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Structural entailment

Brian A. Davey, Miroslav Haviar, and Ross Willard

Abstract. We give a number of characterizations of structural entailment. In particular, we show that an alter ego $\underset{\sim}{\mathbf{M}}$ structurally entails an algebraic relation s on a finite algebra $\underline{\mathbf{M}}$ if and only if s can be obtained via a local construct from $\underset{\sim}{\mathbf{M}}$. We show, via a range of applications, that, whereas entailment is important in the study of dualisability, structural entailment is important in the study of full and strong dualisability. We also give an application to the transfer of strong dualities that connects this paper to our earlier paper [9] on full versus strong duality.

The concept of *structural entailment* has been around since the birth of the theory of natural dualities in 1980 (see Davey and Werner [11]). Nevertheless, until now its weaker cousin, *entailment*, has received all of the attention. It is significant that, until this paper and its companion [9], the concept did not even have a name. In their seminal paper [8] on the syntax and semantics of entailment, Davey, Haviar and Priestley characterised entailment but did not consider structural entailment at all. In this paper we redress the balance. Once we state and prove the Structural Entailment Theorem (in Section 2) and consider its many applications (in Sections 3 and 4) it becomes clear why structural entailment hardly rated a mention in earlier work on entailment. Previously, authors considering entailment have been motivated by questions concerned with *dualisability*. Our results make it clear that, while structural entailment is important in connection with dualisability, its most important applications arise in connection with *strong dualisability* and the more inscrutable *full dualisability*.

In Section 1, we give a brief introduction to natural duality theory tailored to our particular needs. Those familiar with dualisability, full dualisability, strong dualisability and entailment may wish to begin with Section 2, where structural entailment is defined and characterised, referring to Section 1 as needed.

1. Dualisability and all that it entails

Here we shall give a very brief refresher on the basics of natural dualities. The definitions of dualisability, full dualisability and strong dualisability will be recalled,

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but those wanting further details and an account of the theorems related to these concepts are referred to the monograph Clark and Davey [2].

Consider a finite algebra $\underline{\mathbf{M}}$ and let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ be the quasi-variety generated by $\underline{\mathbf{M}}$. A natural duality provides us with a representation of the algebras in $\boldsymbol{\mathcal{A}}$ as algebras of structure-preserving maps. We begin with a topological structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ on the same underlying set as $\underline{\mathbf{M}}$, where

- G is a set of *algebraic operations on* $\underline{\mathbf{M}}$, that is, each $g \in G$ is a homomorphism $g\colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$, for some $n \in \mathbb{N} \cup \{0\}$,
- H is a set of *algebraic partial operations on* $\underline{\mathbf{M}}$, that is, each $h \in H$ is a homomorphism $h\colon \mathbf{D} \to \underline{\mathbf{M}}$, for some $n \in \mathbb{N}$ and some subalgebra $\mathbf{D}$ of $\underline{\mathbf{M}}^n$,
- R is a set of *algebraic relations on* $\underline{\mathbf{M}}$, that is, each $r \in R$ is the underlying set of a subalgebra of $\underline{\mathbf{M}}^n$, for some $n \in \mathbb{N}$, and
- $\mathcal{T}$ is the discrete topology on M.

The structure $\underset{\sim}{\mathbf{M}}$ is called an *alter ego of* $\underline{\mathbf{M}}$. Throughout the remainder of this paper $\underline{\mathbf{M}}$ will be a finite algebra and $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ will be an alter ego of $\underline{\mathbf{M}}$.

Define $\mathcal{X} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$ to be the class of all isomorphic copies of closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$. The morphisms of the category $\mathcal{X}$ are the continuous structure preserving maps. In the presence of partial operations certain well known and loved results from topological algebra are no longer true: for example, the image of a substructure under a morphism need not be a substructure and the topological closure of a substructure need not be a substructure. We begin with some elementary results that we need here and *are* true. The proofs are an easy exercise for the reader.

Lemma 1.1. *Let $\mathbf{X}, \mathbf{Y} \in \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{M}})$, let $\mathbf{Z}$ be a closed substructure of $\mathbf{Y}$ and let $\varphi, \psi\colon \mathbf{X} \to \mathbf{Y}$ be morphisms.*

(i) *$\varphi^{-1}(Z)$ is a closed substructure of $\mathbf{X}$.*
(ii) *$\mathrm{eq}(\varphi, \psi) := \{\, x \in X \mid \varphi(x) = \psi(x) \,\}$ is a closed substructure of $\mathbf{X}$.*
(iii) *If U generates $\mathbf{X}$ (that is, $U \subseteq X$ and the only closed substructure of $\mathbf{X}$ that contains U is $\mathbf{X}$ itself), and $\varphi{\restriction}_U = \psi{\restriction}_U$, then $\varphi = \psi$.*

There is a natural pair of contravariant functors $\mathrm{D}\colon \boldsymbol{\mathcal{A}} \to \mathcal{X}$ and $\mathrm{E}\colon \mathcal{X} \to \boldsymbol{\mathcal{A}}$. For every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, define $\mathrm{D}(\mathbf{A})$ to be the homset $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ regarded as a closed substructure of $\underset{\sim}{\mathbf{M}}^A$. The structure $\mathrm{D}(\mathbf{A})$ is called the *dual of* $\mathbf{A}$, for each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. For every $\mathbf{X} \in \mathcal{X}$, define its *dual*, $\mathrm{E}(\mathbf{X})$, to be the homset $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ regarded as a subalgebra of $\underline{\mathbf{M}}^X$. We define D and E on morphisms in the natural way: for $\varphi\colon \mathbf{A} \to \mathbf{B}$ in $\boldsymbol{\mathcal{A}}$, define $\mathrm{D}(\varphi)\colon \mathrm{D}(\mathbf{B}) \to \mathrm{D}(\mathbf{A})$ by $\mathrm{D}(\varphi)(x) := x \circ \varphi$, and for $\psi\colon \mathbf{X} \to \mathbf{Y}$ in $\mathcal{X}$, define $\mathrm{E}(\psi)\colon \mathrm{E}(\mathbf{Y}) \to \mathrm{E}(\mathbf{X})$ by $\mathrm{E}(\psi)(\alpha) := \alpha \circ \psi$.

For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, there is a embedding $e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$, given by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in A$ and $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Similarly, for each $\mathbf{X} \in \mathcal{X}$, we can define an embedding $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ by $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$, for all $x \in X$ and all $\alpha \in \mathfrak{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$. If $e_{\mathbf{A}}$ is an isomorphism, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathbf{A}$. If $\underset{\sim}{\mathbf{M}}$ yields a duality on every $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\boldsymbol{\mathcal{A}}$ or, more briefly, that $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$. If $\underset{\sim}{\mathbf{M}}$ yields a duality on every finite algebra $\mathbf{A}$ in $\boldsymbol{\mathcal{A}}$, then we say that $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$ *at the finite level*. If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$, then we have a representation for $\boldsymbol{\mathcal{A}}$: each algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ is isomorphic to the algebra $\mathrm{ED}(\mathbf{A})$ of all morphisms from its dual $\mathrm{D}(\mathbf{A})$ into $\underset{\sim}{\mathbf{M}}$. If $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ are isomorphisms, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and $\mathbf{X} \in \mathfrak{X}$, then we say that $\underset{\sim}{\mathbf{M}}$ *yields a full duality on* $\boldsymbol{\mathcal{A}}$ or that $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$. In this case, the categories $\boldsymbol{\mathcal{A}}$ and $\mathfrak{X}$ are dually equivalent. Full duality at the finite level is defined in the obvious way.

As the results of Davey, Haviar and Willard [9] show, full dualities are rather mysterious. There is a stronger notion, simpler to prove, that is often used instead. Given any set I, an I-ary *algebraic partial operation on* $\underline{\mathbf{M}}$ is a homomorphism $h\colon \mathbf{D} \to \underline{\mathbf{M}}$, where $\mathbf{D}$ is a subalgebra of $\underline{\mathbf{M}}^I$. Given a non-empty set T and a subset X of M^T, we say that X is *closed under* h if X is closed under the pointwise extension of h to M^T, that is, if $x_i \in X$, for all $i \in I$, and $\langle x_i(t)\rangle_{i\in I} \in D$, for all $t \in T$, then $h^{M^T}(\langle x_i\rangle_{i\in I}) \in X$, where $h^{M^T}(\langle x_i\rangle_{i\in I})(t) := h(\langle x_i(t)\rangle_{i\in I})$, for all $t \in T$. The subset X of M^T is called *hom-closed* if it is closed under every I-ary algebraic partial operation on $\underline{\mathbf{M}}$, for all sets I.

It is known that $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\boldsymbol{\mathcal{A}}$ if and only if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and every closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$ is isomorphic to a hom-closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ (see [4], [2, 3.1.7]). We say that $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\boldsymbol{\mathcal{A}}$, or that $\underset{\sim}{\mathbf{M}}$ *strongly dualises* $\underline{\mathbf{M}}$, if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ and every closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$ is hom-closed (see Clark and Davey [1], [2]). Thus, every strong duality is also a full duality. At present, it is not known whether every full duality is strong, nor is it known if every finite algebra that is fully dualisable is also strongly dualisable.

It is occasionally convenient to work with an alternative description of hom-closed subsets of powers on M. Let T be a non-empty set. A set $X \subseteq M^T$ is *term-closed* if, for all $y \in M^T\backslash X$, there exist T-ary term functions σ and τ such that $\sigma{\restriction}_X = \tau{\restriction}_X$ and $\sigma(y) \neq \tau(y)$. We shall require the following result just once.

Theorem 1.2. (Closure Theorem [4], [2, 3.1.3]) *If T is non-empty and $X \subseteq M^T$, then X is hom-closed in M^T if and only if X is term closed in M^T.*

It follows easily from this result that if X is a hom-closed subset of M^T, for some non-empty set T, then X is the underlying set of a closed substructure of $\underset{\sim}{\mathbf{M}}^T$, for every choice of the alter ego $\underset{\sim}{\mathbf{M}}$. When working at the finite level, we require the finite-level version of the Closure Theorem. Its proof is a simple modification of the proof of the Closure Theorem [2, 3.1.3]. If $X \subseteq M^T$ is closed under every finitary algebraic partial operation on $\underline{\mathbf{M}}$, then we say that X is *finitely hom-closed*.

Theorem 1.3. (Finite-level Closure Theorem) *Let $\underline{\mathbf{M}}$ be a finite algebra, let $n \in \mathbb{N}$ and let $X \subseteq M^n$. Then the following are equivalent:*

(1) *X is term closed;*
(2) *X is hom-closed;*
(3) *X is finitely hom-closed.*

We say that $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on $\boldsymbol{\mathcal{A}}$ at the finite level* if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ at the finite level and, for all $n \in \mathbb{N}$, the equivalent conditions of the Finite-level Closure Theorem hold for every substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^n$. The First Strong Duality Theorem [2, 3.2.4] tells us that $\underset{\sim}{\mathbf{M}}$ yields a strong duality [at the finite level] if and only if $\underset{\sim}{\mathbf{M}}$ yields a full duality [at the finite level] and $\underset{\sim}{\mathbf{M}}$ is injective in $\boldsymbol{\mathfrak{X}}$ [in $\boldsymbol{\mathfrak{X}}_{\mathrm{fin}}$].

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathfrak{T}\rangle$ be an alter ego of a finite algebra $\underline{\mathbf{M}}$, let I be a set and let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$. Given a closed substructure $\mathbf{X}$ of a non-zero power of $\underset{\sim}{\mathbf{M}}$, we say that $G \cup H \cup R$, or simply $\underset{\sim}{\mathbf{M}}$, *entails s on $\mathbf{X}$* if each morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ preserves s. We say that $\underset{\sim}{\mathbf{M}}$ entails a (partial) operation h on X provided $\underset{\sim}{\mathbf{M}}$ entails graph(h) on X. Note that if $\underset{\sim}{\mathbf{M}}$ entails s on $\mathbf{X}$, then, for all non-empty sets T' and all closed substructures $\mathbf{X}'$ of $\underset{\sim}{\mathbf{M}}^{T'}$, each morphism $\varphi\colon \mathbf{X} \to \mathbf{X}'$ preserves s. As far as duality is concerned we are interested only in entailment on structures $\mathbf{X}$ of the form $\mathrm{D}(\mathbf{A})$, for $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Thus we say that $\underset{\sim}{\mathbf{M}}$ *entails s* if it entails s on every structure of the form $\mathrm{D}(\mathbf{A})$, for $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$.

Recall that $\mathbf{s}$ denotes the subalgebra of $\underline{\mathbf{M}}^I$ corresponding to the I-ary algebraic relation s on $\underline{\mathbf{M}}$, and, for each $i \in I$, we define $\rho_i := \pi_i{\restriction}_s\colon s \to M$, where $\pi_i\colon M^I \to M$ is the natural projection. The following result is fundamental to the study of entailment. It is usually stated and proved for finitary algebraic relations s. For all items but the last this restriction is unnecessary. A formula in the language of $\underset{\sim}{\mathbf{M}}$ is called *primitive positive* if it is an existential conjunct of atomic formulæ.

Theorem 1.4. ([8], [2, 8.1.3, 9.1.2]) *Let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, for some set I. Then the following are equivalent:*

(1) *$\underset{\sim}{\mathbf{M}}$ entails s on every hom-closed subset of each non-zero power of $\underset{\sim}{\mathbf{M}}$;*
(2) *$\underset{\sim}{\mathbf{M}}$ entails s;*
(3) *$\underset{\sim}{\mathbf{M}}$ entails s on $\mathrm{D}(\mathbf{s})$;*
(4) *every morphism $\alpha\colon \mathrm{D}(\mathbf{s}) \to \underset{\sim}{\mathbf{M}}$ satisfies $\langle \alpha(\rho_i)\rangle_{i\in I} \in \mathbf{s}$;*
(5) *$s = \{\, \langle \alpha(\rho_i)\rangle_{i\in I} \mid \alpha \in \mathrm{ED}(\mathbf{s}) \,\}$.*

If s is n-ary, for some $n \in \mathbb{N}$, then we can add

(6) *s may be obtained from $G \cup H \cup R$ via a* primitive positive construct*, that is,*
$$s = \{\, (c_1, \dots, c_n) \in M^n \;\big|\; \underset{\sim}{\mathbf{M}} \models \Phi(c_1, \dots, c_n) \,\}$$
and $\mathrm{D}(\mathbf{s})$ satisfies $\Phi(\rho_1, \dots \rho_n)$, where $\Phi(x_1, \dots, x_n)$ is a primitive positive formula in the language of $\underset{\sim}{\mathbf{M}}$.

Of course, entailment and duality are intimately connected. The following result is a simple consequence of the theorem above.

Lemma 1.5.

(i) *Let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, for some set I. In order to prove that $\underset{\sim}{\mathbf{M}}$ entails s it suffices to prove that $e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ is an isomorphism for some isomorphic copy $\mathbf{A}$ of the algebra $\mathbf{s}$.*

(ii) *If $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, then $\underset{\sim}{\mathbf{M}}$ entails every I-ary algebraic relation on $\underline{\mathbf{M}}$, for all sets I.*

We refer the reader to Davey, Haviar and Priestley [8] and Chapters 8 and 9 of Clark and Davey [2] for further details on entailment.

2. Structural entailment

In practice, rather than prove that $\underset{\sim}{\mathbf{M}}$ entails s, we usually establish a stronger condition (see [2, 2.4.4]). We shall say that $\underset{\sim}{\mathbf{M}}$ *structurally entails* s if $\underset{\sim}{\mathbf{M}}$ entails s on every closed substructure of each non-zero power of $\underset{\sim}{\mathbf{M}}$. Clearly, structural entailment implies entailment. The converse fails. Let $\underline{\mathbf{M}}$ be the three-element chain regarded as a bounded distributive lattice and let $\underset{\sim}{\mathbf{M}} := \langle M; f, g, \mathcal{T}\rangle$, where f and g are the non-identity endomorphisms of $\underline{\mathbf{M}}$. While it is well known (see [7], [5], [2, 9.5.1]) that $\underset{\sim}{\mathbf{M}}$ entails $\leq_{\{0,1\}}$, it is easily seen that it does not structurally entail $\leq_{\{0,1\}}$; indeed, $\{0,1\}$ is a substructure of $\underset{\sim}{\mathbf{M}}$ and the complementation map $'\colon \{0,1\} \to \{0,1\} \subseteq M$ is a morphism which does not preserve $\leq_{\{0,1\}}$.

A subalgebra of $\underline{\mathbf{M}}^I$ corresponding to an algebraic relation or to the domain of an algebraic partial operation will be denoted by $\mathbf{s}$ while a subalgebra of $\underline{\mathbf{M}}^I$ thought of as an algebra in the quasi-variety $\mathbb{ISP}(\underline{\mathbf{M}})$ will be denoted by $\mathbf{A}$.

Let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, for some set I. We shall denote by $\mathbf{Y}_s$ the closed substructure of $\underset{\sim}{\mathbf{M}}^s$ generated by $\{\, \rho_i \mid i \in I \,\}$. Since each ρ_i is a homomorphism, the structure $\mathbf{Y}_s$ is a substructure of $\mathrm{D}(\mathbf{s})$. Note that $\langle \rho_i \rangle_{i\in I} \in s^{Y_s}$. Indeed, the following theorem shows that this occurrence of the relation s is the universal occurrence of s in the class $\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. This theorem is closely related to the corresponding result concerning the occurrence of the relation s in hom-closed substructures of powers of $\underset{\sim}{\mathbf{M}}$: see Davey, Haviar and Priestley [8] and [2, 8.1.1]. The role played there by $\mathrm{D}(\mathbf{s})$ is now played by its substructure $\mathbf{Y}_s$. In the proofs of the next two results we shall use the elementary fact that if $X \subseteq M^S$ and $Y \subseteq M^T$ and both X and Y are closed under a partial operation h, then a map $\varphi\colon X \to Y$ preserves the partial operation h if and only if it preserves the relation $\mathrm{graph}(h)$.

Theorem 2.1. *Let I be a set, let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, let $\mathbf{X}$ be a closed substructure of $\underset{\sim}{\mathrm{M}}^T$, for some non-empty set T, and let $x_i \in X$, for all $i \in I$. Then the following are equivalent:*

(1) $\langle x_i \rangle_{i\in I} \in s^X$;
(2) *there exists a* (*necessarily unique*) *continuous map $\varphi\colon Y_s \to X$ such that φ preserves every algebraic relation on $\underline{\mathbf{M}}$* (*of arbitrary arity*) *and satisfies $\varphi(\rho_i) = x_i$, for all $i \in I$;*
(3) *there exists a* (*necessarily unique*) *morphism $\varphi\colon \mathbf{Y}_s \to \mathbf{X}$ that preserves the relation s and satisfies $\varphi(\rho_i) = x_i$, for all $i \in I$;*
(4) *there exists an s-preserving map $\varphi\colon Y_s \to X$ such that $\varphi(\rho_i) = x_i$, for all $i \in I$.*

Moreover, the maps guaranteed by (2) *and* (3) *are equal.*

Proof. Certainly (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) since $\langle \rho_i \rangle_{i\in I} \in s^{Y_s}$. If $\varphi\colon Y_s \to X$ preserves every algebraic relation on $\underline{\mathbf{M}}$, then φ preserves every operation and partial operation in $G \cup H$ (since it preserves their graphs) and consequently φ is a morphism. Thus the uniqueness claims in (2) and (3) follow from Lemma 1.1(iii). Now let $x_i \in X$, for all $i \in I$, and assume that $\langle x_i \rangle_{i\in I} \in s^X$. Recall that $\mathbf{F}_{\underline{\mathbf{M}}}(T)$ denotes the algebra freely generated by T in the variety generated by $\underline{\mathbf{M}}$. There is a natural isomorphism ρ_T from $\mathrm{D}(\mathbf{F}_{\underline{\mathbf{M}}}(T))$ to $\underset{\sim}{\mathrm{M}}^T$ (see [2, 2.2.1]). Since ρ_T preserves every algebraic relation on $\underline{\mathbf{M}}$, we have

$$\langle \rho_T^{-1}(x_i) \rangle_{i\in I} \in s^{\mathrm{D}(\mathbf{F}_{\underline{\mathbf{M}}}(T))}.$$

Thus, $z\colon \mathbf{F}_{\underline{\mathbf{M}}}(T) \to \mathbf{s}$, given by $z(a) := \langle \rho_T^{-1}(x_i)(a) \rangle_{i\in I}$, is a well-defined homomorphism. The map $\psi := \rho_T \circ \mathrm{D}(z)\colon \mathrm{D}(\mathbf{s}) \to \underset{\sim}{\mathrm{M}}^T$ is continuous and preserves every algebraic relation on $\underline{\mathbf{M}}$ and hence, as above, ψ is a morphism. Thus, by Lemma 1.1(i), $\psi^{-1}(X)$ is a closed substructure of $\mathrm{D}(\mathbf{s})$. Moreover, since ψ satisfies

$$\psi(\rho_i) = \rho_T(\mathrm{D}(z)(\rho_i)) = \rho_T(\rho_i \circ z) = \rho_T(\rho_T^{-1}(x_i)) = x_i,$$

for all $i \in I$, it follows that $\psi^{-1}(X)$ contains $\{\, \rho_i \mid i \in I \,\}$. Thus $Y_s \subseteq \psi^{-1}(X)$ and consequently $\psi(Y_s) \subseteq X$. Thus $\varphi := \psi{\restriction}_{Y_s}\colon Y_s \to X$ is well defined and preserves every algebraic relation on $\underline{\mathbf{M}}$. Hence (1) implies (2). Finally, it is clear that the map given by (2) has the properties required by the map in (3) so the maps are one and the same. □

The map given by (4) of this theorem need not be unique. Amongst all the maps satisfying (4), there is a unique one satisfying both (2) and (3).

The following result is a typical and useful application of the previous theorem. We say that $\underset{\sim}{\mathrm{M}}$ *hom-entails* an I-ary algebraic partial operation h if every closed substructure of a non-zero power of $\underset{\sim}{\mathrm{M}}$ is closed under h (see Theorem 3.5).

Lemma 2.2. *Let I be a set, let h be an I-ary algebraic partial operation on $\underline{\mathbf{M}}$ and assume that $\underset{\sim}{\mathbf{M}}$ hom-entails h. If $\mathbf{X}$ is a closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$ and $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ is a morphism that preserves* $\mathrm{dom}(h)$*, then α preserves h.*

Proof. (Since $\underset{\sim}{\mathbf{M}}$ hom-entails h, the structure $\mathbf{X}$ is closed under h and so it makes sense to ask whether a map $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ preserves h. We shall also use the fact that $\mathbf{Y}_s$ is closed under h.) Let $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ be a morphism that preserves $\mathrm{dom}(h)$ and let $\langle x_i \rangle_{i\in I} \in \mathrm{dom}(h^{\mathbf{X}})$. Since α preserves $\mathrm{dom}(h)$, we have $\langle \alpha(x_i)\rangle_{i\in I} \in \mathrm{dom}(h)$, so it remains to show that $\alpha(h(\langle x_i\rangle_{i\in I})) = h(\langle \alpha(x_i)\rangle_{i\in I})$.

Let $s := \mathrm{dom}(h)$. Since $\langle x_i\rangle_{i\in I} \in s^{\mathbf{X}}$, by the previous theorem there exists a unique continuous map $\varphi\colon Y_s \to X$ that preserves every algebraic relation (of arbitrary arity) on $\underline{\mathbf{M}}$ and satisfies $\varphi(\rho_i) = x_i$, for all $i \in I$. As α preserves s, we have $\langle \alpha(x_i)\rangle_{i\in I} \in s$. Again by the previous theorem there exists a unique continuous map $\psi\colon Y_s \to M$ that preserves every algebraic relation (of arbitrary arity) on $\underline{\mathbf{M}}$ and satisfies $\psi(\rho_i) = \alpha(x_i)$, for all $i \in I$. Since $\alpha \circ \varphi\colon \mathbf{Y}_s \to \underset{\sim}{\mathbf{M}}$ is a morphism that preserves s and maps ρ_i to $\alpha(x_i)$, for all $i \in I$, the uniqueness claim at the end of the previous lemma gives $\alpha \circ \varphi = \psi$. Since $\mathbf{Y}_s$ is closed under h and since both φ and ψ preserve $\mathrm{graph}(h)$, they both preserve h. Hence,

$$\begin{aligned}
\alpha(h(\langle x_i\rangle_{i\in I})) &= \alpha(h(\langle \varphi(\rho_i)\rangle_{i\in I})) \\
&= (\alpha\circ\varphi)(h(\langle \rho_i\rangle_{i\in I})) && \text{as } \mathbf{Y}_s \text{ is closed under } h \text{ and } \varphi \text{ preserves } h \\
&= \psi(h(\langle \rho_i\rangle_{i\in I})) && \text{as } \alpha\circ\varphi = \psi \\
&= h(\langle \psi(\rho_i)\rangle_{i\in I}) && \text{as } \psi \text{ preserves } h \\
&= h(\langle \alpha(x_i)\rangle_{i\in I}),
\end{aligned}$$

whence α preserves h. □

In the case that s is finitary, we can give an explicit and very useful description of the elements of Y_s. The enriched partial clone generated by $G \cup H$ is denoted by $[G \cup H]$ and is referred to as the *enriched partial clone of* $\underset{\sim}{\mathbf{M}}$. For all $n \geqslant 0$, the set of all n-ary maps in $[G \cup H]$ is denoted by $[G \cup H]_n$. (See pages 21, 64 and 278–283 of Clark and Davey [2] for a discussion of the enriched partial clone of $\underset{\sim}{\mathbf{M}}$. The difference between a clone and an enriched partial clone is that in the latter partial maps are allowed and the set is enriched by allowing nullary operations.)

Lemma 2.3. *Let $n \in \mathbb{N}$ and let s be an n-ary algebraic relation on $\underline{\mathbf{M}}$. The underlying set of the structure $\mathbf{Y}_s$ consists of all homomorphisms from* $\mathbf{s}$ *to $\underline{\mathbf{M}}$ that have an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$. Thus,*

$$Y_s = \{\, t{\restriction}_s\colon s \to M \mid t \in [G\cup H]_n \ \ \&\ \ s \subseteq \mathrm{dom}(t)\,\}.$$

Proof. It is easy to verify that

$$Y_s = \left\{\, t^{\mathrm{D}(\mathbf{s})}(\rho_1,\dots,\rho_n) \mid t \in [G\cup H]_n \ \ \&\ \ (\rho_1,\dots,\rho_n) \in \mathrm{dom}(t^{\mathrm{D}(\mathbf{s})})\,\right\}.$$

Thus, we must check that, for all $t \in [G \cup H]_n$, we have $s \subseteq \operatorname{dom}(t)$ if and only if $(\rho_1, \ldots, \rho_n) \in \operatorname{dom}(t^{\mathrm{D}(\mathbf{s})})$. We must also check that, for all $t \in [G \cup H]_n$, we have $t^{\mathrm{D}(\mathbf{s})}(\rho_1, \ldots, \rho_n) = t{\restriction_s}$. These simple calculations, which are left to the reader, use only the fact that $(\rho_1(c), \ldots, \rho_n(c)) = c$, for all $c \in s$. □

Clark and Davey [2, p. 174] define two alter egos $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ of a finite algebra $\underline{\mathbf{M}}$ to be *structurally equivalent* if $\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$ and $\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2)$ are equal as categories, that is, they have the same objects and morphisms. The concept of structural entailment allows us to give an internal characterization of structural equivalence. In fact, we can do a little more. Let $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ be alter egos of a finite algebra $\underline{\mathbf{M}}$. We say that $\underset{\sim}{\mathbf{M}}_1$ is a *structural reduct* of $\underset{\sim}{\mathbf{M}}_2$ if, whenever T is a non-empty set, $\mathbf{X}$ is a closed substructure of $\underset{\sim}{\mathbf{M}}_2^T$ and $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}_2$ is an $\underset{\sim}{\mathbf{M}}_2$-morphism, it follows that $\mathbf{X}$ is a closed substructure of $\underset{\sim}{\mathbf{M}}_1^T$ and that $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}_1$ is an $\underset{\sim}{\mathbf{M}}_1$-morphism. It is an easy exercise to show that $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ are structurally equivalent if and only if each is a structural reduct of the other.

Lemma 2.4. *Let* $\underset{\sim}{\mathbf{M}}_1 = \langle M; G_1, H_1, R_1, \mathcal{T}\rangle$ *and* $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, H_2, R_2, \mathcal{T}\rangle$ *be alter egos of a finite algebra* $\underline{\mathbf{M}}$*. Then* $\underset{\sim}{\mathbf{M}}_1$ *is a structural reduct of* $\underset{\sim}{\mathbf{M}}_2$ *if and only if*

(a) *each relation* $r \in R_1$ *is structurally entailed by* $\underset{\sim}{\mathbf{M}}_2$*, and for each* $h \in H_1$*, the relation* $\operatorname{dom}(h)$ *is structurally entailed by* $\underset{\sim}{\mathbf{M}}_2$*,*

(b) *each operation* $g \in G_1$ *is in the enriched partial clone of* $\underset{\sim}{\mathbf{M}}_2$*, and*

(c) *each partial operation* $h \in H_1$ *has an extension in the enriched partial clone of* $\underset{\sim}{\mathbf{M}}_2$*.*

Proof. It is easy to see that if (a), (b) and (c) hold, then $\underset{\sim}{\mathbf{M}}_1$ is a structural reduct of $\underset{\sim}{\mathbf{M}}_2$. We shall prove the converse. Assume that $\underset{\sim}{\mathbf{M}}_1$ is a structural reduct of $\underset{\sim}{\mathbf{M}}_2$. Then certainly $\underset{\sim}{\mathbf{M}}_2$ structurally entails every relation $r \in R_1$. Let $h \in H_1$. Since $\underset{\sim}{\mathbf{M}}_1$ structurally entails $\operatorname{dom}(h)$, it follows that $\underset{\sim}{\mathbf{M}}_2$ structurally entails $\operatorname{dom}(h)$. Thus (a) holds. By assumption, for every non-empty set T, each closed substructure of $\underset{\sim}{\mathbf{M}}_2^T$ is a substructure of $\underset{\sim}{\mathbf{M}}_1^T$, whence $\underset{\sim}{\mathbf{M}}_2$ hom-entails each $h \in G_1 \cup H_1$. It follows (see [2, Lemma 9.4.1] and Theorem 3.5 below) that each $h \in G_1 \cup H_1$ has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}_2$, whence (b) and (c) hold. □

We can also give a functorial characterization of when $\underset{\sim}{\mathbf{M}}_1$ is a structural reduct of $\underset{\sim}{\mathbf{M}}_2$. It is straightforward to show that $\underset{\sim}{\mathbf{M}}_1$ is a structural reduct of $\underset{\sim}{\mathbf{M}}_2$ if and only if there is a functor $F\colon \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_2) \to \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_1)$ that preserves arbitrary products and closed substructures, preserves underlying sets and underlying set maps, and satisfies $F(\underset{\sim}{\mathbf{M}}_2) = \underset{\sim}{\mathbf{M}}_1$.

The following characterization of structural equivalence follows immediately from the previous lemma. Define $[G \cup H]_{\max}$ to be all partial maps in the enriched partial clone $[G \cup H]$ that have no proper extension in $[G \cup H]$. In particular, $[G \cup H]_{\max}$ contains all total operations in $[G \cup H]$.

Lemma 2.5. *Let* $\underset{\sim}{\mathbf{M}}_1 = \langle M; G_1, H_1, R_1, \mathcal{T}\rangle$ *and* $\underset{\sim}{\mathbf{M}}_2 = \langle M; G_2, H_2, R_2, \mathcal{T}\rangle$ *be alter egos of a finite algebra* $\underline{\mathbf{M}}$*. Then* $\underset{\sim}{\mathbf{M}}_1$ *is structurally equivalent to* $\underset{\sim}{\mathbf{M}}_2$ *if and only if*

(a) *each relation* $r \in R_1$ *is structurally entailed by* $\underset{\sim}{\mathbf{M}}_2$*, and for each* $h \in H_1$*, the relation* $\mathrm{dom}(h)$ *is structurally entailed by* $\underset{\sim}{\mathbf{M}}_2$*,*
(b) *each relation* $r \in R_2$ *is structurally entailed by* $\underset{\sim}{\mathbf{M}}_1$*, and for each* $h \in H_2$*, the relation* $\mathrm{dom}(h)$ *is structurally entailed by* $\underset{\sim}{\mathbf{M}}_1$*, and*
(c) $[G_1 \cup H_1]_{\max} = [G_2 \cup H_2]_{\max}$.

We now give a number of characterizations of structural entailment. These are closely related to the characterizations of entailment given in Davey, Haviar and Priestley [8]: see Theorem 1.4 above.

We shall say that a subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}^I$ is *hom-minimal* if the only homomorphisms from $\mathbf{A}$ to $\underline{\mathbf{M}}$ are the restrictions of the projections, that is, the mappings ρ_i, for $i \in I$. (If, in addition, $\rho_i \neq \rho_j$ for all $i \neq j$, that is, A has no repetition of coordinates, then $\mathbf{A}$ is called *balanced*.) An n-ary algebraic relation s is called *hom-minimal* or *balanced* if the corresponding subalgebra $\mathbf{s}$ of $\underline{\mathbf{M}}^n$ is. Hom-minimal and balanced closed substructures of powers of $\underset{\sim}{\mathbf{M}}$ are defined in an analogous way. Let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$, let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and let $\mathbf{X}$ be a substructure of $\mathrm{D}(\mathbf{A})$. There is a natural map $e^{\mathbf{X}}_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{E}(\mathbf{X})$ given by $e^{\mathbf{X}}_{\mathbf{A}}(a) := e_{\mathbf{A}}(a){\restriction}_{\mathbf{X}}$, for all $a \in A$. Clearly, $e^{\mathbf{X}}_{\mathbf{A}}$ is a homomorphism which is an embedding if and only if X separates the points of A and is surjective if and only if $\mathbf{X}$ is hom-minimal. In particular, $e^{\mathbf{X}}_{\mathbf{A}}$ is an isomorphism if and only if $\mathbf{X}$ is a balanced. In the important special case in which $\mathbf{X} = \mathbf{Y}_s$, for some I-ary algebraic relation s, we abbreviate the cumbersome notation $e^{\mathbf{Y}_s}_{\mathbf{A}}$ to $e^{\flat}_{\mathbf{s}} \colon \mathbf{s} \to \mathrm{E}(\mathbf{Y}_s)$. Since Y_s contains the projections $\rho_i \colon s \to M$, for all $i \in I$, and since the projections separate the points of s, the map $e^{\flat}_{\mathbf{s}}$ is an embedding.

Structural Entailment Theorem 2.6. *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ *be an alter ego of a finite algebra* $\underline{\mathbf{M}}$*. Let* I *be a set, let* s *be an* I*-ary algebraic relation on* $\underline{\mathbf{M}}$ *and let* $\mathbf{Y}_s$ *be the closed substructure of* $\underset{\sim}{\mathbf{M}}^s$ *generated by the set* $\{\, \rho_i \colon s \to M \mid i \in I \,\}$ *of projections. Then the following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}$ *structurally entails* s*;*
(2) $\underset{\sim}{\mathbf{M}}$ *entails* s *on* $\mathbf{Y}_s$*;*
(3) $\varepsilon_{\mathbf{Y}_s} \colon \mathbf{Y}_s \to \mathrm{DE}(\mathbf{Y}_s)$ *preserves* s*;*
(4) *the embedding* $e^{\flat}_{\mathbf{s}} \colon \mathbf{s} \to \mathrm{E}(\mathbf{Y}_s)$ *is surjective and hence is an isomorphism;*
(5) *every morphism from* $\mathbf{Y}_s$ *to* $\underset{\sim}{\mathbf{M}}$ *is the restriction of a projection, that is, the substructure* $\mathbf{Y}_s$ *of* $\underset{\sim}{\mathbf{M}}^s$ *is hom-minimal* (*and therefore balanced*)*;*
(6) *every morphism* $\alpha \colon \mathbf{Y}_s \to \underset{\sim}{\mathbf{M}}$ *satisfies* $\langle \alpha(\rho_i)\rangle_{i\in I} \in s$*;*
(7) $s = \{\, \langle \alpha(\rho_i)\rangle_{i\in I} \mid \alpha \in \mathrm{E}(\mathbf{Y}_s) \,\}$.

If s *is* n*-ary, for some* $n \in \mathbb{N}$*, then we can add:*

(8) *s may be obtained from $G \cup H \cup R$ via a* local construct*, that is,*

$$s = \{\, (c_1, \dots, c_n) \in M^n \mid \underset{\sim}{\mathbf{M}} \models \Phi(c_1, \dots, c_n) \,\}$$

for some conjunct of atomic formulæ, $\Phi(x_1, \dots, x_n)$, in the language of $\underset{\sim}{\mathbf{M}}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial. Assume (3) and let $\alpha\colon \mathbf{Y}_s \to \underset{\sim}{\mathbf{M}}$. Since we have $\langle \rho_i \rangle_{i \in I} \in s^{\mathbf{Y}_s}$, it follows that $\langle \varepsilon_{\mathbf{Y}_s}(\rho_i) \rangle_{i \in I} \in s^{\mathrm{DE}(\mathbf{Y}_s)}$, by (3). Since s is defined pointwise on $\mathrm{DE}(\mathbf{Y}_s)$, we find

$$c := \langle \alpha(\rho_i) \rangle_{i \in I} = \langle \varepsilon_{\mathbf{Y}_s}(\rho_i)(\alpha) \rangle_{i \in I} \in s.$$

For all $i \in I$, we have $\alpha(\rho_i) = c_i = \rho_i(c) = e^\flat_{\mathbf{s}}(c)(\rho_i)$. Since α and $e^\flat_{\mathbf{s}}(c)$ agree on the generators of $\mathbf{Y}_s$ we conclude that $\alpha = e^\flat_{\mathbf{s}}(c)$. Thus (3) $\Rightarrow$ (4). As $e^\flat_{\mathbf{s}}(c)$ is another name for $\pi_c{\restriction}_{\mathbf{Y}_s}$, the implication (4) $\Rightarrow$ (5) is trivial. Since the projections preserve every relation (as the relations are defined pointwise on the powers of M), the implication (5) $\Rightarrow$ (6) follows from the fact that $\langle \rho_i \rangle_{i \in I} \in s^{\mathbf{Y}_s}$. Define

$$s' := \{\, \langle \alpha(\rho_i) \rangle_{i \in I} \mid \alpha \in \mathrm{E}(\mathbf{Y}_s) \,\}.$$

Then $s \subseteq s'$ by Theorem 2.1 while (6) yields $s' \subseteq s$, whence (6) $\Rightarrow$ (7). Now assume that (7) holds. In order to prove (1), let $\mathbf{X}$ be a closed substructure of $\underset{\sim}{\mathbf{M}}^T$, let $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ be a morphism and let $\langle x_i \rangle_{i \in I} \in s^{\mathbf{X}}$. By Theorem 2.1 there exists a morphism $\varphi\colon \mathbf{Y}_s \to \mathbf{X}$ with $\varphi(\rho_i) = x_i$, for all $i \in I$. Thus, $\alpha \circ \varphi \in \mathrm{E}(\mathbf{Y}_s)$ whence

$$\langle \alpha(x_i) \rangle_{i \in I} = \langle (\alpha \circ \varphi(\rho_i) \rangle_{i \in I} \in s,$$

by (7). Hence α preserves s and consequently $\underset{\sim}{\mathbf{M}}$ structurally entails s, as required.

Finally, assume that s is n-ary for some $n \in \mathbb{N}$. It is a very easy exercise to show that (8) $\Rightarrow$ (1). We now prove (7) $\Rightarrow$ (8). Since $\mathbf{Y}_s$ is generated by $\{\, \rho_i \mid i \in I \,\}$, the set Y_s consists of elements $\rho_1, \dots, \rho_n$, $\tau_1 = h_1(\rho_1, \dots, \rho_n), \dots, \tau_m = h_m(\rho_1, \dots, \rho_n)$, for some $h_1, \dots, h_m \in [G \cup H]$. Since Y_s and M are finite, there are finite sets $G' \subseteq G$, $H' \subseteq H$ and $R' \subseteq R$ such that (i) $h_1, \dots, h_m \in [G' \cup H']$, and (ii) a map $\varphi\colon \mathbf{Y}_s \to M$ is a morphism if and only if φ preserves the operations, partial operations and relations in $G' \cup H' \cup R'$. Let $K := G' \cup H' \cup R' \cup \{=\}$. For each $k \in K$ there is a quantifier-free formula $\Phi_k(x_1, \dots, x_n, y_1, \dots, y_m)$ in the language of $\underset{\sim}{\mathbf{M}}$ which is a conjunct of atomic formulæ and precisely describes k in $\mathbf{Y}_s$ via the correspondence $x_i \leftrightarrow \rho_i$ and $y_j \leftrightarrow \tau_j$. Let

$$\Phi(x_1, \dots, x_n) := \&\{\, \Phi_k(x_1, \dots, x_n, h_1(x_1, \dots, x_n), \dots, h_m(x_1, \dots, x_n)) \mid k \in K \,\}.$$

By construction, $\mathbf{Y}_s \models \Phi(\rho_1, \dots, \rho_n)$. Define

$$s'' := \{\, (c_1, \dots, c_n) \in M^n \mid \underset{\sim}{\mathbf{M}} \models \Phi(c_1, \dots, c_n) \,\}.$$

We shall prove $s = s''$ by showing that $s' = s''$, where s' is as defined above. Let $(\alpha(\rho_1), \dots, \alpha(\rho_n)) \in s'$ for some $\alpha \in \mathrm{E}(\mathbf{Y}_s)$. As $\mathbf{Y}_s \models \Phi(\rho_1, \dots, \rho_n)$ and α

preserves Φ, we obtain $\underset{\sim}{\mathrm{M}} \models \Phi(\alpha(\rho_1), \dots, \alpha(\rho_n))$, whence $(\alpha(\rho_1), \dots, \alpha(\rho_n)) \in s''$. Let now $(c_1, \dots, c_n) \in s''$, thus

$$\underset{\sim}{\mathrm{M}} \models \Phi_k(c_1, \dots, c_n, h_1(c_1, \dots, c_n), \dots, h_m(c_1, \dots, c_n))$$

for all $k \in K$. Consequently, we can define a map $\alpha \in \mathrm{E}(\mathbf{Y}_s)$ by putting $\alpha(\rho_i) = c_i$ and $\alpha(\tau_j) = h_j(c_1, \dots, c_n)$. This gives $(c_1, \dots, c_n) \in s'$, as required. □

We note in passing that local constructs were referred to as *concrete* in [8]. A list of natural constructions, like product and intersection, that together are sufficient to yield any algebraic relation locally constructible from $G \cup H \cup R$ is given in [8] (see Chapter 9 of [2] and Proposition 9.2.3 in particular). Although local constructs were introduced and studied in [8], the connection given here between local constructs and structural entailment is new.

3. Applications of the structural entailment theorem

The characterization of structural entailment, with its emphasis on the role played by the structure $\mathbf{Y}_s$, allows us to see results from some earlier papers in their proper context. In many applications the condition $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$ is fundamental. This condition guarantees that, as far as s is concerned, there is no difference between entailment and structural entailment.

Lemma 3.1. *Let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, for some set I, and assume that $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$. Then $\underset{\sim}{\mathrm{M}}$ structurally entails s if and only if $\underset{\sim}{\mathrm{M}}$ entails s.*

Proof. This follows at once from the equivalence of (1) and (2) in Theorem 2.6 and the equivalence of (2) and (3) in Theorem 1.4. □

Corollary 3.2. *Assume that $\underset{\sim}{\mathrm{M}}$ dualises $\underline{\mathbf{M}}$. Then $\underset{\sim}{\mathrm{M}}$ structurally entails every algebraic relation s on $\underline{\mathbf{M}}$ (of arbitrary arity) that satisfies $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$.*

Proof. If $\underset{\sim}{\mathrm{M}}$ dualises $\underline{\mathbf{M}}$, then, by Lemma 1.5(ii), $\underset{\sim}{\mathrm{M}}$ entails every algebraic relation s on $\underline{\mathbf{M}}$, whence the result follows immediately from the previous lemma. □

The next result was first proved in [8, Proposition 4.1] (see also [2, 9.3.1]).

Lemma 3.3. *Let s be a finitary algebraic relation on $\underline{\mathbf{M}}$. Assume that $\underset{\sim}{\mathrm{M}}$ entails s and that every element of $\mathrm{D}(s)$ has an extension in $[G \cup H]$. Then s is structurally entailed by $\underset{\sim}{\mathrm{M}}$ and hence can be obtained from $G \cup H \cup R$ via a local construct.*

Proof. Assume that $\underset{\sim}{\mathrm{M}}$ entails s and that every element of $\mathrm{D}(\mathbf{s})$ has an extension in $[G \cup H]$. Then, by Lemma 2.3, we have $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$ and hence $\underset{\sim}{\mathrm{M}}$ entails s on $\mathbf{Y}_s$. The result follows by the equivalence of (1), (2) and (8) in Theorem 2.6. □

One of the easiest ways to ensure that $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$ is to insist that $\mathrm{D}(\mathbf{s})$ consists only of the projections, that is, that $\mathbf{s}$ is hom-minimal. Hyndman and Willard [12] gave a direct proof of the following result under the stronger assumption that $\mathbf{s}$ is balanced.

Corollary 3.4. *Let $n \in \mathbb{N}$, let s be an n-ary algebraic relation on $\underline{\mathbf{M}}$ and assume that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathbf{s}$. If s is hom-minimal, then s is structurally entailed by $\underset{\sim}{\mathbf{M}}$ and hence can be obtained from $G \cup H \cup R$ via a local construct.*

Proof. Since $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathbf{s}$, it follows that $\underset{\sim}{\mathbf{M}}$ entails s, by Lemma 1.5. If $\mathbf{s}$ is hom-minimal, then every element of $\mathrm{D}(\mathbf{s})$ is a projection and so has an extension in $[G \cup H]$. Thus the result follows at once from the previous lemma. □

The equivalence of (1) and (4) in the finitary version of the following result is due to Davey, Haviar and Priestley [8], see also [2, Lemma 9.4.1].

Theorem 3.5. *Let $h\colon \mathbf{s} \to \underline{\mathbf{M}}$ be a homomorphism with $\mathbf{s}$ a subalgebra of $\underline{\mathbf{M}}^I$, for some set I. Then the following are equivalent:*

(1) *$\underset{\sim}{\mathbf{M}}$ hom-entails h,*
(2) *$\mathbf{Y}_s$ is closed under h;*
(3) *$h \in Y_s$.*

If s is n-ary, for some $n \in \mathbb{N}$, then we can add:

(4) *h has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$.*

Proof. (1) ⇒ (2) is trivial and (2) ⇒ (3) follows from the fact that $\rho_i \in Y_s$, for all $i \in I$, and $h^{\mathbf{Y}_s}(\langle \rho_i \rangle_{i \in I}) = h$. Assume that $h \in Y_s$ and let $\mathbf{X}$ be a closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$. We shall show that $\mathbf{X}$ is closed under h. Let $x_i \in X$, for all $i \in I$, with $\langle x_i \rangle_{i \in I} \in s^{\mathbf{X}}$. By Theorem 2.1 there exists a continuous map $\varphi\colon \mathbf{Y}_s \to \mathbf{X}$ that preserves every algebraic relation on $\underline{\mathbf{M}}$ (of arbitrary arity) and satisfies $\varphi(\rho_i) = x_i$, for all $i \in I$. Since $\langle \langle \rho_i \rangle_{i \in I}, h \rangle \in \mathrm{graph}(h)^{\mathbf{Y}_s}$, we have

$$\langle \langle x_i \rangle_{i \in I}, \varphi(h) \rangle = \langle \langle \varphi(\rho_i) \rangle_{i \in I}, \varphi(h) \rangle \in \mathrm{graph}(h)^{\mathbf{X}},$$

and hence $h^{\mathbf{X}}(\langle x_i \rangle_{i \in I}) = \varphi(h) \in X$, as required. Hence, (3) ⇒ (1). It is clear that (4) ⇒ (1), and (3) ⇒ (4) follows directly from Lemma 2.3. □

The following result may be viewed as connecting structural entailment and hom-entailment in the presence of a full duality or may be viewed as connecting full duality and strong duality via structural entailment. The proof is of interest because it takes a specific failure of hom-closedness in a structure $\mathbf{X}$ and creates from it a specific instance of the non-surjectivity of the map $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$.

Lemma 3.6. *Let h be an I-ary algebraic partial operation on $\underline{\mathbf{M}}$, for some set I. If $\mathbf{X}$ is a closed substructure of a non-zero power of $\underset{\sim}{\mathbf{M}}$ such that $\underset{\sim}{\mathbf{M}}$ entails $\mathrm{dom}(h)$ on $\mathbf{X}$ and the map $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, then $\mathbf{X}$ is closed under h.*

Proof. Assume that $\underset{\sim}{\mathbf{M}}$ entails $\mathrm{dom}(h)$ on $\mathbf{X}$ and $\mathbf{X}$ is not closed under h. We shall show that $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is not surjective. As $\mathbf{X}$ is not closed under h, there exists $x_i \in X$, for all $i \in I$, such that $\langle x_i \rangle_{i\in I} \in \mathrm{dom}(h)$ and $h(\langle x_i \rangle_{i\in I}) \notin X$. Define a homomorphism $k\colon \mathrm{E}(\mathbf{X}) \to \underline{\mathbf{M}}$ by

$$k(\alpha) = h(\langle \alpha(x_i) \rangle_{i\in I}) \text{ for all } \alpha \in \mathrm{E}(\mathbf{X}).$$

We note that, as $\langle x_i \rangle_{i\in I} \in \mathrm{dom}(h)$ and α preserves $\mathrm{dom}(h)$ (since $\underset{\sim}{\mathbf{M}}$ entails $\mathrm{dom}(h)$ on $\mathbf{X}$), we have $\langle \alpha(x_i) \rangle_{i\in I} \in \mathrm{dom}(h)$. Hence $k\colon \mathrm{E}(\mathbf{X}) \to \underline{\mathbf{M}}$ is a well-defined homomorphism and consequently $k \in \mathrm{DE}(\mathbf{X})$.

Suppose that $k = \varepsilon_{\mathbf{X}}(x)$ for some $x \in X$. Then for every $\alpha \in \mathrm{E}(\mathbf{X})$, we have

$$\alpha(x) = \varepsilon_{\mathbf{X}}(x)(\alpha) = k(\alpha) = h(\langle \alpha(x_i) \rangle_{i\in I}).$$

Since $\mathbf{X} \leqslant \underset{\sim}{\mathbf{M}}^T$, for some non-empty set T, we may apply this equation in the case that α is the projection $\rho_t\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$. Hence, for each $t \in T$, we have

$$x(t) = \rho_t(x) = h(\langle \rho_t(x_i) \rangle_{i\in I}) = h(\langle x_i(t) \rangle_{i\in I}) = h(\langle x_i \rangle_{i\in I})(t),$$

whence $h(\langle x_i \rangle_{i\in I}) = x$. But this yields $h(\langle x_i \rangle_{i\in I}) \in X$, a contradiction. Thus, $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is not surjective. □

The case of the previous lemma in which $\mathbf{X} = \mathbf{Y}_s$, where $s = \mathrm{dom}(h)$, is particularly important. Before we present this result we give a preliminary lemma which is of interest in its own right. An instance of it was used in the proof of [9, Lemma 5.6].

Lemma 3.7. *Let $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$ and let $\mathbf{X}$ be a balanced substructure of $\mathrm{D}(\mathbf{A})$. If $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism, then $\mathbf{X} = \mathrm{D}(\mathbf{A})$.*

Proof. Since $\mathbf{X}$ is balanced, the natural map $e_{\mathbf{A}}^{\mathbf{X}}\colon \mathbf{A} \to \mathrm{E}(\mathbf{X})$ is an isomorphism whose inverse we shall denote by u. Let $y\colon \mathbf{A} \to \underline{\mathbf{M}}$ be a homomorphism. We must show that $y \in X$. Since $y \circ u\colon \mathrm{E}(\mathbf{X}) \to \underset{\sim}{\mathbf{M}}$ is a homomorphism and since $\varepsilon_{\mathbf{X}}$ is an isomorphism, there exists $x \in X$ such that $\varepsilon_{\mathbf{X}}(x) = y \circ u$, whence

$$y = \varepsilon_{\mathbf{X}}(x) \circ u^{-1} = \varepsilon_{\mathbf{X}}(x) \circ e_{\mathbf{A}}^{\mathbf{X}}.$$

For each $a \in A$, we have

$$y(a) = \varepsilon_{\mathbf{X}}(x)(e_{\mathbf{A}}^{\mathbf{X}}(a)) = e_{\mathbf{A}}^{\mathbf{X}}(a)(x) = e_{\mathbf{A}}(a){\restriction}_{\mathbf{X}}(x) = x(a),$$

whence $y = x \in X$. Thus, $\mathbf{X} = \mathrm{D}(\mathbf{A})$. □

Theorem 3.8. *Let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, for some set I. The following conditions are related by* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(1) $\underset{\sim}{\mathbf{M}}$ *structurally entails s and the map $\varepsilon_{\mathbf{Y}_s} \colon \mathbf{Y}_s \to \mathrm{DE}(\mathbf{Y}_s)$ is an isomorphism;*
(2) $\underset{\sim}{\mathbf{M}}$ (*structurally*) *entails s and $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$;*
(3) $\underset{\sim}{\mathbf{M}}$ (*structurally*) *entails s and $\underset{\sim}{\mathbf{M}}$ hom-entails h, for every algebraic partial operation h on $\underline{\mathbf{M}}$ with domain s;*
(4) *for every algebraic partial operation h on $\underline{\mathbf{M}}$ with domain s, the structure $\underset{\sim}{\mathbf{M}}$ hom-entails h and, for each closed substructure $\mathbf{X}$ of a non-zero power of $\underset{\sim}{\mathbf{M}}$, every morphism $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ preserves h.*

Moreover, if $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, then all four conditions are equivalent and are also equivalent to

(5) $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$.

Proof. Assume that $\underset{\sim}{\mathbf{M}}$ structurally entails s and $\varepsilon_{\mathbf{Y}_s} \colon \mathbf{Y}_s \to \mathrm{DE}(\mathbf{Y}_s)$ is an isomorphism. By the equivalence of (1) and (5) in Theorem 2.6, $\mathbf{Y}_s$ is a balanced substructure of $\mathrm{D}(\mathbf{s})$, whence $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$, by the previous lemma. Since structural entailment is stronger than entailment, it follows that (1) implies (2).

The stronger version of (2) is equivalent to the weaker version by Lemma 3.1. Clearly, $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$ if and only if Y_s contains every algebraic partial operation with domain s. Hence (2) and (3) are equivalent by Theorem 3.5. That (4) follows from the weaker version of (3) is an immediate consequence of Lemma 2.2. Assume that (4) holds. Since any map $\alpha \colon X \to M$ that preserves h also preserves $\mathrm{dom}(h)$, it follows that $\underset{\sim}{\mathbf{M}}$ structurally entails $\mathrm{dom}(h)$, whence the stronger version of (3) holds. Finally, if $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, then $\varepsilon_{\mathrm{D}(\mathbf{A})}$ is an isomorphism for all $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$ (see [2, Lemma 3.1.1]) and $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$ implies that $\underset{\sim}{\mathbf{M}}$ structurally entails s by Corollary 3.2. Hence, if $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, then (5) implies (1). $\square$

The following corollary, which gives a partial converse to Lemma 3.3, was proved directly in our companion paper [9].

Corollary 3.9. *Assume that $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ is an isomorphism for all finite structures $\mathbf{X}$ in $\mathfrak{X}$.*

(i) *Let $n \in \mathbb{N}$ and let s be an n-ary algebraic relation on $\underline{\mathbf{M}}$. If $\underset{\sim}{\mathbf{M}}$ structurally entails s, then every n-ary algebraic partial operation on $\underline{\mathbf{M}}$ with domain s has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$.*
(ii) *Every total algebraic operation on $\underline{\mathbf{M}}$ is in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$.*

Proof. The global assumptions along with the assumptions in (i) guarantee, by the finitary version of the previous theorem, that $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$. Hence (i) follows at once from Lemma 2.3. It is trivial that $\underset{\sim}{\mathbf{M}}$ structurally entails M^n, whence (ii) follows immediately from (i). $\square$

We turn now to the consequences of our results for full and strong dualities. The following lemma is important as it says that adding extra algebraic relations to the type of $\utilde{\mathbf{M}}$ cannot destroy a full duality provided that the added relations are structurally entailed by $\utilde{\mathbf{M}}$.

Lemma 3.10. *Let $\utilde{\mathbf{M}}'$ be obtained by adding an extra set R' of finitary algebraic relations to the type of $\utilde{\mathbf{M}}$. If each relation $s \in R'$ is structurally entailed by $\utilde{\mathbf{M}}$, and in particular if each relation $s \in R'$ is hom-minimal, then $\utilde{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ if and only if $\utilde{\mathbf{M}}'$ fully dualises $\underline{\mathbf{M}}$.*

Proof. Assume that $\utilde{\mathbf{M}}'$ is obtained from $\utilde{\mathbf{M}}$ by adding extra algebraic relations that are structurally entailed by $\utilde{\mathbf{M}}$. First, assume that $\utilde{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$. Adding extra algebraic relations can never destroy a duality, so $\utilde{\mathbf{M}}'$ still dualises $\underline{\mathbf{M}}$. Since adding a relation that is structurally entailed by $\utilde{\mathbf{M}}$ changes neither the underlying sets of the objects nor the underlying set-maps of the morphisms in the category $\mathbb{IS}_c\mathbb{P}^+(\utilde{\mathbf{M}})$, the structure $\utilde{\mathbf{M}}'$ fully dualises $\underline{\mathbf{M}}$. The argument that shows that $\utilde{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ provided $\utilde{\mathbf{M}}'$ does is similar. (To show that taking out from $\utilde{\mathbf{M}}'$ relations entailed by $\utilde{\mathbf{M}}$ does not destroy a duality uses [2, 2.4.2].) The particular case of hom-minimal relations follows from Corollary 3.4. $\square$

Lemma 3.11. *Let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{M}}^I$, for some set I, and let $X \subseteq \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Then the following are equivalent:*

(1) *X is hom-closed in M^A and contains the projection ρ_i, for all $i \in I$;*
(2) *X is hom-closed in M^A and separates the points of A;*
(3) *$X = \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$.*

Moreover, if $\mathbf{A}$ is finite, then in both (1) *and* (2)*, 'hom-closed' may be replaced by 'finitely hom-closed'.*

Proof. Since $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ is hom-closed in M^A, we have (3) $\Rightarrow$ (1), while (1) $\Rightarrow$ (2) is trivial. Assume that X is hom-closed in M^A and separates the points of A. Suppose that $z \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})\backslash X$. Since X is hom-closed, it is term-closed, by the Closure Theorem 1.2, and consequently there exist $a_1, \dots, a_n \in A$ and A-ary term functions $t_1(\pi_{a_1}, \dots, \pi_{a_n})$ and $t_2(\pi_{a_1}, \dots, \pi_{a_n})$ that agree on X and differ at z. Define $b := t_1(a_1, \dots, a_n)$ and $c := t_2(a_1, \dots, a_n)$. Then, for all $x \in X$, we have

$$\begin{aligned} x(b) &= x(t_1(a_1, \dots, a_n)) = t_1(x(a_1), \dots, x(a_n)) = t_1(\pi_{a_1}(x), \dots, \pi_{a_n}(x)) \\ &= t_1(\pi_{a_1}, \dots, \pi_{a_n})(x) = t_2(\pi_{a_1}, \dots, \pi_{a_n})(x) = t_2(\pi_{a_1}(x), \dots, \pi_{a_n}(x)) \\ &= t_2(x(a_1), \dots, x(a_n)) = x(t_2(a_1, \dots, a_n)) = x(c), \end{aligned}$$

while a similar argument gives $z(b) \neq z(c)$. It follows that $b \neq c$, but b and c are not separated by the maps in X, a contradiction. Thus, $X = \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ and hence (2) $\Rightarrow$ (3). If $\mathbf{A}$ is finite, then our use of the Closure Theorem 1.2 may be replaced by an application of the Finite-level Closure Theorem 1.3 and hence we need only that X is finitely hom-closed. □

By combining the previous result with Theorem 3.8, we obtain an important result due to Clark, Idziak, Sabourin, Szabó and Willard [3]. Indeed, our proof amounts to a careful dissection of theirs.

Theorem 3.12. *The following are equivalent:*

(1) *every closed substructure of every non-zero power of $\underset{\sim}{\mathbf{M}}$ is hom-closed;*
(2) *for all sets I and all I-ary algebraic relations s on $\underline{\mathbf{M}}$, the substructure $\mathbf{Y}_s$ of $\underset{\sim}{\mathbf{M}}^s$ is hom-closed;*
(3) *for all sets I and all I-ary algebraic relations s on $\underline{\mathbf{M}}$, we have $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$;*
(4) *for all algebras $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$, the only closed substructure of $\mathrm{D}(\mathbf{A})$ that separates the points of A is $\mathrm{D}(\mathbf{A})$ itself.*

Proof. Clearly, (1) ⇒ (2) is trivial, (2) ⇒ (3) follows from (1) ⇒ (3) of the previous lemma, and (3) ⇒ (1) by Theorem 3.5. Finally, (1) ⇒ (4) by (2) ⇒ (3) of the previous lemma, and (4) ⇒ (3) is trivial since $\mathbf{Y}_s$ contains the projections and hence separates the points of s. □

When studying full versus strong dualities at the finite level, we require the finite-level version of this theorem whose proof uses the finite-level part of Lemma 3.11.

Theorem 3.13. *The following are equivalent:*

(1) *every closed substructure of every finite non-zero power of $\underset{\sim}{\mathbf{M}}$ is hom-closed;*
(1)′ *every closed substructure of every finite non-zero power of $\underset{\sim}{\mathbf{M}}$ is finitely hom-closed;*
(2) *for each $n \in \mathbb{N} \cup \{0\}$ and every n-ary algebraic relation s on $\underline{\mathbf{M}}$, the substructure $\mathbf{Y}_s$ of $\underset{\sim}{\mathbf{M}}^s$ is hom-closed;*
(3) *for each $n \in \mathbb{N} \cup \{0\}$ and every n-ary algebraic relation s on $\underline{\mathbf{M}}$, we have $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$;*
(4) *for every finite algebra $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$, the only closed substructure of $\mathrm{D}(\mathbf{A})$ that separates the points of A is $\mathrm{D}(\mathbf{A})$ itself.*

As our final important application of Theorem 2.6 and its consequences, we give another characterization of strong dualisability.

Theorem 3.14. *The following are equivalent:*

(1) *$\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$;*
(2) *$\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$ and $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$, for all sets I and all I-ary algebraic relations s on $\underline{\mathbf{M}}$;*
(3) *$\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ structurally entails every I-ary algebraic relation on $\underline{\mathbf{M}}$, for all sets I.*

Proof. The equivalence of (1) and (2) follows at once from the equivalence of (1) and (3) in Theorem 3.12. Assume that $\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$. Then certainly $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$. By (1) $\Rightarrow$ (3) of Theorem 3.12 we have $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$, for all sets I and all I-ary algebraic relations s on $\underline{\mathbf{M}}$, and hence $\underset{\sim}{\mathbf{M}}$ structurally entails s by Theorem 3.8. Conversely, assume that $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ and structurally entails every algebraic relation on $\underline{\mathbf{M}}$ of arbitrary arity. Let s be an I-ary algebraic relation on $\underline{\mathbf{M}}$, for some set I. By Theorem 3.8 we have $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$. Hence $\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$ by Theorem 3.12. □

Once again, there is a finite-level version of the theorem.

Theorem 3.15. *The following are equivalent:*

(1) $\underset{\sim}{\mathbf{M}}$ *strongly dualises* $\underline{\mathbf{M}}$ *at the finite level;*
(2) $\underset{\sim}{\mathbf{M}}$ *dualises* $\underline{\mathbf{M}}$ *at the finite level and, for each* $n \in \mathbb{N} \cup \{0\}$ *and every* n*-ary algebraic relation* s *on* $\underline{\mathbf{M}}$*, we have* $\mathbf{Y}_s = \mathrm{D}(\mathbf{s})$*;*
(3) $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$ *at the finite level and* $\underset{\sim}{\mathbf{M}}$ *structurally entails every* n*-ary algebraic relation on* $\underline{\mathbf{M}}$*, for all* $n \in \mathbb{N} \cup \{0\}$*.*

4. An application to the transfer of strong dualities

In our considerations below we assume that $\underline{\mathbf{D}}$ is a finite algebra and $\underline{\mathbf{M}}$ is a finite algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{D}})$ which has $\underline{\mathbf{D}}$ as a subalgebra. It follows that $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\underline{\mathbf{M}})$. Our assumptions guarantee the existence of a set $\{\omega_1, \dots, \omega_n\}$ of endomorphisms of $\underline{\mathbf{M}}$ which satisfy:

(E1) $\omega_i(M) \subseteq D$ for all i, and
(E2) $\omega_1, \dots, \omega_n$ separate the points of M.

Indeed, if $\nu\colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n$ is an embedding and $\pi_i\colon \underline{\mathbf{D}}^n \to \underline{\mathbf{D}}$ is the ith projection, then $\{\pi_1 \circ \nu, \dots, \pi_n \circ \nu\}$ is such a set.

Now let

$$\omega := \omega_1 \sqcap \cdots \sqcap \omega_n \colon \underline{\mathbf{M}} \to \underline{\mathbf{D}}^n \leqslant \underline{\mathbf{M}}^n,$$

so that $\omega(a) = (\omega_1(a), \dots, \omega_n(a))$. Since the maps $\omega_1, \dots, \omega_n$ separate the points of M, the homomorphism ω is an embedding. Let $M_r := \omega(M) \subseteq D^n \subseteq M^n$ and let $\sigma\colon \mathbf{M}_r \to \underline{\mathbf{M}}$ be the inverse of ω regarded as an n-ary algebraic partial operation on $\underline{\mathbf{M}}$. Hence, $\sigma(\omega_1(a), \dots, \omega_n(a)) = a$, for all $a \in M$.

Remark 4.1. The example considered in our earlier paper [9] fits this scenario: the algebra $\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ was the two-element bounded distributive lattice while the algebra $\underline{\mathbf{M}} = \langle \{0,a,1\}; \vee, \wedge, 0, 1\rangle$ was a three-element bounded distributive lattice (with $0 < a < 1$). In this case, we have $n = 2$. In [9] the endomorphisms ω_1 and ω_2 were denoted by f and g and the partial operation σ was denoted by m. The domain, M_r, of the partial operation m is the order relation $\leq$ on $\underline{\mathbf{D}}$. When proving that every full duality based on the three-element chain $\underline{\mathbf{M}}$ must be strong we proved that if $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$, then the partial operation m has an extension in the enriched partial clone on $\underset{\sim}{\mathbf{M}}$. The results below (particularly Corollary 4.4) show that we really had to prove this or something equivalent to it.

The partial operation σ was introduced under the name 'schizophrenic operation' in Davey and Haviar [6] where it was shown to play a vital role in transferring a strong duality from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$. Assume that $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ is an alter ego of $\underline{\mathbf{D}}$. Let G_D and H_D denote the sets of partial operations on M that arise when we regard the sets G and H of total operations and partial operations on D as partial operations on M. Similarly, let R_D denote the set R of relations on D regarded as relations on M. The main result of [6] states that if $\underset{\sim}{\mathbf{D}}$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ based on $\underline{\mathbf{D}}$, then

$$\underset{\sim}{\mathbf{M}} := \langle M; \omega_1, \dots, \omega_n, G_D \cup H_D \cup \{\sigma\}, R_D \cup \{D\}, \mathcal{T}\rangle$$

yields a strong duality on $\boldsymbol{\mathcal{A}}$ based on $\underline{\mathbf{M}}$. The essence of the proof of this result is part (i) of the following lemma which is proved though not stated on page 219 of [6]. Note that (ii) and (iii) are easy consequences of (i). In the following three results, the unrestricted version is obtained by ignoring the words in square brackets, while the finite-level version is obtained by adding the words in square brackets.

Lemma 4.2. *Let* $\underline{\mathbf{D}}$ *be a finite algebra, let* $\underline{\mathbf{M}}$ *be a finite algebra in* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{D}})$ *which has* $\underline{\mathbf{D}}$ *as a subalgebra and let* $\omega_1, \dots, \omega_n$ *be endomorphisms of* $\underline{\mathbf{M}}$ *which satisfy* (E1) *and* (E2) *and let* $\sigma\colon M_r \to M$ *be the corresponding schizophrenic operation. Let* $\underset{\sim}{\mathbf{D}} = \langle D; G, H, \mathcal{T}\rangle$ *be an alter ego of* $\underline{\mathbf{D}}$ *and let* $\underset{\sim}{\mathbf{M}} = \langle M; G', H', \mathcal{T}\rangle$ *be an alter ego of* $\underline{\mathbf{M}}$ *with* $\{\omega_1, \dots, \omega_n\} \subseteq G'$ *and* $\sigma \in H'$.

(i) *If* $\mathbf{X}$ *is a closed substructure of* $\underset{\sim}{\mathbf{M}}^T$*, for some non-empty set* T*, and* $X \cap D^T$ *is hom-closed* (*with respect to partial operations on* $\underline{\mathbf{D}}$)*, then* X *is hom-closed* (*with respect to partial operations on* $\underline{\mathbf{M}}$)*.*

(ii) *Assume that* $X \cap D^T$ *is a substructure of* $\underset{\sim}{\mathbf{D}}^T$*, for every* [*finite*] *non-empty set* T *and every closed substructure* $\mathbf{X}$ *of* $\underset{\sim}{\mathbf{M}}^T$*. If every closed substructure of a* [*finite*] *non-zero power of* $\underset{\sim}{\mathbf{D}}$ *is hom-closed, then every closed substructure of a* [*finite*] *non-zero power of* $\underset{\sim}{\mathbf{M}}$ *is hom-closed.*

(iii) *If* $\underset{\sim}{\mathbf{M}} = \langle M; \omega_1, \dots, \omega_n, G_D \cup H_D \cup \{\sigma\}, \mathcal{T}\rangle$ *and every closed substructure of a* [*finite*] *non-zero power of* $\underset{\sim}{\mathbf{D}}$ *is hom-closed, then every closed substructure of a* [*finite*] *non-zero power of* $\underset{\sim}{\mathbf{M}}$ *is hom-closed.*

By combining this lemma with results from the previous section we obtain a necessary and sufficient condition for certain full dualities based upon $\underline{\mathbf{M}}$ to be strong.

Theorem 4.3. *Let $\underline{\mathbf{D}}$ be a finite algebra, let $\underline{\mathbf{M}}$ be a finite algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{D}})$ that has $\underline{\mathbf{D}}$ as a subalgebra, let $\omega_1, \dots, \omega_n$ be endomorphisms of $\underline{\mathbf{M}}$ that satisfy* (E1) *and* (E2) *and let $\sigma\colon M_r \to M$ be the corresponding schizophrenic operation. Assume that $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ is an alter ego of $\underline{\mathbf{D}}$ that yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level] based on $\underline{\mathbf{D}}$ and let $\underset{\sim}{\mathbf{M}} := \langle M; G', H', R', \mathcal{T}\rangle$ be an alter ego of $\underline{\mathbf{M}}$ that yields a full duality on $\boldsymbol{\mathcal{A}}$ [at the finite level] based on $\underline{\mathbf{M}}$. Assume that, for every non-empty [finite] set T and every closed substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^T$, the set $X \cap D^T$ is a substructure of $\underset{\sim}{\mathbf{D}}^T$. Then the following are equivalent:*

(1) *$\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level];*
(2) *$\underset{\sim}{\mathbf{M}}$ structurally entails M_r;*
(3) *σ has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$.*

Proof. Assume that $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\boldsymbol{\mathcal{A}}$ [at the finite level] based on $\underline{\mathbf{M}}$. By Corollary 3.9(ii), we may assume that $\{\omega_1, \dots, \omega_n\} \subseteq G'$. If $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level], then $\underset{\sim}{\mathbf{M}}$ structurally entails M_r by Theorem 3.14. Assume that $\underset{\sim}{\mathbf{M}}$ structurally entails M_r. By Corollary 3.9(i) the schizophrenic operation σ has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$. Finally, if σ has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$, then we may assume without loss of generality that σ belongs to H'. Since $\underset{\sim}{\mathbf{D}}$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level], every closed substructure of a [finite] non-zero power of $\underset{\sim}{\mathbf{D}}$ is hom-closed and so, by (ii) of the previous lemma, every closed substructure of a [finite] non-zero power of $\underset{\sim}{\mathbf{M}}$ is hom-closed. Consequently, $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level]. □

Corollary 4.4. *Let $\underline{\mathbf{D}}$ be a finite algebra, let $\underline{\mathbf{M}}$ be a finite algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{D}})$ that has $\underline{\mathbf{D}}$ as a subalgebra, let $\omega_1, \dots, \omega_n$ be endomorphisms of $\underline{\mathbf{M}}$ that satisfy* (E1) *and* (E2) *and let $\sigma\colon M_r \to M$ be the corresponding schizophrenic operation. Assume that $\underset{\sim}{\mathbf{D}} = \langle D; G, H, R, \mathcal{T}\rangle$ is an alter ego of $\underline{\mathbf{D}}$ that yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level] based on $\underline{\mathbf{D}}$ and let $\underset{\sim}{\mathbf{M}} := \langle M; G', H', R', \mathcal{T}\rangle$ be an alter ego of $\underline{\mathbf{M}}$, with $G_D \cup H_D \subseteq H'$, that yields a full duality on $\boldsymbol{\mathcal{A}}$ [at the finite level] based on $\underline{\mathbf{M}}$. Then the following are equivalent:*

(1) *$\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\boldsymbol{\mathcal{A}}$ [at the finite level];*
(2) *$\underset{\sim}{\mathbf{M}}$ structurally entails M_r;*
(3) *σ has an extension in the enriched partial clone of $\underset{\sim}{\mathbf{M}}$.*

Example 4.5. Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\underline{\mathbf{2}})$ be the variety of bounded distributive lattices and let $\underline{\mathbf{M}} = \underline{\mathbf{2}}^n$ be a finite Boolean lattice. Let σ be the schizophrenic operation corresponding to $\underline{\mathbf{M}}$ with domain M_r. It was shown in [7] that $\underline{\mathbf{2}}$ is dualised by $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}, M_r, \mathcal{T}\rangle$ and that $(M_r)_D$ cannot be omitted from the dualising structure $\underset{\sim}{\mathbf{M}} = \langle M; \omega_1, \dots, \omega_n, (M_r)_D, \mathcal{T}\rangle$. By the Theorem above, every full duality on $\mathbb{ISP}(\underline{\mathbf{M}}) = \boldsymbol{\mathcal{A}}$ given by any extension of $\underset{\sim}{\mathbf{M}}$ must be strong. We note that this is in accordance with the fact that $\underline{\mathbf{M}}$ is an injective algebra and any $\underset{\sim}{\mathbf{M}}$ yielding a full duality is therefore injective in $\boldsymbol{\mathfrak{X}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ (see [2, 3.2.4 and 3.2.10]).

References

[1] D. M. Clark and B. A. Davey, *The quest for strong dualities*, J. Austral. Math. Soc. (Series A) **58** (1995), 248–280.

[2] D. M. Clark and B. A. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, 1998.

[3] D. M. Clark, P. M. Idziak, L. R. Sabourin, Cs. Szabó and R. Willard, *Natural dualities for quasi-varieties generated by a finite commutative ring*, Algebra Universalis **46** (2001), 285–320.

[4] D. M. Clark and P. H. Krauss, *Topological quasivarieties*, Acta Sci. Math. (Szeged) **47** (1984), 3–39.

[5] B. A. Davey, *Dualisability in general and endodualisability in particular*, in: Logic and algebra (Pontignano, 1994), (A. Ursini and P. Aglianò, eds), Lecture Notes in Pure and Appl. Math. **180**, Dekker, New York, 1996, pp. 437–455.

[6] B. A. Davey and M. Haviar, *A schizophrenic operation which aids the efficient transfer of strong dualitites*, Houston Math. J. **26** (2000), 215–222.

[7] B. A. Davey, M. Haviar and H. A. Priestley, *Endoprimal distributive lattices are endodualisable*, Algebra Universalis **34** (1995), 444–453.

[8] B. A. Davey, M. Haviar and H. A. Priestley, *The syntax and semantics of entailment in duality theory*, J. Symbolic Logic **60** (1995), 1087–1114.

[9] B. A. Davey, M. Haviar and R. Willard, *Full does not imply strong, does it?*, Algebra Universalis **54** (2005), 1–22.

[10] B. A. Davey and J. G. Pitkethly, *Endoprimal algebras*, Algebra Universalis **38** (1997), 266–288.

[11] B. A. Davey and H. Werner, *Dualities and equivalences for varieties of algebras*, in: Contributions to Lattice Theory (Szeged, 1980), (A.P. Huhn and E.T. Schmidt, eds) Coll. Math. Soc. János Bolyai **33**, North-Holland, Amsterdam, 1983, pp. 101–275.

[12] J. Hyndman and R. Willard, *An algebra that is dualizable but not fully dualizable*, J. Pure Appl. Alg. **151** (2000), 31–42.

Brian A. Davey

Department of Mathematics, La Trobe University, Victoria 3086, Australia
e-mail: B.Davey@latrobe.edu.au

Miroslav Haviar

Department of Mathematics, Matej Bel University, PdF, Ružová 13, 974 11 Banská Bystrica, Slovak Republic
e-mail: mhaviar@pdf.umb.sk

Ross Willard

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
e-mail: rdwillar@gillian.math.uwaterloo.ca