general shape of the lattice of finite-level dualities $\mathcal{D}_{\mathbf{M}}$; see Figure 4.2. For a particular structure $\mathbf{M}$, the shape of the lattice $\mathcal{D}_{\mathbf{M}}$ may be a 'collapsed' version of that shown in the figure.

EXAMPLE 4.7. Consider the two-element bounded lattice $\mathbf{2}$. We have already seen that $|\mathcal{A}_{\mathbf{2}}| = 2$, in Example 2.9, and that $|\mathcal{D}_{\mathbf{2}}| = 1$, in Example 3.5.

We can use the three-element bounded lattice $\mathbf{3}$ to show that the five 'different' alter egos labelled in our general picture really can be different.

EXAMPLE 4.8. Let $\mathbf{3} = \langle\{0, a, 1\}; \vee, \wedge, \underline{0}, \underline{1}\rangle$ be the three-element bounded lattice, with $0 < a < 1$. Define the two endomorphisms f and g of $\mathbf{3}$ and the binary compatible partial operations σ and h on $\mathbf{3}$ as in Figure 4.3. Using results from the literature, we can find an alter ego of $\mathbf{3}$ to represent each of the labelled elements in Figure 4.2.

- *top*: f, g, σ yield a strong duality, by Davey and Haviar [**7**].
- *smallest finite-level full*: f, g, h yield a finite-level full duality, by Davey, Haviar and Willard [**13**]; as f and g are total and the domain of h is definable via $g(v_1) = f(v_2)$, it follows by Lemma 4.3 that f, g, h are structural functions of every alter ego that fully dualises $\mathbf{3}$ at the finite level.

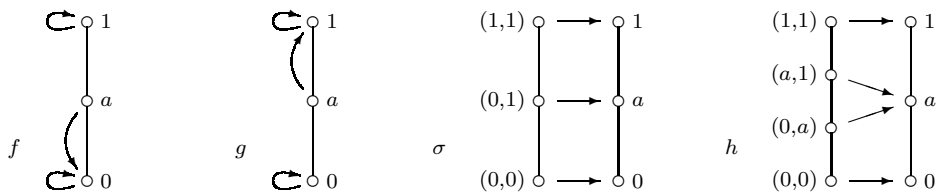

FIGURE 4.3. *The compatible operations f, g, σ and h on* **3**

– *hom-minimals + totals*: f, g yield a duality, by Davey, Haviar and Priestley [**11**]; every compatible total operation on **3** is essentially unary, and so f, g generate the clone of all compatible total operations on **3**.
– *hom-minimals*: graph(σ) yields a duality, as it entails the two relations graph(f) and graph(g) on duals.
– *brute force*: every compatible relation on **3** is conjunct-atomic definable from the binary compatible relations.

The structural-reduct relation between these five alter egos is exactly that shown implicitly in Figure 4.2.

4.1. *Symmetry breakdown*

Beyond the finite level, the topological side is no longer discrete and so the symmetry breaks down. We can use the three-element bounded lattice **3** as an illustration. Since the alter ego $\underset{\sim}{\mathbf{3}}_h := \langle \{0, a, 1\}; f, g, h, \mathcal{T} \rangle$ fully dualises **3** at the finite level, it follows by the Full Duality Lemma 4.3 that $\underset{\sim}{\mathbf{3}}$ fully dualises the partial algebra $\mathbf{3}_h$ at the finite level. By the Full Duality Lifting Theorem [**17**, **6**], this implies that $\underset{\sim}{\mathbf{3}}$ fully dualises $\mathbf{3}_h$. But $\underset{\sim}{\mathbf{3}}_h$ does not fully dualise **3** [**13**], and so full duality is not symmetric.

4.2. *Strong duality and injectivity*

As we are mainly working at the finite level, we shall not really need a definition of strong duality for our expanded setting. But, by analogy with the usual setting, we can say that $\underset{\sim}{\mathbf{M}}_1$ *strongly dualises* $\mathbf{M}_0$ if

– $\underset{\sim}{\mathbf{M}}_1$ dualises $\mathbf{M}_0$ and,
– for each non-empty set S and topologically closed substructure $\mathbf{X}$ of $(\underset{\sim}{\mathbf{M}}_1)^S$, the S-ary relation X on M is conjunct-atomic definable from $\mathbf{M}_0$.

(As in the extension of the Closure Lemma 3.9, we allow infinite conjunctions.) The proof for the usual setting can be modified to show that every strong duality is full [**5**] (see also [**3**, 3.2.4]).

Injectivity does not play such an important role in our expanded setting as it does in the usual setting. If $\mathbf{M}_0$ is an algebra, then the two equivalent conditions of the IC Lemma 4.5 are also equivalent to:

($*$) $\underset{\sim}{\mathbf{M}}_1$ dualises $\mathbf{M}_0$ at the finite level and $\mathbf{M}_1$ is injective in $\mathbb{ISP}^+_{\text{fin}}(\mathbf{M}_1)$ [**3**, 2.2.5].

This extra equivalence still holds if $\mathbf{M}_0$ is a *total structure*, that is, if every term function of $\mathbf{M}_0$ extends to a total term function of $\mathbf{M}_0$. Indeed, a finite structure $\mathbf{M}_0$ has an alter ego $\underset{\sim}{\mathbf{M}}_1$ that satisfies condition ($*$) above if and only if $\mathbf{M}_0$ is a total structure. So an alter ego that strongly dualises $\mathbf{M}_0$ at the finite level does not have to be injective.

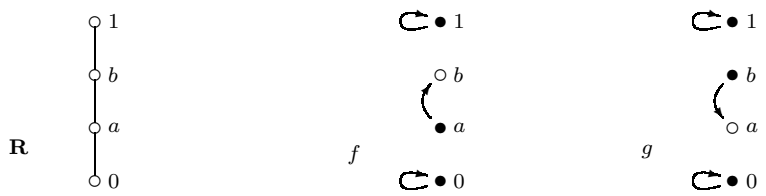

FIGURE 5.4. *The partial automorphisms f and g of* $\mathbf{R}$

In the usual setting, an alter ego $\underset{\sim}{\mathbf{M}}_1$ strongly dualises a finite algebra $\mathbf{M}_0$ if and only if $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ and $\underset{\sim}{\mathbf{M}}_1$ is injective in the dual category it generates [**3**, 3.2.4]. In our symmetric setting, this equivalence extends to the case that $\mathbf{M}_0$ is a total structure, but not beyond.

5. *Full duality at the finite level*

The main focus of this section is to prove that, within the lattice $\mathcal{A}_{\mathbf{M}}$ of all alter egos of a base structure $\mathbf{M}$, those that yield a finite-level full duality form a complete sublattice. We start by looking at enrichment of full dualities.

If $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ are alter egos of a base structure $\mathbf{M}_0$ such that $\underset{\sim}{\mathbf{M}}_1$ is a structural reduct of $\underset{\sim}{\mathbf{M}}_2$, then we will call $\underset{\sim}{\mathbf{M}}_2$ an *enrichment* of $\underset{\sim}{\mathbf{M}}_1$ (relative to $\mathbf{M}_0$). So the enrichments of $\underset{\sim}{\mathbf{M}}_1$ form the principal filter generated by $\underset{\sim}{\mathbf{M}}_1$ in the lattice of alter egos $\mathcal{A}_{\mathbf{M}_0}$. We seek an answer to the question:

($\star$) *if $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ [at the finite level], under what conditions does an enrichment $\underset{\sim}{\mathbf{M}}_2$ of $\underset{\sim}{\mathbf{M}}_1$ also fully dualise $\mathbf{M}_0$ [at the finite level]?*

The answer is definitely not 'Always', as shown by the following example.

EXAMPLE 5.1. The first known example of a full but not strong duality, due to Clark, Davey and Willard [**4**], is built on the four-element lattice-based algebra

$$\mathbf{R} := \langle \{0, a, b, 1\}; t, \vee, \wedge, \underline{0}, \underline{1} \rangle,$$

where $0 < a < b < 1$ and the operation t is the ternary discriminator. Define the following two alter egos of $\mathbf{R}$:

$$\underset{\sim}{\mathbf{R}}_\perp := \langle \{0, a, b, 1\}; \mathrm{graph}(f), \mathcal{T} \rangle \quad \text{and} \quad \underset{\sim}{\mathbf{R}}_\top := \langle \{0, a, b, 1\}; f, g, \mathcal{T} \rangle,$$

where the partial automorphisms f and g of $\mathbf{R}$ are shown in Figure 5.4. Then $\underset{\sim}{\mathbf{R}}_\top$ strongly dualises $\mathbf{R}$, by the Quasi-primal Strong Duality Theorem [**15**] (see also [**3**, 3.3.13]). As $\underset{\sim}{\mathbf{R}}_\perp$ and $\underset{\sim}{\mathbf{R}}_\top$ are clearly not equivalent, the alter ego $\underset{\sim}{\mathbf{R}}_\perp$ cannot strongly dualise $\mathbf{R}$. Nevertheless Clark, Davey and Willard showed that $\underset{\sim}{\mathbf{R}}_\perp$ fully dualises $\mathbf{R}$. Since $\mathrm{graph}(f)$ is a hom-minimal relation on $\mathbf{R}$, the alter ego $\underset{\sim}{\mathbf{R}}_\perp$ represents the bottom element of the lattice $\mathcal{F}_{\mathbf{R}}$ of finite-level full dualities.

What happens if we enrich $\underset{\sim}{\mathbf{R}}_\perp$? The alter ego $\underset{\sim}{\mathbf{R}}_1 := \langle R; \mathrm{graph}(f), \mathrm{dom}(f), \mathcal{T} \rangle$, for example, does not satisfy condition (iv)(b) of the Full Duality Lemma 4.3, and so does not fully dualise $\mathbf{R}$ even at the finite level. If we want to add $\mathrm{dom}(f)$ to the type of $\underset{\sim}{\mathbf{R}}_\perp$ and retain a full duality, we must also add the partial operation f:

$$\underset{\sim}{\mathbf{R}}_2 := \langle R; f, \mathcal{T} \rangle \equiv \langle R; f, \mathrm{graph}(f), \mathrm{dom}(f), \mathcal{T} \rangle.$$

But is this enough? We haven't checked that *every* compatible partial operation on $\mathbf{R}$ with domain conjunct-atomic definable from $\mathbf{R}_2$ extends to a term function of $\mathbf{R}_2$. We'll see in this section that we don't need to.

The key tool in our study of the enrichment of full dualities will be entailment. We shall extract yet another theorem from our general entailment lemma. In this theorem, we consider two alter egos $\underset{\sim}{\mathbf{M}}_1 \sqsubseteq \underset{\sim}{\mathbf{M}}_2$ of a finite base structure $\mathbf{M}_0$. Using Lemma 3.3, we know that a concrete structure $\mathbf{X}$ in $\mathcal{X}_2 := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ has the same underlying set as a concrete structure $\mathbf{X}^\flat$ in $\mathcal{X}_1 := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$. So, for a compatible relation r on $\mathbf{M}_0$, we can say that $\underset{\sim}{\mathbf{M}}_1$ *entails* r *on* $\mathbf{X}$ to mean that $\underset{\sim}{\mathbf{M}}_1$ entails r on $\mathbf{X}^\flat$. For each $i \in \{1,2\}$, we shall use $\mathrm{D}_i : \mathcal{A} \to \mathcal{X}_i$ and $\mathrm{E}_i : \mathcal{X}_i \to \mathcal{A}$ for the dual functors with respect to the alter ego $\underset{\sim}{\mathbf{M}}_i$.

Full Entailment Theorem 5.2. *Let $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ be alter egos of a base structure $\mathbf{M}_0$, with $\underset{\sim}{\mathbf{M}}_1$ a structural reduct of $\underset{\sim}{\mathbf{M}}_2$. Let r be a compatible relation on $\mathbf{M}_0$. Then the following are equivalent*:

(i) *$\underset{\sim}{\mathbf{M}}_1$ entails r on every concrete structure in $\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$*;
(ii) *$\underset{\sim}{\mathbf{M}}_1$ entails r on the substructure of $\mathrm{D}_2(\mathbf{r})$ generated by the projections*;
(iii) *r is primitive-positive definable from $\mathbf{M}_1$ with existence witnessed by $\mathbf{M}_2$.*

Proof. Say that $\underset{\sim}{\mathbf{M}}_i = \langle M; H_i, R_i, \mathcal{T}\rangle$, for each $i \in \{1,2\}$. To prove (ii) $\Rightarrow$ (iii), we use Lemma 1.9 with $F_i := H_i \cup (\Pi{\restriction}R_i)$; see Remark 3.7. □

The previous theorem generalises the Dual Entailment Theorem 3.6 (take $\underset{\sim}{\mathbf{M}}_2$ to be $\underset{\sim}{\mathbf{M}}_1$ enriched with $\hom(\mathbf{r}, \mathbf{M}_0)$) and the Structural Entailment Theorem 3.8 (take $\underset{\sim}{\mathbf{M}}_2 = \underset{\sim}{\mathbf{M}}_1$).

Our next theorem describes exactly when enrichment of an alter ego preserves full duality and finite-level full duality. (Either include or delete both phrases in square brackets. The theorem also holds in the usual setting.)

Full Enrichment Theorem 5.3. *Let $\underset{\sim}{\mathbf{M}}_1 = \langle M; F_1, R_1, \mathcal{T}\rangle$ and $\underset{\sim}{\mathbf{M}}_2 = \langle M; F_2, R_2, \mathcal{T}\rangle$ be alter egos of a base structure $\mathbf{M}_0$, with $\underset{\sim}{\mathbf{M}}_1$ a structural reduct of $\underset{\sim}{\mathbf{M}}_2$. Define $F_\delta := F_2 \backslash F_1$ and $R_\delta := R_2 \backslash R_1$. If $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ [at the finite level], then the following are equivalent*:

(i) *$\underset{\sim}{\mathbf{M}}_2$ fully dualises $\mathbf{M}_0$ [at the finite level]*;
(ii) *for each $r \in R_\delta \cup \mathrm{dom}(F_\delta)$, every compatible partial operation on $\mathbf{M}_0$ with domain r extends to a term function of $\mathbf{M}_2$*;
(iii) *for each $r \in R_\delta \cup \mathrm{dom}(F_\delta)$, the dual $\mathrm{D}_2(\mathbf{r})$ is generated by the projections.*

Proof. Assume that $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ [at the finite level]. The implication (i) $\Rightarrow$ (ii) follows by the Full Duality Lemma 4.3. The equivalence (ii) $\Leftrightarrow$ (iii) is easy; see Remark 3.7. It remains to prove that (iii) $\Rightarrow$ (i).

Assume that (iii) holds. As [finite-level] duality is always preserved by enrichment of the alter ego, we just need to show that $\underset{\sim}{\mathbf{M}}_2$ co-dualises $\mathbf{M}_0$ [at the finite level]. So let $\mathbf{X} \leqslant (\underset{\sim}{\mathbf{M}}_2)^S$, for some [finite,] non-empty set S. It is now enough to show that $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathrm{D}_2\mathrm{E}_2(\mathbf{X})$ is surjective, that is, to show that each homomorphism $\varphi : \mathrm{E}_2(\mathbf{X}) \to \mathbf{M}_0$ is an evaluation.

By Lemma 3.3, every structure $\mathbf{Z}$ in $\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ determines a structure $\mathbf{Z}^\flat$ in $\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$. Since we are assuming that $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ [at the finite level], each homomorphism $\varphi : \mathrm{E}_1(\mathbf{X}^\flat) \to \mathbf{M}_0$ is an evaluation. So it now suffices to show that $\mathrm{E}_2(\mathbf{X}) = \mathrm{E}_1(\mathbf{X}^\flat)$ in $\mathcal{A} := \mathbb{ISP}^+(\mathbf{M}_0)$.

Using Lemma 3.3 again, every continuous homomorphism $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}_2$ is also a continuous homomorphism $\alpha : \mathbf{X}^\flat \to \underset{\sim}{\mathbf{M}}_1$. Now let $\beta : \mathbf{X}^\flat \to \underset{\sim}{\mathbf{M}}_1$. To show that $\beta : \mathbf{X} \to \underset{\sim}{\mathbf{M}}_2$ is a homomorphism, we can show that $\underset{\sim}{\mathbf{M}}_1$ entails s on $\mathbf{X}^\flat$, for each $s \in R_\delta \cup \mathrm{graph}(F_\delta)$.

So let $s \in R_\delta \cup \mathrm{graph}(F_\delta)$. As $\underset{\sim}{\mathbf{M}}_1$ dualises $\mathbf{M}_0$ at the finite level, we know that $\underset{\sim}{\mathbf{M}}_1$ entails s on $\mathrm{D}_1(\mathbf{s}) = \mathrm{D}_2(\mathbf{s})^\flat$. By the Full Entailment Theorem 5.2, it is now enough to show that $\mathrm{D}_2(\mathbf{s})$ is generated by the projections. If $s \in R_\delta$, then this follows directly from assumption (iii). If $s = \mathrm{graph}(f)$, for some $f \in F_\delta$, then this is an easy consequence of our assumption that $\mathrm{D}_2(\mathbf{r})$ is generated by the projections, where $r := \mathrm{dom}(f)$. □

We already knew that condition (ii) in the previous theorem is necessary for enrichment of an alter ego to preserve full duality [at the finite level]. We have now seen that it is sufficient. This effectively answers the question ($\star$) from the beginning of this section. As an application, we now see immediately that the alter ego $\underset{\sim}{\mathbf{R}}_2$ from Example 5.1 fully dualises $\mathbf{R}$.

Corollary 5.4. *Let $\mathbf{M}$ be a base structure. Assume that $\{\, \underset{\sim}{\mathbf{M}}_i \mid i \in I \,\}$ is a non-empty collection of alter egos of $\mathbf{M}$, each of which fully dualises $\mathbf{M}$ [at the finite level]. Then the alter ego $\bigcup_{i \in I} \underset{\sim}{\mathbf{M}}_i$ fully dualises $\mathbf{M}$ [at the finite level].*

Proof. Combine the Full Enrichment Theorem 5.3 and Full Duality Lemma 4.3. □

Corollary 5.5. *Let $\mathbf{M}$ be a base structure.*

(i) *The lattice $\mathcal{F}_{\mathbf{M}}$, of all alter egos that fully dualise $\mathbf{M}$ at the finite level, is a complete sublattice of $\mathcal{A}_{\mathbf{M}}$ and is therefore doubly algebraic.*

(ii) *The alter egos that fully dualise $\mathbf{M}$ form an increasing subset of $\mathcal{F}_{\mathbf{M}}$.*

Proof. For (i), use the previous corollary together with Theorem 2.6, Remark 2.7 and Corollary 4.4. For (ii), use the Full Enrichment Theorem 5.3. □

We know that, even if $\mathbf{M}$ is fully dualisable, not every finite-level full duality lifts to a full duality; see Subsection 4.1.

Question 5.6. Assume that $\mathbf{M}$ is fully dualisable. Do the alter egos that fully dualise $\mathbf{M}$ necessarily form a filter of $\mathcal{F}_{\mathbf{M}}$?

We finish the paper with two applications of our results. First we use the Full Enrichment Theorem 5.3 to draw a particular lattice of finite-level full dualities.

Example 5.7. The lattice $\mathcal{F}_{\mathbf{R}}$ is shown in Figure 5.5, where $\mathbf{R}$ is the quasi-primal algebra from Example 5.1. The elements of $\mathcal{F}_{\mathbf{R}}$ are labelled by sets of relations: the element labelled by a set S is represented by the alter ego

$$\underset{\sim}{\mathbf{R}}_S := \langle R; H_S, \mathcal{T} \rangle, \quad \text{where} \quad H_S := \{\, h : \mathbf{s} \to \mathbf{R} \mid s \in S \,\}.$$

As the bottom element $\underset{\sim}{\mathbf{R}}_\perp$ of $\mathcal{F}_{\mathbf{R}}$ yields a full duality, we see by Corollary 5.5(ii) that every element of $\mathcal{F}_{\mathbf{R}}$ corresponds to a full duality.

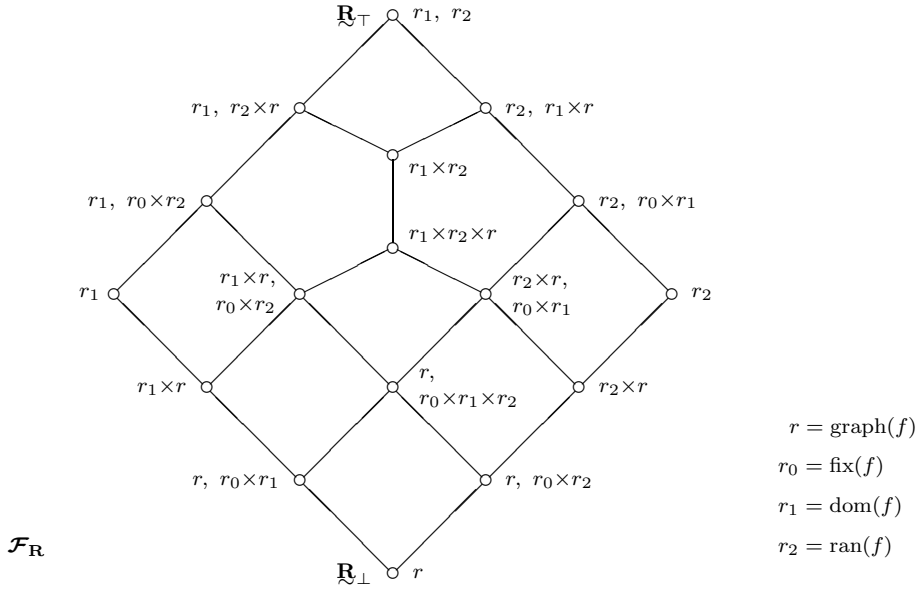

FIGURE 5.5. *The full dualities for* $\mathbf{R} = \langle \{0, a, b, 1\}; t, \vee, \wedge, \underline{0}, \underline{1} \rangle$

Sketch proof. Let $\mathcal{D}$ be the lattice drawn in Figure 5.5, with the labelling sets of relations as its elements. We aim to establish an order-isomorphism $\zeta : \mathcal{D} \to \mathcal{F}_{\mathbf{R}}$, where $\zeta(S)$ is represented by the alter ego $\underset{\sim}{\mathbf{R}}_S$. As both the lattices $\mathcal{D}$ and $\mathcal{F}_{\mathbf{R}}$ are finite (by Example 2.10), the following five checks will suffice:

(i) ζ is well defined;
(ii) ζ is order-preserving;
(iii) ζ preserves joins of irredundant sets of join-irreducibles of $\mathcal{D}$;
(iv) ζ separates each meet-irreducible of $\mathcal{D}$ from its unique upper cover;
(v) $\zeta(\mathcal{D})$ contains all the join-irreducibles of $\mathcal{F}_{\mathbf{R}}$.

The necessary calculations for each of these checks are outlined very briefly below. The symmetry of the lattice $\mathcal{D}$ can be used to cut back some of the calculations.

(i) For each S in $\mathcal{D}$, check that $\mathrm{graph}(f) \in \mathrm{Def}_{\mathrm{ca}}(\mathbf{R}_S)$. As $\underset{\sim}{\mathbf{R}}_\perp$ fully dualises $\mathbf{R}$, it then follows by the Full Enrichment Theorem that $\underset{\sim}{\mathbf{R}}_S$ fully dualises $\mathbf{R}$.
(ii) For each cover $S \prec T$ in $\mathcal{D}$, check that $S \subseteq \mathrm{Def}_{\mathrm{ca}}(\mathbf{R}_T)$. By Check (i), you can then apply the Full Duality Lemma 4.3 to deduce that $\mathbf{R}_S \sqsubseteq \mathbf{R}_T$.
(iii) For each irredundant join of join-irreducibles $S_1 \vee \cdots \vee S_n = T$ in $\mathcal{D}$, check that $T \subseteq \mathrm{Def}_{\mathrm{ca}}(\mathbf{R}_{S_1} \cup \cdots \cup \mathbf{R}_{S_n})$.
(iv) For each meet-irreducible S with upper cover T, find some set $X \subseteq R^n$ that is closed under every $h \in H_S$ but not closed under some $k \in H_T$.
(v) Each join-irreducible of $\mathcal{F}_{\mathbf{R}}$ is equivalent to an alter ego of the form $\underset{\sim}{\mathbf{R}}_{\{r,s\}}$, where $r := \mathrm{graph}(f)$ and s is a compatible relation on $\mathbf{R}$. Indeed, we can assume that the relation s is a product, without any repetition of factors, of the relations r, r_0, r_1 and r_2 defined in Figure 5.5. (See the proof of Example 2.10.) For each such relation s, find some T in $\mathcal{D}$ such that $\mathbf{R}_T \equiv \mathbf{R}_{\{r,s\}}$.

It follows that the drawn lattice $\mathcal{D}$ correctly represents the lattice $\mathcal{F}_{\mathbf{R}}$. □

For the second application of our results, we use our new description of finite-level full duality to show that each finite non-boolean bounded distributive lattice has a full but not strong duality at the finite level (Davey, Haviar, Niven and Perkal [**8**]).

The Full Duality Lemma 4.3 tells us that an alter ego $\underset{\sim}{\mathbf{M}}_1$ that fully dualises a fixed structure $\mathbf{M}_0$ at the finite level is determined (up to structural equivalence) by its set of conjunct-atomic definable relations $\mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_1)$. The following lemma completely characterises these sets of relations.

Definition 5.8. Let r and s be relations on a finite non-empty set M, of arities $k, \ell \geqslant 1$, and let $f_1, \dots, f_\ell : r \to M$. Define the set

$$(f_1, \dots, f_\ell)^{-1}(s) := \big\{\, \vec{a} \in r \;\big|\; (f_1(\vec{a}), \dots, f_\ell(\vec{a})) \in s \,\big\}.$$

If this set is non-empty, then it is a k-ary relation on M that is conjunct-atomic definable from $\{s, f_1, \dots, f_\ell\}$.

Lemma 5.9. *Let R be a set of compatible relations on a base structure $\mathbf{M}_0$. Then $R = \mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_1)$, for some alter ego $\underset{\sim}{\mathbf{M}}_1$ that fully dualises $\mathbf{M}_0$ at the finite level, if and only if*

(i) *R contains the equality relation Δ_M and the total relation M^k, for all $k \geqslant 1$,*
(ii) *R contains all the hom-minimal relations on $\mathbf{M}_0$, and*
(iii) *R contains each relation of the form $(f_1, \dots, f_\ell)^{-1}(s)$, for relations $r, s \in R$ of arities $k, \ell \geqslant 1$ and homomorphisms $f_1, \dots, f_\ell : \mathbf{r} \to \mathbf{M}_0$.*

Proof. First assume that $R = \mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_1)$, for some $\underset{\sim}{\mathbf{M}}_1$ that fully dualises $\mathbf{M}_0$ at the finite level. Then conditions (i) to (iii) hold, by the Full Duality Lemma 4.3.

Now assume that (i) to (iii) hold. Define $\mathbf{M}_R := \langle M; H_R \rangle$, where H_R is the set of all compatible partial operations on $\mathbf{M}_0$ with domain in R. Then $R \subseteq \mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_R)$. We shall show that $\mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_R) \subseteq R$. Using (ii), it will then follow straight from the Full Duality Lemma 4.3 that $\underset{\sim}{\mathbf{M}}_R$ fully dualises $\mathbf{M}_0$ at the finite level.

We first show that R is closed under non-empty, pairwise intersection. Choose $r, s \in R$, both of arity $k \geqslant 1$, such that $r \cap s \neq \varnothing$. Using (iii) with the projections $\rho_1, \dots, \rho_k : \mathbf{r} \to \mathbf{M}_0$, we have

$$r \cap s = \big\{\, \vec{a} \in r \;\big|\; \big(\rho_1(\vec{a}), \dots, \rho_k(\vec{a})\big) \in s \,\big\} = (\rho_1, \dots, \rho_k)^{-1}(s) \in R.$$

Thus R is closed under intersection.

We next show that H_R is a partial clone on M. All the total projections belong to H_R, by assumption (i). Now let $f, g_1, \dots, g_k \in H_R$, where f has arity $k \geqslant 1$, each g_i has arity $\ell \geqslant 1$, and $f(g_1, \dots, g_k)$ is non-empty. We just need to check that $\mathrm{dom}\big(f(g_1, \dots, g_k)\big) \in R$. Since R is closed under intersection, we have $r := \mathrm{dom}(g_1) \cap \dots \cap \mathrm{dom}(g_k) \in R$. As $\mathrm{dom}(f) \in R$, this gives us

$$\mathrm{dom}\big(f(g_1, \dots, g_k)\big) = \big(g_1{\restriction}_r, \dots, g_k{\restriction}_r\big)^{-1}\big(\mathrm{dom}(f)\big) \in R,$$

by (iii). Thus H_R is a partial clone on M.

We want to show that $\mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_R) \subseteq R$. As R is closed under intersection, it suffices to show that R contains every relation that is atomic definable from $\mathbf{M}_R$. So let r_Φ be a relation on M described by a k-variable atomic formula $\Phi(\vec{v})$ of type H_R, for some $k \geqslant 1$. As R contains the equality relation Δ_M and H_R is a partial clone on M, the relation r_Φ is also described by a formula of the form $s\big(f_1(\vec{v}), \dots, f_\ell(\vec{v})\big)$, where $s \in R$ has arity $\ell \geqslant 1$ and $f_1, \dots, f_\ell \in H_R$ have arity k. We have $r := \mathrm{dom}(f_1) \cap \dots \cap \mathrm{dom}(f_\ell) \in R$, since R is closed under intersection. Assumption (iii) gives us

$$r_\Phi = \big\{\, \vec{a} \in r \;\big|\; (f_1(\vec{a}), \dots, f_\ell(\vec{a})) \in s \,\big\} = (f_1{\restriction}_r, \dots, f_\ell{\restriction}_r)^{-1}(s) \in R.$$

Thus $\mathrm{Def}_{\mathrm{ca}}(\mathbf{M}_R) \subseteq R$, as required. □

Since finite-level strong dualities all correspond to the top element of $\mathcal{F}_{\mathbf{M}}$, a base structure $\mathbf{M}$ admits a finite-level full but not strong duality if and only if the lattice $\mathcal{F}_{\mathbf{M}}$ is non-trivial.

Example 5.10. *For each finite non-boolean bounded distributive lattice* $\mathbf{L}$, *the lattice of finite-level full dualities* $\mathcal{F}_{\mathbf{L}}$ *is non-trivial.*

Proof. Let C be the set of all complemented elements of $\mathbf{L} = \langle L; \vee, \wedge, \underline{0}, \underline{1}\rangle$. Then C forms a boolean sublattice of $\mathbf{L}$. For $k \geqslant 1$, we say that a k-ary compatible relation r on $\mathbf{L}$ is *quasi-boolean* if every element of $r \cap C^k$ has a complement in the sublattice $\mathbf{r}$ of $\mathbf{L}^k$. Let R_β denote the set of all quasi-boolean relations on $\mathbf{L}$.

We show that $R_\beta = \mathrm{Def}_{\mathrm{ca}}(\mathbf{L}_\beta)$, for some alter ego $\underset{\sim}{\mathbf{L}}_\beta$ that fully dualises $\mathbf{L}$ at the finite level. This alter ego cannot represent the top of $\mathcal{F}_{\mathbf{L}}$, as not every compatible relation on $\mathbf{L}$ is quasi-boolean. (For example, the binary relation $\{(0,0),(0,1),(1,1)\}$.) To do this, we show that R_β satisfies conditions (i), (ii) and (iii) of the previous lemma. Clearly, the relations Δ_L and L^k are quasi-boolean, for all $k \geqslant 1$. So (i) holds.

For (ii), let r be a k-ary compatible relation on $\mathbf{L}$ that is not quasi-boolean. We want to prove that r is not hom-minimal. There is some $\vec{c} \in r \cap C^k$ with no complement in $\mathbf{r}$. This implies there is a homomorphism $\varphi : \mathbf{r} \to \mathbf{3}$ with $\varphi(\vec{c}) = a \notin \{0,1\}$. (Hint: Use Priestley duality.) Since $\mathbf{L}$ is non-boolean, there is an embedding $\xi : \mathbf{3} \to \mathbf{L}$ with $\xi(a) \notin C$. We now have $x := \xi \circ \varphi : \mathbf{r} \to \mathbf{L}$ with $x(\vec{c}) \notin C$. As $\vec{c} \in C^k$, the homomorphism x cannot be a projection. Thus r is not hom-minimal.

To prove that (iii) holds, let r and s be quasi-boolean relations on $\mathbf{L}$, of arities $k, \ell \geqslant 1$. Let $\psi : \mathbf{r} \to \mathbf{L}^\ell$ be a homomorphism. We need to show that the k-ary relation $\psi^{-1}(s)$ on $\mathbf{L}$ is quasi-boolean. To do this, let $\vec{c} \in \psi^{-1}(s) \cap C^k \subseteq r \cap C^k$. Since r is quasi-boolean, we know there is a complement $\vec{c}^*$ of $\vec{c}$ in $\mathbf{r}$. As ψ is a homomorphism, the elements $\psi(\vec{c})$ and $\psi(\vec{c}^*)$ are complements in $\mathbf{L}^\ell$. Thus $\psi(\vec{c}), \psi(\vec{c}^*) \in C^\ell$. Since $\psi(\vec{c}) \in s$ and s is quasi-boolean, it follows that $\psi(\vec{c}^*) \in s$. Thus $\vec{c}^* \in \psi^{-1}(s)$, and so $\psi^{-1}(s)$ is quasi-boolean. □

6. *Appendix: Alternative settings*

We chose to develop our duality theory within a somewhat restricted setting (excluding empty operations, nullary operations, empty structures and empty-indexed products). We shall now briefly discuss three alternative settings that we could have chosen instead. However, we wish to emphasise that the general theory only needs to be developed in one setting:

– any particular duality in one setting can easily be reformulated in any other setting, by making slight adjustments to the type of the alter ego.

The four settings for duality theory that we consider are summarised in Table 6.1, which will be explained through the course of this section. We note here that the class operator $\mathbb{P}$ allows the empty-indexed product and $\mathbb{P}^+$ does not, while the class operators $\mathbb{S}^0, \mathbb{S}^0_{\mathrm{c}}$ allow the empty structure (if it is a substructure) and $\mathbb{S}, \mathbb{S}_{\mathrm{c}}$ do not.

Each of the four settings arises naturally from a Galois connection.

6.1. *Alternative Galois connections*

Fix a finite non-empty set M. By extending the kinds of partial operations that we consider, we shall define two slight modifications of the Galois connection $F \mapsto F^\Diamond$ introduced in Section 1. One of these modified Galois connections will serve as the basis for the supersymmetric setting, and the other as the basis for the two asymmetric settings.

setting	$\mathbf{M}_0$ $\mathbf{M}_1$	'algebraic' side $\boldsymbol{\mathcal{A}}$	topological side $\boldsymbol{\mathfrak{X}}$
symmetric	$\Diamond : \mathcal{P}_M \rightleftarrows \mathcal{P}_M : \Diamond$	$\mathbb{ISP}^+(\mathbf{M}_0)$	$\mathbb{IS}_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$
usual	$\blacktriangleright : \mathcal{P}^{\mathrm{L}}_M \rightleftarrows \mathcal{P}^{\mathrm{R}}_M : \blacktriangleleft$	$\mathbb{ISP}(\mathbf{M}_0)$	$\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$
co-usual	$\blacktriangleleft : \mathcal{P}^{\mathrm{R}}_M \rightleftarrows \mathcal{P}^{\mathrm{L}}_M : \blacktriangleright$	$\mathbb{IS}^0\mathbb{P}^+(\mathbf{M}_0)$	$\mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{M}}_1)$
supersymmetric	$\blacklozenge : \mathcal{P}^{\sharp}_M \rightleftarrows \mathcal{P}^{\sharp}_M : \blacklozenge$	$\mathbb{IS}^0\mathbb{P}(\mathbf{M}_0)$	$\mathbb{IS}^0_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{M}}_1)$

TABLE 6.1. *Four possible settings for a duality*

DEFINITION 6.1. For all $a \in M$, let $\widehat{a} : M^0 \to M$ denote the nullary operation with value a. For all $k \geqslant 0$, let $\eta_k : \varnothing \to M$ denote the k-ary empty operation. Now define the set of all finitary partial operations

$$\mathcal{P}^{\sharp}_M := \mathcal{P}_M \cup \{\, \widehat{a} \mid a \in M \,\} \cup \{\, \eta_k \mid k \geqslant 0 \,\}.$$

Mimicking Definition 1.2, we extend the definition of *compatibility* to partial operations in $\mathcal{P}^{\sharp}_M$. (To do so, we must allow $k \times \ell$ matrices over M, for all $k, \ell \geqslant 0$.) For example, it is easy to check that η_0 is compatible with η_1 but not with itself. Mimicking Definition 1.4, we now obtain a Galois connection $F \mapsto F^{\blacklozenge}$ from the power-set lattice $\boldsymbol{\wp}(\mathcal{P}^{\sharp}_M)$ to itself.

DEFINITION 6.2. Define the two sets of partial operations

$$\mathcal{P}^{\mathrm{L}}_M = \mathcal{P}_M \cup \{\, \eta_k \mid k \geqslant 1 \,\} \quad \text{and} \quad \mathcal{P}^{\mathrm{R}}_M = \mathcal{P}_M \cup \{\, \widehat{a} \mid a \in M \,\}.$$

By restricting compatibility to $\mathcal{P}^{\mathrm{L}}_M \times \mathcal{P}^{\mathrm{R}}_M$, we obtain a Galois connection $^{\blacktriangleright} : \boldsymbol{\wp}(\mathcal{P}^{\mathrm{L}}_M) \to \boldsymbol{\wp}(\mathcal{P}^{\mathrm{R}}_M)$ and $^{\blacktriangleleft} : \boldsymbol{\wp}(\mathcal{P}^{\mathrm{R}}_M) \to \boldsymbol{\wp}(\mathcal{P}^{\mathrm{L}}_M)$.

We look more closely at these two alternative Galois connections in the final subsection.

6.2. *Alternative duality theories*

We can now identify four natural settings for duality theory; see Table 6.1 again. Each setting is determined by the choice of the 'algebraic' side $\boldsymbol{\mathcal{A}}$, since we follow the rule of thumb:

– empty-indexed powers are permitted on one side if and only if empty substructures are permitted on the other.

To simplify the discussion, we restrict to the case that both $\mathbf{M}_0$ and $\mathbf{M}_1$ are partial algebras. (Note that essentially we are still allowed non-nullary relations in the types of our structures, since a non-nullary relation r is interchangeable with the restricted projection $\pi_1{\restriction}_r$.)

In each of the four settings, the types allowed for the two partial algebras $\mathbf{M}_0$ and $\mathbf{M}_1$ on the set M reflects the construction of the two sides:

– for each $k \geqslant 0$, a k-ary partial operation $f : r \to M$ in the type of $\mathbf{M}_1$ must be a homomorphism $f : \mathbf{r} \to \mathbf{M}_0$, with $\mathbf{r} \leqslant (\mathbf{M}_0)^k$, on the 'algebraic' side $\boldsymbol{\mathcal{A}}$;
– for each $k \geqslant 0$, a k-ary partial operation $f : r \to M$ in the type of $\mathbf{M}_0$ must be a homomorphism $f : \mathbf{r} \to \mathbf{M}_1$, with $\mathbf{r} \leqslant (\mathbf{M}_1)^k$, on the topological side $\boldsymbol{\mathfrak{X}}$.

This corresponds to the compatibility of $\mathbf{M}_0$ and $\mathbf{M}_1$ under the appropriate Galois connection, as indicated in the table.

REMARK 6.3. While the 'usual' setting in Table 6.1 mimics the set-up used by Clark and Davey [**3**], we no longer insist that $\mathbf{M}_0$ is an algebra, but allow non-nullary partial operations

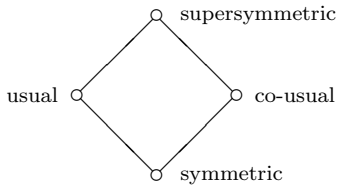

FIGURE 6.6. *The order on the settings*

in its type. Strictly speaking, this new 'usual' setting does not completely generalise the Clark–Davey setting, as we do not allow nullary operations in the type of $\mathbf{M}_0$. But, since the empty structure is excluded from $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M}_0)$, the nullary operation $\widehat{a} : M^0 \to M$ can replace the constant unary operation $\underline{a} : M \to M$ in the type of $\mathbf{M}_0$. (However, as usual [**3**, **6**], the nullary $\widehat{a}$ should be treated as the constant unary $\underline{a}$ in the general theory: for example, in the definition of the partial clone of term functions of $\mathbf{M}_0$.)

We next look at how to transfer a duality from one setting to another, focussing on transfer between the usual and symmetric settings. The four settings have a natural ordering, as shown in Figure 6.6. Dualities can always be 'pushed down' in the obvious way. For example, it is easy to check the following.

LEMMA 6.4. *Let* $(\mathbf{M}_0, \mathbf{M}_1)$ *be a compatible pair in the usual setting. Define the compatible pair* $(\mathbf{M}'_0, \mathbf{M}'_1)$ *in the symmetric setting as follows*:
- *construct* $\mathbf{M}'_0$ *from* $\mathbf{M}_0$ *by removing all empty operations in the type of* $\mathbf{M}_0$, *and*
- *construct* $\mathbf{M}'_1$ *from* $\mathbf{M}_1$ *by replacing each nullary operation* $\widehat{a} : M^0 \to M$ *in the type of* $\mathbf{M}_1$ *with the corresponding constant unary operation* $\underline{a} : M \to M$.

If the original pair $(\mathbf{M}_0, \underset{\sim}{\mathbf{M}}_1)$ *yields a* [*finite-level*] *duality / full duality / strong duality in the usual setting, then the same is true of the new pair* $(\mathbf{M}'_0, \underset{\sim}{\mathbf{M}}'_1)$ *in the symmetric setting.*

Dualities can always be 'pushed up', though not necessarily uniquely. We first look at a concrete example.

EXAMPLE 6.5. Consider the bounded lattice $\mathbf{2} = \langle \{0,1\}; \vee, \wedge, \underline{0}, \underline{1} \rangle$ and the ordered set $\mathbf{2}_{\mathrm{o}} = \langle \{0,1\}; \leqslant \rangle$ from Example 2.9. (Note that $\underline{0}$ and $\underline{1}$ are unary operations.) The pair $(\mathbf{2}, \underset{\sim}{\mathbf{2}}_{\mathrm{o}})$ yields a strong duality in the symmetric setting; this is the 'pushed down' version of Priestley duality [**23**]. So the flipped pair $(\mathbf{2}_{\mathrm{o}}, \underset{\sim}{\mathbf{2}})$ automatically yields a finite-level strong duality in the symmetric setting; see the Strong Duality Lemma 4.6. But in the usual setting, the same pair $(\mathbf{2}_{\mathrm{o}}, \underset{\sim}{\mathbf{2}})$ does not yield a finite-level co-duality (as $\mathrm{E}(\underset{\sim}{\mathbf{2}}) \cong \mathrm{E}(\varnothing)$, where $\varnothing$ denotes the empty substructure of $\underset{\sim}{\mathbf{2}}$).

To transfer the finite-level strong duality based on $(\mathbf{2}_{\mathrm{o}}, \underset{\sim}{\mathbf{2}})$ from the symmetric to the usual setting, we have two options:
- add η_1 to the type of $\mathbf{2}_{\mathrm{o}}$ and leave $\underset{\sim}{\mathbf{2}}$ unchanged, or
- replace $\underline{0}, \underline{1}$ in the type of $\underset{\sim}{\mathbf{2}}$ with the nullaries $\widehat{0}, \widehat{1}$ and leave $\mathbf{2}_{\mathrm{o}}$ unchanged.

In the first case, the 'algebraic' category essentially gains a new terminal element and the topological category gains a new initial element. In the second case, the two categories essentially do not change.

Lemma 6.6. *Let* $(\mathbf{M}_0, \mathbf{M}_1)$ *be a compatible pair in the symmetric setting.*

(i) *If the pair* $(\mathbf{M}_0, \underset{\sim}{\mathbf{M}}_1)$ *yields a* [*finite-level*] *duality in the symmetric setting, then the same is true in the usual setting.*

(ii) *Define a new compatible pair* $(\mathbf{M}'_0, \mathbf{M}'_1)$ *in the usual setting, by choosing either one of the following two options*:
- *add the empty operation* η_1 *to the type of* $\mathbf{M}_0$ *and leave* $\mathbf{M}_1$ *unchanged, or*
- *add the nullary operation* $\widehat{a}$ *to the type of* $\mathbf{M}_1$, *for each* $a \in M$ *that forms a one-element substructure of* $\mathbf{M}_0$, *and leave* $\mathbf{M}_0$ *unchanged.*

If the original pair $(\mathbf{M}_0, \underset{\sim}{\mathbf{M}}_1)$ *yields a* [*finite-level*] *duality / full duality / strong duality in the symmetric setting, then the same is true of the new pair* $(\mathbf{M}'_0, \underset{\sim}{\mathbf{M}}'_1)$ *in the usual setting.*

We conclude this subsection with a cautionary tale about transfer to the supersymmetric setting.

Example 6.7. As in the previous example, we start from Priestley duality in the symmetric setting, based on $\mathbf{2} = \langle \{0,1\}; \vee, \wedge, \underline{0}, \underline{1} \rangle$ and $\underset{\sim}{\mathbf{2}}_{\mathrm{o}} = \langle \{0,1\}; \leqslant \rangle$. But we now shift to the supersymmetric setting. Then $\underset{\sim}{\mathbf{2}}_{\mathrm{o}}$ does not dualise $\mathbf{2}$ at the finite level (as $\mathrm{D}(\mathbf{2}) \cong \mathrm{D}(\varnothing)$, where $\varnothing$ denotes the empty substructure of $\mathbf{2}$).

To obtain a full duality based on $\mathbf{2}$, we can use the alter ego $\underset{\sim}{\mathbf{2}}'_{\mathrm{o}} := \langle \{0,1\}; \eta_1, \leqslant, \mathcal{T} \rangle$. But the empty substructure of $(\underset{\sim}{\mathbf{2}}'_{\mathrm{o}})^0$ is not conjunct-atomic definable from $\mathbf{2}$. To obtain a strong duality, we need to use the alter ego $\langle \{0,1\}; \eta_0, \leqslant, \mathcal{T} \rangle$.

So, in the supersymmetric setting, the structure $\mathbf{2}$ admits a duality that is full but not strong! But this duality is not strong only at the 'trivial' level, and thus this example is very different from the 'genuine' solution of the Full versus Strong Problem given by Clark, Davey and Willard [**4**].

6.3. *The general theory in alternative settings*

As illustrated in the previous subsection, we can choose to develop all our general theory in the symmetric setting (for example), and then reformulate any particular concrete duality that we obtain in another setting if it is more appropriate. We can also translate general results from one setting to another.

We finish this paper by giving a few hints and warnings to an intrepid reader who would like to translate any part of our general theory into another setting. In the discussion, we shall associate each setting in Table 6.1 with the set of partial operations in its '$\mathbf{M}_0$' column.

Definition 6.8. Each of the four sets of partial operations $\mathcal{P}^{?}_{M}$ has an associated set of relations $\mathcal{R}^{?}_{M} := \mathrm{dom}(\mathcal{P}^{?}_{M})$. For the supersymmetric setting, we now extend the definition of *preservation* to $\mathcal{P}^{\sharp}_{M} \times \mathcal{R}^{\sharp}_{M}$. (The matrix-based definition is the easiest to extend.) For the two asymmetric settings, restrict to $\mathcal{P}^{\mathrm{L}}_{M} \times \mathcal{R}^{\mathrm{R}}_{M}$ and $\mathcal{P}^{\mathrm{R}}_{M} \times \mathcal{R}^{\mathrm{L}}_{M}$.

Let $F \subseteq \mathcal{P}^{?}_{M}$. Define $\mathrm{Clo}^{?}_{\mathrm{p}}(F)$ to be the smallest subset of $\mathcal{P}^{?}_{M}$ containing $F \cup \Pi$ that is closed under composition, and define $\mathrm{Def}^{?}_{\mathrm{ca}}(F)$ to be the set of all relations in $\mathcal{R}^{?}_{M}$ that are conjunct-atomic definable from F. (Note that, in the supersymmetric and co-usual settings, the nullary relation M^0 is always definable from F, via the empty conjunction in 0 variables.)

To illustrate the translation process, we state Lemma 1.6 in the usual setting.

Lemma 6.9. *Let $F \cup \{g\} \subseteq \mathcal{P}^{\mathrm{L}}_M$, for some finite non-empty set M. Then the following are equivalent*:

(i) *every relation in $\mathcal{R}^{\mathrm{R}}_M$ that is closed under all $f \in F$ is also closed under g*;
(ii) *the relation $\mathrm{Clo}^{\mathrm{L}}_{\mathrm{p}}(F) \restriction \{r\}$ on M is closed under g, where $r := \mathrm{dom}(g)$*;
(iii) *g has an extension in $\mathrm{Clo}^{\mathrm{L}}_{\mathrm{p}}(F)$*;
(iv) *g preserves the graph of every partial operation in $F^{\blacktriangleright}$.*

Almost every lemma, theorem and corollary in Section 1 remains true when translated into each of the three other settings. (The one exception in discussed in Warning 6.10 below.) In particular, the description of closed sets $F^{\Diamond\Diamond} \subseteq \mathcal{P}_M$ in Theorem 1.11 translates directly into descriptions of the closed sets $F^{\blacklozenge\blacklozenge} \subseteq \mathcal{P}^{\sharp}_M$, $F^{\blacktriangleright\blacktriangleleft} \subseteq \mathcal{P}^{\mathrm{L}}_M$ and $F^{\blacktriangleleft\blacktriangleright} \subseteq \mathcal{P}^{\mathrm{R}}_M$.

Warning 6.10. The supersymmetric setting involves a few annoying technicalities. When Lemma 1.6 is re-interpreted in the supersymmetric setting, the equivalence of (i), (ii) and (iii) still holds. But the equivalence of (iii) and (iv) only holds if $g \neq \eta_0$. Nevertheless, the dependent Theorem 1.11 still holds in the supersymmetric setting: the operation η_0 must be dealt with as a special case in its proof.

The definition of a base structure in Section 2 has a natural modification for each of the three new settings. By way of example, we consider a base structure $\mathbf{M}_0 = \langle M; F\rangle$ in the usual setting:

– F is a subset of $\mathcal{P}^{\mathrm{L}}_M$;
– the *term functions* of $\mathbf{M}_0$ form the subset $\mathrm{Clo}^{\mathrm{L}}_{\mathrm{p}}(F)$ of $\mathcal{P}^{\mathrm{L}}_M$;
– the *structural functions* of $\mathbf{M}_0$ form the closed subset $F^{\blacktriangleright\blacktriangleleft}$ of $\mathcal{P}^{\mathrm{L}}_M$;
– the *compatible partial operations* on $\mathbf{M}_0$ form the closed subset $F^{\blacktriangleright}$ of $\mathcal{P}^{\mathrm{R}}_M$;
– a *compatible* base structure $\mathbf{M}_1$ must belong to the co-usual setting.

To illustrate the translation process for the duality-theoretic results in Section 3–5, we very carefully translate part of the Full Duality Lemma 4.3 into the usual setting.

Lemma 6.11. *Let $(\mathbf{M}_0, \mathbf{M}_1)$ be a pair of compatible base structures in the usual setting. Then $\underset{\sim}{\mathbf{M}}_1$ fully dualises $\mathbf{M}_0$ at the finite level if and only if both the following hold*:

(i) *every hom-minimal relation $r \in \mathcal{R}^{\mathrm{R}}_M$ on $\mathbf{M}_0$ is conjunct-atomic definable from $\mathbf{M}_1$*;
(ii) *for all $r \in \mathrm{Def}^{\mathrm{R}}_{\mathrm{ca}}(\mathbf{M}_1)$, every compatible partial operation on $\mathbf{M}_0$ with domain r extends to a term function of $\mathbf{M}_1$.*

Sketch proof. Let $(\mathbf{M}'_0, \mathbf{M}'_1)$ denote the 'pushed down' version of $(\mathbf{M}_0, \mathbf{M}_1)$ in the symmetric setting, as defined in Lemma 6.4.

First assume that $(\mathbf{M}_0, \underset{\sim}{\mathbf{M}}_1)$ yields a finite-level full duality in the usual setting. By Lemma 6.4, the same is true of $(\mathbf{M}'_0, \underset{\sim}{\mathbf{M}}'_1)$ in the symmetric setting. Now we can use the original Full Duality Lemma 4.3 applied to $(\mathbf{M}'_0, \mathbf{M}'_1)$ to obtain conditions (i) and (ii) above. To see that (i) holds, note that the only relation in $\mathcal{R}^{\mathrm{R}}_M \backslash \mathcal{R}_M$ is M^0, which is always conjunct-atomic definable from $\mathbf{M}_1$.

For (ii), we need to check that $\mathbf{M}_1$ has enough nullaries. To see this, we can assume that $\mathbf{M}_0$ has no empty term functions. So, up to isomorphism, there is only one one-element structure in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\mathbf{M}_0)$. Now define

$$K_0 := \{\, a \in M \mid \widehat{a} \in \mathrm{Clo}^{\mathrm{R}}_{\mathrm{p}}(\mathbf{M}_1) \,\} \quad \text{and} \quad K_1 := \{\, a \in M \mid \underline{a} \in \mathrm{Clo}^{\mathrm{R}}_{\mathrm{p}}(\mathbf{M}_1) \,\},$$

where $K_0 \subseteq K_1$. We must have $|\mathrm{E}(\mathbf{K}_0)| = 1 = |\mathrm{E}(\mathbf{K}_1)|$. Therefore $\mathbf{K}_0 \cong \mathbf{K}_1$, and so $K_0 = K_1$. Since $\mathbf{M}_1'$ satisfies condition 4.3(iv)(b), we know that $a \in K_1$ whenever $\underline{a} : M \to M$ is a compatible operation on $\mathbf{M}_0$. It follows that $\mathbf{M}_1$ has enough nullaries, whence (ii) holds.

Next, assume that conditions (i) and (ii) hold. By the Full Duality Lemma 4.3, the pair $(\mathbf{M}_0', \underset{\sim}{\mathbf{M}}_1')$ yields a finite-level full duality in the symmetric setting. As M^0 is conjunct-atomic definable from $\mathbf{M}_1$, condition (ii) ensures that $\mathbf{M}_1$ has enough nullaries. It now follows by Lemma 6.6 that $(\mathbf{M}_0, \underset{\sim}{\mathbf{M}}_1)$ yields a finite-level full duality in the usual setting. □

The asymmetry of the types allowed on the two sides in the usual setting causes annoying, trivial complications when we try to 'flip' a duality. The complete version of the Full Duality Lemma 4.3 (like the Co-duality Lemma 4.2 and the Strong Duality Lemma 4.6) is most naturally stated in the two symmetric settings.

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