29 / The uniform space and the uniform topology

Abstract. *This section serves as a brief introduction to uniform spaces. Given a non-empty set S we define a family of subsets, $\mathscr{U}$, of $\mathscr{P}(S \times S)$ which satisfies a set of axioms. The family, $\mathscr{U}$, is called a uniformity on S and the pair $(S, \mathscr{U})$, is called a uniform space. We then show that any metric space, (S, ρ), generates a uniform space, $(S, \mathscr{U}_\rho)$, which inherits some of its metric space properties. Even though $(S, \mathscr{U})$ is not a topological space, the uniformity $\mathscr{U}$, generates a topology $\tau_{\mathscr{U}}$, on S. A topological space, (S, τ), which can be obtained from a uniformity, $\mathscr{U}$, is called a uniformizable topological space.*

29.1 Introduction.

A topological space generalizes many concepts and properties associated to metric spaces. But, some properties closely associated to the structure of a metric space do not transfer easily to a less structured topological space. For example, notions associated to "Cauchy sequences" or "uniform continuity" are not meaningful in the context of a topological space even though such tools can be useful in describing some topological properties. To see this, witness how the function $g(x) = 1/x$ on $\mathbb{R}\backslash\{0\}$ maps $\mathbb{R}\backslash\{0\}$ homeomorphically onto itself. The sequence $T = \{1, 1/2, 1/3, \ldots,\}$ is a Cauchy sequence in the domain of g. But its image, $g[T] = \{1, 2, 3, \ldots,\}$, is not Cauchy in $\mathbb{R}\backslash\{0\}$. So being "Cauchy" is not a topological property and so cannot be adequately expressed in this mathematical context.

In this section, we discuss a space which, like topological spaces, generalizes metric spaces, but in a slightly different way.

We begin by reviewing a few basic notions associated with relations.

Basic definitions and notation. Recall that for a given non-empty set S, a subset U of $S \times S$ is referred to as a (binary) *relation* on S. If U is a relation on S,

$$\begin{aligned} \text{dom } U &= \{x \in S : \text{ there is a } y \in S \text{ such that } (x, y) \in U\} \\ \text{im } U &= \{y \in S : (x, y) \in U \text{ for some } x \in S\} \end{aligned}$$

If U is a relation on S, then its inverse, U^{-1}, is defined as,

$$U^{-1} = \{(x, y) : (y, x) \in U\}$$

Composition of relations. If U and V are relations on S

$$U \circ V = \{(a, b) : (a, c) \in V \text{ and } (c, b) \in U \text{ for some } c \in \text{im } V\}$$

The relation

$$\Delta(S) = \{(x, x) : x \in S\}$$

is referred to as the *diagonal* of $S \times S$, or the *identity relation.*

Symmetric relations. The relation, U, is said to be *symmetric* if and only if $U = U^{-1}$. Alternatively, U is symmetric whenever...

"$(a, b) \in U$ if and only if $(b, a) \in U$"

If S is a set and $\mathscr{U}$ is a subfamily of $\mathscr{P}(S \times S)$, then the elements of $\mathscr{U}$ are subsets of $S \times S$ of the form, $U = \{(a, b) : a, b \in S\}$.

We now introduce the formal definition of a *uniform space.*

Definition 29.1 *Uniform space.* Let S be a non-empty set (not topologized). A *uniform space* is a pair $(S, \mathscr{U})$ where $\mathscr{U}$ is a non-empty subfamily of $\mathscr{P}(S \times S)$ which satisfies the following five axioms:

- U1. $\Delta(S) \subseteq U$ for all $U \in \mathscr{U}$
- U2. If $U, V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$. That is, $\mathscr{U}$ is closed under finite intersections.
- U3. If $U \in \mathscr{U}$, then $V \circ V \subseteq U$ for some $V \in \mathscr{U}$.
- U4. If $U \in \mathscr{U}$, then $E^{-1} \subseteq U$ for some $E \in \mathscr{U}$.
- U5. If $U \in \mathscr{U}$ and $U \subseteq V \in \mathscr{P}(S \times S)$, then $V \in \mathscr{U}$. That is, $\mathscr{U}$ is closed under supersets.

A subset, $\mathscr{U}$, of $\mathscr{P}(S \times S)$ which satisfies these five axioms is called a *uniformity* for S.

Given a set S, the uniformity $\mathscr{U}$ describes a particular family of subsets of $S \times S$ which satisfies the five listed properties, one of which dictates that each of its elements must contain the diagonal, ΔS.

In spite of its nomenclature, a uniform space, $(S, \mathscr{U})$, is *not* a topological space. We will however soon define a topology, $\tau_{\mathscr{U}}$, on S which is derived from the structure, $\mathscr{U}$. Remember that a topology τ on a set S is characterized by specific properties of a subset of $\mathscr{P}(S)$ while a uniformity $\mathscr{U}$ on S is characterized by properties of a subset of $\mathscr{P}(S \times S)$.

Like a topology on S, a uniformity, $\mathscr{U}$, for S is not unique. For a given set S, there can be many different subfamilies of $\mathscr{P}(S \times S)$ which satisfy the five described properties.

Also, note that, since $\mathscr{U}$ is non-empty and is closed both under finite intersections and supersets (U2 and U5), it is a filter of sets on $S \times S$.

Example 1. The largest uniformity on S, is the family

$$\mathscr{U}_d = \{U \in \mathscr{P}(S \times S) : \Delta(S) \subseteq U\}$$

It is called the *discrete uniformity.*

Example 2. The smallest uniformity on S is the singleton set,

$$\mathscr{U}_i = \{S \times S\}$$

It is called the *trivial uniformity* or the *indiscrete uniformity.*

If U is member of $\mathscr{U}$, then $\mathscr{U}$ must also contain its inverse, U^{-1}, as we shall see in the proposition below. That is, if $U = \{(a, b) : a \in A, b \in B\}$ belongs to $\mathscr{U}$, then $U^{-1} = \{(b, a) : b \in B, a \in A\}$ is also a member of $\mathscr{U}$. So,

> *. . . the uniformity, $\mathscr{U}$, is "closed" under finite intersections, supersets and inverses.*

Proposition 29.2 Let S be a non-empty set and let $\mathscr{U} \subseteq \mathscr{P}(S \times S)$ be a uniformity on S. If $U \in \mathscr{U}$, then U^{-1} also belongs to $\mathscr{U}$ on S.

Proof: We are given a set S and an element, U, of a uniformity, $\mathscr{U}$.

By property U4, $\mathscr{U}$ contains an element V such that $V^{-1} \subseteq U$. Then $V = (V^{-1})^{-1} \subseteq U^{-1}$. Since $V \in \mathscr{U}$ and $V \subseteq U^{-1}$, then, by U5, $U^{-1} \in \mathscr{U}$, as required.

It follows from the above proposition that, to each element U in $\mathscr{U}$, we can associate another element $V \in \mathscr{U}$ which is symmetric. To see this, suppose $U \in \mathscr{U}$ and $V = U \cap U^{-1}$. If $(a, b) \in U$, then $(b, a) \in U^{-1}$. Then, if $(a, b) \in V$, $(a, b) \in U$ so $(b, a) \in V$. Then $V = U \cap U^{-1}$ is a symmetric element of $\mathscr{U}$ which is contained in U. So,

$$\text{for all } U \in \mathscr{U}, U^{-1} \in \mathscr{U} \text{ hence } U \cap U^{-1} \in \mathscr{U}$$

The uniformity axiom U3 (which one must admit is a bit more difficult to visualize or to predict its purpose) states that a uniformity, $\mathscr{U}$, contains a family of elements which behaves a bit like a base.

To "crudely" visualize this, suppose, for example, that $S = \mathbb{R}$, and $\mathscr{U}$ is the set bounded by $f(x) = e^x$ and $g(x) = \ln x$. Let V be the set bounded by $x+1/2$ and $x-1/2$. Then $i(x)$ is a subset of $V \circ V \subseteq U$.

With this example in mind, we now discuss the notion of a "base" $\mathscr{B}$ for a uniformity $\mathscr{U}$ on a set S. A base $\mathscr{B}$ for $\mathscr{U}$ will be seen to be a particular subset of $\mathscr{U}$ which "generates" $\mathscr{U}$.

29.2 A base for a uniformity.

Recall that, given any (untopologized) set S and $\mathscr{B} \subseteq \mathscr{P}(S)$, the property,

> "... if $\mathscr{B}$ covers S and for $A, B \in \mathscr{B}$, if $x \in A \cap B$, there is $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

characterizes a base for a topology on S. By this we mean that, if $\mathscr{B}$ satisfies this property, it generates a topology, τ, on S by collecting the union of all subsets of $\mathscr{B}$. More specifically,

$$\tau = \{U \in \mathscr{P}(S) : U = \cup\{C : C \in \mathscr{C}\} \text{ for some subset } \mathscr{C} \text{ of } \mathscr{B}\}$$

The definition we provide for the base of a uniformity will, in some ways, be similar to the one given for a base for a topology.

Definition 29.3 Let S be a non-empty set and $\mathscr{U} \subseteq \mathscr{P}(S \times S)$ be a uniformity on S.

A subfamily $\mathscr{B}$ of $\mathscr{U}$ is a *base* for $\mathscr{U}$, if,

> ... *for every $U \in \mathscr{U}$, there is a $B \in \mathscr{B}$ such that $B \subseteq U$*

A subfamily, $\mathscr{S}$, of $\mathscr{U}$ is a *subbase* for $\mathscr{U}$, if the family of finite intersections of the members of $\mathscr{S}$ forms a base for $\mathscr{U}$.

The characterization we will obtain for the uniformity base, $\mathscr{B}$, is a slightly weakened version of axioms U1 to U4. The uniformity, $\mathscr{U}$, is then generated from $\mathscr{B}$ by applying the Axiom U5 to $\mathscr{B}$.

Theorem 29.4 Let S be a non-empty set and $\mathscr{B} \subseteq \mathscr{P}(S \times S)$.
The family, $\mathscr{B}$, is a base for some uniformity on S if and only if, $\mathscr{B}$ satisfies U1, U2*, U3 and U4 where U2* is a slightly weaker version of U2:

U2* : "If $U, V \in \mathscr{B}$, there is some $W \in \mathscr{B}$ such that $W \subseteq U \cap V \in \mathscr{U}$"

Proof: We are given a set S.

($\Leftarrow$) Suppose $\mathscr{B}$ is a subset of $\mathscr{P}(S \times S)$ which satisfies properties, U1, U2*, U3 and U4. We are required to show that $\mathscr{B}$ is a base for some uniformity on S.

Let

$$\mathscr{U}_{\mathscr{B}} = \{V \in \mathscr{P}(S \times S) : V \text{ contains some } B \in \mathscr{B}\}$$

- By definition of $\mathscr{U}_{\mathscr{B}}$ every V in $\mathscr{U}_{\mathscr{B}}$ contains an element of $\mathscr{B}$.

- Furthermore, if $B \in \mathscr{B}$, $B \subseteq B$ so $\mathscr{B}$ is a subfamily of $\mathscr{U}_{\mathscr{B}}$.

It now suffices to show that the family, $\mathscr{U}_{\mathscr{B}}$ generated by $\mathscr{B}$, is a uniformity on S. We do this by verifying that it satisfies the properties U1 to U5.

U1: Since $\Delta(S) \subseteq B$ for all $B \in \mathscr{B}$ and $B \subseteq V$ for some V in $\mathscr{U}_{\mathscr{B}}$, then every element of $\mathscr{U}_{\mathscr{B}}$ contains $\Delta(S)$. So $\mathscr{U}_{\mathscr{B}}$ satisfies U1.

U2: Let $U, V \in \mathscr{U}_{\mathscr{B}}$. Then $B \subseteq U$, $W \subseteq V$ for some $B, W \in \mathscr{B}$. There exists $Z \in \mathscr{B}$ such that $Z \subseteq B \cap W$ (by U2*). Then $Z \subseteq U \cap V$. By definition of $\mathscr{U}_{\mathscr{B}}$, $U \cap V \in \mathscr{U}_{\mathscr{B}}$. So $\mathscr{U}_{\mathscr{B}}$ satisfies U2.

U3: Let $U \in \mathscr{U}_{\mathscr{B}}$. Then $B \subseteq U$ for some $B \in \mathscr{B}$. Since $\mathscr{B}$ satisfies U3, $W \circ W \subseteq B$ for some $W \in \mathscr{U}_{\mathscr{B}}$. So $W \circ W \subseteq U$. We conclude that $\mathscr{U}_{\mathscr{B}}$ satisfies U3.

U4: Let $U \in \mathscr{U}_{\mathscr{B}}$. Then $B \subseteq U$ for some $B \in \mathscr{B}$. Since $\mathscr{B}$ satisfies U4, $W^{-1} \subseteq B$ for some $W \in \mathscr{B}$ where $W \in \mathscr{U}_{\mathscr{B}}$. Then $W^{-1} \subseteq U$. So $\mathscr{U}_{\mathscr{B}}$ satisfies U4.

U5: Let $U \in \mathscr{U}_{\mathscr{B}}$ and $V \in \mathscr{P}(S \times S)$ such that $U \subseteq V$. Then $B \subseteq U \subseteq V$ for some $B \in \mathscr{B}$. Then $B \subseteq V$. By definition of $\mathscr{U}_{\mathscr{B}}$, $V \in \mathscr{U}_{\mathscr{B}}$. So $\mathscr{U}_{\mathscr{B}}$ satisfies U5.

We conclude that, if $\mathscr{B}$ is a subset of $\mathscr{P}(S \times S)$ which satisfies properties, U1, U2*, U3 and U4, then it is a base for the uniformity, $\mathscr{U}_{\mathscr{B}}$.

$(\Rightarrow)$ Suppose now that $\mathscr{B}$ is a base for the uniformity $\mathscr{U} \subseteq \mathscr{P}(S \times S)$ on S. Then $\mathscr{B}$ is a subfamily of $\mathscr{U}$ where for every $V \in \mathscr{U}$ there is a $B \in \mathscr{B}$ such that $B \subseteq V$. It suffices to show that $\mathscr{B}$ satisfies U1, U2*, U3, U4.

That $\mathscr{B}$ satisfies U1 and U2* is immediate.

Suppose $B \in \mathscr{B}$. Since $B \in \mathscr{U}$ there is a $V \in \mathscr{U}$ such that $V \circ V \subseteq B$ for some $V \in \mathscr{U}$. Since $\mathscr{B}$ is a base of $\mathscr{U}$ there is a $W \in \mathscr{B}$ such that $W \subseteq V$. Then $W \circ W \subseteq V \circ V \subseteq B$. So $\mathscr{B}$ satisfies U3.

Suppose $B \in \mathscr{B}$. Since $B \in \mathscr{U}$ there is a $V \in \mathscr{U}$ such that $V^{-1} \subseteq B$ for some $V \in \mathscr{U}$. Since $\mathscr{B}$ is a base of $\mathscr{U}$ there is a $W \in \mathscr{B}$ such that $W \subseteq V$. Then $W^{-1} \subseteq V^{-1} \subseteq B$. So $\mathscr{B}$ satisfies U4.

We are done.

The above result guarantees that ...

> ... *if* $\mathscr{B} \subseteq \mathscr{P}(S \times S)$ *satisfies* U1, U2*, U3 *and* U4 *then the collection,* $\mathscr{U}_{\mathscr{B}}$*, of all supersets of elements of* $\mathscr{B}$ *is a uniformity on* S.

There can be many bases to a uniformity. The following result shows that, for any uniformity $\mathscr{U}$ on S with a base $\mathscr{B}$, the family of all symmetric relations in $\mathscr{U}$ also forms a base for $\mathscr{U}$.

Theorem 29.5 Suppose $\mathscr{B}$ is a base for the uniformity, $\mathscr{U}$, on S. Let

$$\mathscr{B}_{\text{sym}} = \{V \in \mathscr{U} : V \text{ is symmetric}\}$$

Then $\mathscr{B}_{\text{sym}}$ forms a base for the uniformity $\mathscr{U}$.

Proof: Let $\mathscr{U}$ be a uniformity on S with base $\mathscr{B}$. To show that $\mathscr{B}_{\text{sym}}$ forms a base for $\mathscr{U}$, we have to show that for any V in $\mathscr{U}$ there is an element of $\mathscr{B}_{\text{sym}}$ which is contained in V.

Let $V \in \mathscr{U}$. There is $U \in \mathscr{B}$ such that $U \subseteq V$. By proposition 29.2, there exists $U^{-1} \in \mathscr{U}$. Let $B = U \cap U^{-1}$. By U2, $B \in \mathscr{U}$.

We claim that $B \in \mathscr{B}_{\text{sym}}$. If $(a,b) \in U \cap U^{-1}$, then $(a,b) \in U$ and so $(b,a) \in U^{-1}$. Since $(a,b) \in U^{-1}$, then $(b,a) \in U$. Then $(b,a) \in U \cap U^{-1}$. So $B = U \cap U^{-1}$ is symmetric. Since $B \subseteq U \subseteq V$, then $B \subseteq V$. So $\mathscr{B}_{\text{sym}}$ forms a base for the uniformity $\mathscr{U}$, as required.

So when hypothesizing a base for $\mathscr{U}$, we can always (when useful) be more specific by choosing $\mathscr{B}_{\text{sym}}$ as a base for $\mathscr{U}$.

29.3 A uniform space, $(S, \mathscr{U})$, generated by a metric, ρ, on S.

We have seen that, given any metric space (S, ρ), we can define a corresponding topology τ_ρ which allows us to view it as a topological space (S, τ_ρ). We are then able to apply topological tools to solve particular problems related to these. We also have seen that not every topological space is metrizable, so topological spaces are "strict" generalizations of metric spaces.

Example 3. Consider the metric space (S, ρ). For $\kappa > 0$, let

$$\overline{B}_\kappa = \{(a,b) \in S \times S : \rho(a,b) \leq \kappa\} \in \mathscr{P}(S \times S)$$

Verify that the collection,

$$\overline{\mathscr{B}}_\rho = \{\overline{B}_\kappa : \kappa > 0\} \subseteq \mathscr{P}(S \times S)$$

forms a base for some uniformity on S.

Solution: We verify that $\overline{\mathscr{B}}_\rho$ satisfies the four base properties for a uniformity.

U1. Since $(x,x) \in \overline{B}_\kappa$ for all κ, then $\overline{\mathscr{B}}_\rho$ satisfies U1.

U2*. For κ_1 and κ_2 larger than 0, let $\kappa_3 = \min\{\kappa_1, \kappa_2\}$. If $(x,y) \in \overline{B}_{\kappa_3}$, then (x,y) are strictly within a distance of κ_3 from each other. So $(x,y) \in \overline{B}_{\kappa_1} \cap \overline{B}_{\kappa_2}$. Then $\overline{B}_{\kappa_3} \subseteq \overline{B}_{\kappa_1} \cap \overline{B}_{\kappa_2}$. It follows that $\overline{\mathscr{B}}_\rho$ satisfies U2*.

U3. Let $\overline{B}_\kappa \in \overline{\mathscr{B}}_\rho$. We claim there exists λ such that $\overline{B}_\lambda \circ \overline{B}_\lambda \subseteq \overline{B}_\kappa$. Let $\lambda = \kappa/4$. Recall that

$$U \circ V = \{(u,v) : (u,y) \in V \text{ and } (y,v) \in U \quad \text{for some } y \in \text{im } V.\}$$

Let $(u,v) \in \overline{B}_{\kappa/4} \circ \overline{B}_{\kappa/4}$. We claim that $\rho(u,v) < \kappa$ (and so $\overline{B}_{\kappa/4} \circ \overline{B}_{\kappa/4} \subseteq \overline{B}_\kappa$).

See that $(u,v) \in \overline{B}_{\kappa/4} \circ \overline{B}_{\kappa/4}$ implies that there exists $z \in \operatorname{im} \overline{B}_{\kappa/4}$ such that $(u,z) \in \overline{B}_{\kappa/4}$ and $(z,v) \leq \overline{B}_{\kappa/4}$.

Then $\rho(u,z) \leq \kappa/4$ and $\rho(z,v) \in \kappa/4$. We have,

$$\begin{aligned} \rho(u,v) &\leq \rho(u,z) + \rho(z,v) \\ &\leq \kappa/4 + \kappa/4 \\ &= \kappa/2 < \kappa \end{aligned}$$

Then $(u,v) \in \overline{B}_\kappa$. So $\overline{B}_{\kappa/4} \circ \overline{B}_{\kappa/4} \subseteq \overline{B}_\kappa$, as claimed.

We conclude that $\overline{\mathscr{B}}_\rho$ satisfies U3.

U4. Let $\overline{B}_\kappa \in \overline{\mathscr{B}}$. Since $\overline{B}_\kappa = \{(x,y) : \rho(x,y) \leq \kappa\} = \{(x,y) : \rho(y,x) \leq \kappa\} = \overline{B}_\kappa^{-1}$, then $\overline{\mathscr{B}}$ satisfies U4.

Then the subset of $\mathscr{P}(S \times S)$

$$\overline{\mathscr{B}}_\rho = \{\overline{B}_\kappa : \kappa > 0\}$$

is a base which generates a uniformity on S. Then the uniformity on S corresponding to (S,ρ) is

$$\mathscr{U} = \{V \in \mathscr{P}(S \times S) : \overline{B}_\kappa \subseteq V \text{ for some } \kappa > 0\}$$

Remark. Note that we have purposely used the closed subsets of the form $\{(a,b) \in S \times S : \rho(a,b) \leq \kappa\}$ to generate the base for $\mathscr{U}_\rho$. We could have also used "$\rho(a,b) < \kappa$" to obtain a base for $\mathscr{U}$.

Definition 29.6 If we define

$$B_\kappa = \{(a,b) \in \mathbb{R} \times \mathbb{R} : |a-b| < \kappa\}$$

we can easily verify (simply my mimicking the steps in the example above) that the collection

$$\mathscr{B}_\rho = \{B_\kappa : \kappa > 0\} \subseteq \mathscr{P}(\mathbb{R} \times \mathbb{R})$$

also forms a base for some uniformity on $\mathbb{R}$.
The uniformity on $\mathbb{R}$ generated by the base, $\mathscr{B}_\rho = \{B_\kappa : \kappa > 0\}$, is called the *usual* or the *standard* uniformity on $\mathbb{R}$.

Note that the base generated for the uniformities on $\mathbb{R}$ described above with the help of a metric ρ is neither open nor closed since the uniform space that ρ generates is not a topological space. The example illustrates how we can use a metric on a metric space S to construct a base $\mathscr{B}$ for a uniformity $\mathscr{U} \subseteq \mathscr{P}(S \times S)$.

For example, since the norm $\|\ \|_\infty$ on $C^*(\mathbb{R})$ can be viewed as a metric $\rho(f,g) = \|f-g\|_\infty$, we can use it as a mechanism to construct a base for a uniformity on $C^*(\mathbb{R})$ as follows. For each $\kappa > 0$ define

$$B_\kappa = \{(f,g) \in C^*(\mathbb{R}) \times C^*(\mathbb{R}) : \|f-g\|_\infty < \kappa\}$$

to obtain the base

$$\mathscr{B}_{\|\ \|_\infty} = \{B_\kappa : \kappa > 0\} \subseteq \mathscr{P}(C^*(\mathbb{R}) \times C^*(\mathbb{R}))$$

for a uniformity $\mathscr{U}$ on $C^*(\mathbb{R})$.

Definition 29.7 Suppose (S, ρ) is a metric space and

$$B_\kappa = \{(x,y) \in S \times S : \rho(x,y) < \kappa\}$$

Let $\mathscr{B}_\rho = \{B_\kappa : \kappa > 0\}$. Then

$$\mathscr{U}_\rho = \{V \in \mathscr{P}(S \times S) : V \text{ contains some } B_\kappa \in \mathscr{B}_\rho\}$$

is called the *metric uniformity generated by ρ.*

29.4 Using a uniformity, $\mathscr{U}$, on S to topologize S.

It is important to keep in mind that a uniformity, $\mathscr{U}$, on a set is not, by itself, a topology on S. It is not even a subset of $\mathscr{P}(S)$. The examples presented above show how we have used a set equipped with a metric ρ to construct a uniform space $(S, \mathscr{U}_\rho)$. That is,

$$(S, \rho) \longrightarrow (S, \mathscr{U}_\rho)$$

In general, when we speak of a uniform space, $(S, \mathscr{U})$, we do not assume a topology has been defined on S.

We will show, however, that we can use the elements of a uniformity, $\mathscr{U}$, on S to define a corresponding topology, $\tau_{\mathscr{U}}$, on S to obtain the topological space, $(S, \tau_{\mathscr{U}})$. That is,

$$(S, \mathscr{U}) \longrightarrow (S, \tau_{\mathscr{U}})$$

Recall that, if B is an element of a uniformity $\mathscr{U}$ on S, then $B : S \to S$ is a subset of $S \times S$ which contains $\Delta(S)$. So, for each $B \in \mathscr{U}$, the image of $\{x\}$ under the relation B is a subset of S which contains the point x. Let $B(x)$ denote the image of $\{x\}$. The collection

$$\{B(x) : B \in \mathscr{U}\}$$

is a family of subsets of S. Since, for each $B \in \mathscr{U}$, the image of B at x is

$$B(x) = \{y \in S : (x, y) \in B\}$$

The set $B(x)$ can also be viewed as

$$B(x) = \pi_2\,[\,(\,\{x\} \times S\,) \,\cap\, B\,]$$ [1]

Theorem 29.8 Let S be a non-empty set, $\mathscr{U}$ be a uniformity on S and $\mathscr{B}$ be a base for $\mathscr{U}$. For $x \in S$, and B in $\mathscr{B}$, let $B(x)$ denote the image of $\{x\}$ (in S) under the relation, B. Let

$$\tau_{\mathscr{U}} = \{U \in \mathscr{P}(S) : \forall\ x \in U\ \exists\ B \in \mathscr{B} \text{ such that } B(x) \subseteq U\}$$

Then $\tau_{\mathscr{U}}$ is a topology on S.

Proof: We are given a set S and a uniformity $\mathscr{U}$ on S and the family, $\tau_{\mathscr{U}}$, of subsets of S. We are required to show that $\tau_{\mathscr{U}}$ is indeed a topology on S. It suffices to show that $\tau_{\mathscr{U}}$ satisfies the three basic topological axioms.

O1. Clearly $\varnothing \in \tau_{\mathscr{U}}$. Let $x \in S$. Since $S \times S \in \mathscr{U}$, $x \in (S \times S)(x) = S \subseteq S$. So $S \in \tau_{\mathscr{U}}$.

O2. Let $U, V \in \tau_{\mathscr{U}}$. We claim $U \cap V \in \tau_{\mathscr{U}}$. Suppose $x \in U \cap V$. It suffices to show there is a $W \in \mathscr{B}$ such that $W(x) \subseteq U \cap V$.

---

[1] Where $\pi_2(a, b) = b$.

Since $U, V \in \tau_{\mathscr{U}}$, there exist B_1, B_2 in $\mathscr{B}$ such that $x \in B_1(x) \subseteq U$ and $x \in B_2(x) \subseteq V$. Since B_1, $B_2 \in \mathscr{B}$, by U2* there exists $W \in \mathscr{B}$ such that $W \subseteq B_1 \cap B_2$ (see Theorem 29.4).

$$\begin{aligned} z \in W(x) &\Rightarrow z \in (B_1 \cap B_2)(x) \\ &\Rightarrow (x, z) \in B_1 \cap B_2 \\ &\Rightarrow z \in B_1(x) \text{ and } z \in B_2(x) \\ &\Rightarrow z \in B_1(x) \cap B_2(x) \\ &\Rightarrow z \in U \cap V \end{aligned}$$

Then $w(x) \subseteq U \cap V$. Since $W \in \mathscr{B}$, $U \cap V \in \tau_{\mathscr{U}}$. So O2 is satisfied.

O3. Let $\{U_i : i \in A\}$ be a family of elements of $\tau_{\mathscr{U}}$.

If $x \in \cup\{U_i : i \in A\}$, then $x \in U_j$ for some $j \in A$. Since $U_j \in \tau_{\mathscr{U}}$ there exists $B \in \mathscr{B}$ such that $x \in B(x) \subseteq U_j \subseteq \cup\{U_i : i \in A\}$. Then $\cup\{U_i : i \in A\} \in \tau_{\mathscr{U}}$. So $\tau_{\mathscr{U}}$ is closed under arbitrary unions.

So $\tau_{\mathscr{U}}$ is a topology on S, as claimed.

Definition 29.9 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. The topology, $\tau_{\mathscr{U}}$, on S is called the

uniform topology generated by the uniformity, $\mathscr{U}$

Let (S, τ) be a topological space. If

$$\tau = \tau_{\mathscr{U}}$$

for some uniformity, $\mathscr{U}$, on S, then (S, τ) is said to be a *uniformizable* topological space.

We have shown that, given a uniformity $\mathscr{U}$ on S with base $\mathscr{B}$, we can define the uniform topology, $\tau_{\mathscr{U}}$, on S as one whose elements are those subsets, U, of S such that

... for each point $x \in U$ there is a $B \in \mathscr{B}$ such that $B(x) \subseteq U$.

This may lead one to suspect that, the set

$$\{B(x) : x \in S,\ B \in \mathscr{B}\}$$

forms an open base for $\tau_{\mathscr{U}}$. The problem here is that the elements in $\{B(x) : x \in S,\ B \in \mathscr{B}\}$ need not be open in $(S, \tau_{\mathscr{U}})$. To see this, suppose that, in Example 3, we let $S = \mathbb{R}$ and $\rho(a, b) = |a - b|$ and $|a - b| \leq \kappa$. It is obvious in this case that $B(x)$ cannot be an open subset of $\mathbb{R}$.

To cast a bit more light on this question, we prove the following lemma in which we describe the interior of a set with respect to the topology $\tau_{\mathscr{U}}$. It provides the tools necessary to prove a consequential theorem.

Lemma 29.10 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S and $\mathscr{B}$ be a base. Let $\tau_{\mathscr{U}}$ be the topology on S generated by $\mathscr{U}$. If $T \subseteq S$, then

$$\mathrm{int}_{\tau_{\mathscr{U}}} T = \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$$

Proof: We are given a set S and a uniformity $\mathscr{U}$ on S with base $\mathscr{B}$. For each $B \in \mathscr{B}$,

$$B(x) = \pi_2[\,(\{x\} \times S) \cap B\,]$$

Let $T \subseteq S$

We first show $\mathrm{int}_{\tau_{\mathscr{U}}} T \subseteq \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$:

Let $x \in \mathrm{int}_{\tau_{\mathscr{U}}} T$. Since $\mathrm{int}_{\tau_{\mathscr{U}}} T$ is open with respect to to $\tau_{\mathscr{U}}$ there exists $B \in \mathscr{B}$ such that $x \in B(x) \subseteq \mathrm{int}_{\tau_{\mathscr{U}}} T \subseteq T$.

So $x \in \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$.

Hence $\mathrm{int}_{\tau_{\mathscr{U}}} T \subseteq \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$.

Next we show that $\{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\} \subseteq \mathrm{int}_{\tau_{\mathscr{U}}} T$:

Since $\{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\} \subseteq T$, it suffices to show that this set is open with respect to $\tau_{\mathscr{U}}$.

Let $x \in \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$.

To show this set is open, we must show that there is some $W \in \mathscr{B}$ such that $W(x) \subseteq \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$. Since $x \in \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$, then there exists $B \in \mathscr{B}$ such that $x \in B(x) \subseteq T$. By property U3, there exists $W \in \mathscr{B}$ such that $W \circ W \subseteq B$. Then $(W \circ W)(x) = W[W(x)] \subseteq B(x) \subseteq T$.

We claim $W(x) \subseteq T$.

Let $u \in W(x)$. Then we have $W(u) \subseteq W[W(x)]$. We then obtain the chain of inclusions,

$$\begin{array}{rcl} u & \in & W(u) \\ & \subseteq & (W \circ W)(x) \\ & \subseteq & B(x) \\ & \subseteq & T \end{array}$$

So $u \in W(x)$ implies $u \in T$. Then $W(x) \subseteq T$, as claimed.

Then the set $\{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$ is open with respect to $\tau_{\mathscr{U}}$.

So $\{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\} \subseteq \text{int}_{\tau_{\mathscr{U}}} T$.

We conclude that

$$\text{int}_{\tau_{\mathscr{U}}} T = \{x \in T : x \in B(x) \subseteq T \text{ for some } B \in \mathscr{B}\}$$

as required.

Note that it is possible that, for the subset T of S, $\text{int}_{\tau_{\mathscr{U}}} T$ is empty.

Recall (from Definition 5.1) that, for a given $x \in S$, the set,

$$\mathscr{U}_x = \{A \in \mathscr{P}(S) : A \text{ is a neighborhood of } x\}$$

is called a *neighborhood system of x with respect to τ*. A subfamily, $\mathscr{B}_x$, of a neighborhood system, $\mathscr{U}_x$, such that, for any set, $A \in \mathscr{U}_x$, there exists $B_x \in \mathscr{B}_x$ such that $x \in B_x \subseteq A$, is called a

base for the neighborhood system at x

The elements of a neighborhood base, $\mathscr{B}_x$ at x, are called *basic neighborhoods*.[1]

---

[1]The elements of a neighborhood base, $\mathscr{B}_x$ at x, are called *basic neighborhoods*. The basic neighborhoods need not be open in the space (S, τ); but, if every element of the neighborhood base, $\mathscr{B}_x$, belongs to τ, then we specify this by declaring $\mathscr{B}_x$ to be an *open neighborhood base at x*.

Theorem 29.11 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S with base $\mathscr{B}$. Let $\tau_{\mathscr{U}}$ be the topology on S generated by the uniformity, $\mathscr{U}$. Then, for each $x \in S$,

$$\mathscr{B}(x) = \{B(x) : B \in \mathscr{B}\}$$

forms a base for the neighborhood system of x, with respect to $\tau_{\mathscr{U}}$. (This neighborhood base need not necessarily be an open neighborhood base.)

Proof : Let $x \in S$ and N_x be a neighborhood of x, with respect to $\tau_{\mathscr{U}}$. Then, by definition, $x \in \text{int}_S N_x$. To show that $\mathscr{B}(x) = \{B(x) : B \in \mathscr{B}\}$ forms a neighborhood base at x, we must show that there is some $B(x) \in \mathscr{B}(x)$ such that $x \in B(x) \subseteq N_x$.

We have just shown that

$$\text{int}_{\tau_{\mathscr{U}}} N_x = \{x \in N_x : x \in B(x) \subseteq N_x \text{ for some } B \in \mathscr{B}\}$$

Since $x \in \text{int}_{\tau_{\mathscr{U}}} N_x$, there exists $B \in \mathscr{B}$ such that $B(x) \subseteq N_x$. Since $B(x) \in \mathscr{B}(x)$, $\mathscr{B}(x)$ forms a neighborhood base at x (with respect to the topology, $\tau_{\mathscr{U}}$, on S).

We have shown that $\mathscr{B}(x) = \{B(x) : B \in \mathscr{B}\}$ forms a base for the neighborhood system $\mathscr{U}_x$ of x, with respect to $\tau_{\mathscr{U}}$. Keep in mind that the elements of $\mathscr{B}(x)$ are not necessarily open in S with respect to $\tau_{\mathscr{U}}$.

Also, note that for each $x \in S$ and $B \in \mathscr{B}$, $\text{int}_{\tau_{\mathscr{U}}} B(x) \neq \varnothing$. Since, if $B_1 \in \mathscr{B}$ and $x \in S$, for any $B \in \mathscr{B}$, there exists $W \in \mathscr{B}$ such that $W \subseteq B_1 \cap B$, so $W(x) \subseteq (B \cap B_1)(x) \subseteq B_1(x)$. Then the set,

$$\text{int}_{\tau_{\mathscr{U}}} B_1(x) = \{x \in B_1(x) : x \in B(x) \subseteq B_1(x) \text{ for some } B \in \mathscr{B}\}$$

is non-empty.

Then for for the topological space, $(S, \tau_{\mathscr{U}})$, generated by $(S, \mathscr{U})$ we can write, $A \in \tau_{\mathscr{U}}$ is and only if

$$A = \cup_{x \in A}\{\text{int}_{\tau_{\mathscr{U}}} B(x) : B \in \mathscr{B} \text{ and } B(x) \subseteq A\}$$

A word of caution. In the literature, the expression "uniform space" is sometimes used to refer to both $(S, \mathscr{U})$ and the space, $(S, \tau_{\mathscr{U}})$ (where $(S, \tau_{\mathscr{U}})$ is, in fact, a uniformizable topological space). It is best to avoid this. This is akin to referring to both space (S, τ_ρ) and (S, ρ) as a metric spaces.

Theorem 29.8 guarantees that given a uniformity $\mathscr{U}$ on a set S we can generate a corresponding topology, $\tau_{\mathscr{U}}$ on S:

$$S \longrightarrow (S, \mathscr{U}) \longrightarrow (S, \tau_{\mathscr{U}})$$

On uniformizable spaces. By Definition 29.9, a topological space, (S, τ), derived from some uniformity, $\mathscr{U}$, on S (that is, where $\tau = \tau_{\mathscr{U}}$) is said to be "uniformizable". Are there any topological spaces that are not uniformizable? That is, is there a topology, τ, on S for which there is no uniformity, $\mathscr{U}$, such that $\tau = \tau_{\mathscr{U}}$? The answer is yes. We will show in Theorem 29.18 that a topological space derived from a uniformity is a regular space (i.e., $\mathrm{T}_1 + \mathrm{T}_3$).[3] So a non-regular space is not uniformizable.

The following result exhibits a way to express the closure of a set in $(S, \tau_{\mathscr{U}})$.

Notation. Let $\mathscr{U}$ be a uniformity on S. If $T \subseteq S$ and $U \in \mathscr{U}$, then

$$U[T] = \cup_{x \in T}\{U(x)\}$$

Theorem 29.12 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. Let $\tau_{\mathscr{U}}$ be the topology on S generated by $\mathscr{U}$. Let $T \subseteq S$. Then the closure, $\mathrm{cl}_{\tau_{\mathscr{U}}} T$, of T with respect to the uniform topology $\tau_{\mathscr{U}}$ can be expressed as,

$$\mathrm{cl}_{\tau_{\mathscr{U}}} T = \bigcap_{U \in \mathscr{U}} \{U[T]\}$$

Proof : We are given a set S and a uniformity $\mathscr{U}$ on S from which we derive the topological space, $(S, \tau_{\mathscr{U}})$, equipped with the uniform topology. Let $T \subseteq S$.

We are required to show that $\bigcap_{U \in \mathscr{U}}\{U[T]\} = \mathrm{cl}_{\tau_{\mathscr{U}}} T$.

The element $x \in \mathrm{cl}_{\tau_{\mathscr{U}}} T$ if and only if the set $U(x)$ intersects T for all $U \in \mathscr{U}$.

The set $U(x)$ intersects T for all $U \in \mathscr{U}$ if and only if the set $B(x)$ intersects T for all $B \in \mathscr{B}_{\text{sym}}$.

---

[3] In fact, a uniformizable space is always completely regular. We will not prove this in this book. For a proof the reader is referred to theorem 38.2 of Willard where it is shown that, for (S, τ), "uniformizable $\Leftrightarrow$ completely regular".

The set $B(x)$ intersects T for all $B \in \mathscr{B}_{\text{sym}}$ if and only if $x \in B^{-1}[T]$ for all $B \in \mathscr{B}_{\text{sym}}$.

The element $x \in B^{-1}[T]$ for all $B \in \mathscr{B}_{\text{sym}}$ if and only if $x \in B[T]$ for all $B \in \mathscr{B}_{\text{sym}}$ (Since B is symmetric $B = B^{-1}$).

The element $x \in B[T]$ for all $B \in \mathscr{B}_{\text{sym}}$ if and only if $x \in U[T]$ for all $U \in \mathscr{U}$.

The element $x \in U[T]$ for all $U \in \mathscr{U}$ if and only if $x \in \cap_{U\in\mathscr{U}} U[T]$.

We then have

$$\text{“}x \in \text{cl}_{\tau_{\mathscr{U}}} T \;\Leftrightarrow\; x \in \cap_{U\in\mathscr{U}} U[T]\text{”}$$

We then conclude that $\bigcap_{U\in\mathscr{U}}\{U[T]\} = \text{cl}_{\tau_{\mathscr{U}}} T$.

Recall that the U3 axiom for uniformities states that “. . . for $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V{\circ}V \subseteq B \subseteq U$”. We slightly extend this property in the following proposition. This fact will be a useful tool to prove a theorem to come.

Proposition 29.13 For any $M \in \mathscr{U}$, there exists $W \in \mathscr{B}_{\text{sym}}$ such that $W{\circ}W{\circ}W \subseteq M$.

Proof : We are given a set S and a uniformity $\mathscr{U}$ on S.

Let $M \in \mathscr{U}$. There exists $B \in \mathscr{B}_{\text{sym}}$ such that $B \subseteq M$. Then, by uniformity axiom U3, there exists $V \in \mathscr{U}$ such that $V{\circ}V \subseteq B$. Then there exists $D \in \mathscr{B}_{\text{sym}}$ such that $D \subseteq V$. Since $D \subseteq V$ and $V{\circ}V \subseteq B$, then $D{\circ}D \subseteq V{\circ}V \subseteq B \subseteq M$. Then

$$(D \cap B){\circ}(D \cap B) \subseteq D{\circ}D \subseteq B \subseteq M$$

So

$$(D \cap B){\circ}(D \cap B){\circ}(D \cap B) \subseteq (D \cap B){\circ}B \subseteq B{\circ}B \subseteq M$$

By U2*, there exists $W \in \mathscr{B}_{\text{sym}}$ such that $W \subseteq D \cap B$. Then there exists $W \in \mathscr{B}_{\text{sym}}$ such that

$$W{\circ}W{\circ}W \subseteq M$$

As required.

29.5 Examples.

Example 4. Consider the usual uniformity, $\mathscr{U}$, on $\mathbb{R}$ which is generated by the base, $\mathscr{B} = \{B_\kappa : \kappa > 0\}$ where $B_\kappa = \{(a,b) \in \mathbb{R}\times\mathbb{R} : |a-b| < \kappa\}$. By theorem, 29.8, the uniformity, $\mathscr{U}$, generates a topology $\tau_{\mathscr{U}}$ on $\mathbb{R}$. Verify that $\tau_{\mathscr{U}}$ is the usual topology, τ, on $\mathbb{R}$. That is,

$$\mathbb{R} \longrightarrow (\mathbb{R}, \mathscr{U}) \longrightarrow (\mathbb{R}, \tau_{\mathscr{U}}) = (\mathbb{R}, \tau)$$

Solution: Let $\mathscr{U}$ be the usual uniformity on $\mathbb{R}$ and τ denote the usual topology on $\mathbb{R}$. Then $\mathscr{U}$ is generated by the base, $\mathscr{B} = \{B_\kappa : \kappa > 0\}$, where B_κ represents all pairs (a,b) such that $|a-b| < \kappa$.

Claim #1: We claim that $\tau_{\mathscr{U}} \subseteq \tau$.

Proof of claim: Let U be an open neighborhood of $x \in \mathbb{R}$ with respect to $\tau_{\mathscr{U}}$. That is, let $U \in \tau_{\mathscr{U}}$. It suffices to show that $U \in \tau$. By definition of $\tau_{\mathscr{U}}$, there exists $B_\kappa \in \mathscr{B}$ such that $x \in (B_\kappa)(x) \subseteq U$.

See that $(B_\kappa)(x) = (x - \kappa/2, x + \kappa/2)$. So U is a union of basic open intervals. So $U \in \tau$. We conclude that $\tau_{\mathscr{U}} \subseteq \tau$, as claimed.

Claim #2: We claim that $\tau \subseteq \tau_{\mathscr{U}}$.

Proof of claim: Let A be a non-empty open subset in $\mathbb{R}$. It suffices to show that $A \in \tau_{\mathscr{U}}$.

Let $x \in A$. Then there exists ε such that $(x-\varepsilon, x+\varepsilon) \subseteq A$. Consider $B_{\varepsilon/2}$, a base element of $\mathscr{U}$. Then $(B_{\varepsilon/2})(x) \in \tau_{\mathscr{U}}$. By claim #1, $\tau_{\mathscr{U}} \subseteq \tau$; then $(B_{\varepsilon/2})(x) \in \tau$.

See that,

$$\begin{array}{rcl} x & \in & (x-\varepsilon/4, x+\varepsilon/4) \\ & = & \pi_2\left[\ (\{x\}\times S\ ) \cap B_{\varepsilon/2}\right] \\ & = & (B_{\varepsilon/2})(x) \\ & \subseteq & (x-\varepsilon, x+\varepsilon) \\ & \subseteq & A \end{array}$$

Then $x \in (B_{\varepsilon/2})(x) \subseteq (x-\varepsilon, x+\varepsilon) \subseteq A$. So A is the union of $\tau_{\mathscr{U}}$-base elements. Then $A \in \tau_{\mathscr{U}}$.

We conclude that $\tau \subseteq \tau_{\mathscr{U}}$, as claimed.

So $\tau_{\mathscr{U}} = \tau$.

We now examine, in a more general context, how the open subsets of the topology, $\tau_{\mathscr{U}}$, relate to the uniformity, $\mathscr{U}$, which generates it.

Example 5. Suppose $(S, \mathscr{U})$ is the uniform space equipped with the discrete uniformity, $\mathscr{U} = \{U \in \mathscr{P}(S \times S) : \Delta(S) \subseteq U\}$. Show that the corresponding topological space is equipped with the discrete topology, $\tau_{\mathscr{U}} = \mathscr{P}(S)$.

Solution: By definition, $\mathscr{U} = \{U \in \mathscr{P}(S \times S) : \Delta(S) \subseteq U\}$, is the discrete uniformity. By definition, a base for $\mathscr{U}$ is a subfamily, $\mathscr{B}$, such that for every $U \in \mathscr{U}$, there is a $B \in \mathscr{B}$ such that $B \subseteq U$. We choose,

$$\mathscr{B} = \{\Delta(S)\}$$

By definition, for the given $\mathscr{B}$,

$$\mathscr{B}_\tau = \{B(x) : x \in S \text{ and } B \in \mathscr{B} = \{\Delta(S)\} \text{ the base for } \mathscr{U}\}$$

forms a base for $(S, \tau_{\mathscr{U}})$.

For each $x \in S$,

$$\begin{aligned} B(x) &= \pi_2[\,(\{x\} \times S) \cap B\,] \\ &= \pi_2[\,(\{x\} \times S) \cap \Delta(S)\,] \\ &= \{x\} \end{aligned}$$

Then $\{\{x\} : x \in S\}$ forms a base for the uniform topology. This base generates $\tau_{\mathscr{U}} = \mathscr{P}(S)$.

Example 6. Suppose $(S, \mathscr{U})$ is the uniform space equipped with the single element uniformity, $\mathscr{U} = \{S \times S\}$. Show that the corresponding topological space is equipped with the trivial topology, $\tau_{\mathscr{U}} = \{\varnothing, S\}$.

Solution: We are given that $\mathscr{U} = \{S \times S\}$, so $\mathscr{B} = \{S \times S\}$ is a base for $\mathscr{U}$. Now, $\mathscr{B}_\tau = \{B(x) : x \in S \text{ and } B \in \mathscr{B} \text{ a base for } \mathscr{U}\}$. For each $x \in S$, $B(x) = \pi_2[\,(\{x\} \times S) \cap (S \times S)\,] = S$. So $\tau_{\mathscr{U}} = \{\varnothing, S\}$.

Example 7. For each $\kappa \in \mathbb{R}$, let

$$B_\kappa = \Delta(\mathbb{R}) \cup [\kappa, \infty) \times [\kappa, \infty)$$

Suppose $(\mathbb{R}, \mathscr{U})$ is the uniform space where $\mathscr{U}$ has as base,

$$\mathscr{B} = \{B_\kappa : \kappa \in \mathbb{R}\}$$

Show that the corresponding topological space, $(\mathbb{R}, \tau_{\mathscr{U}})$, is the discrete space.

Solution: We are given the family of subsets $\mathscr{B} = \{B_\kappa : \kappa \in \mathbb{R}\}$ is a base for $\mathscr{U}$. It is easily verified that $\mathscr{B}$ satisfies the properties required for it to serve as a base for some uniformity, $\mathscr{U}$, on S.

Now $\tau_{\mathscr{U}} = \{U \in \mathscr{P}(S) : \forall\, x \in U\ \exists\, B \in \mathscr{U}$ such that $B(x) \subseteq U\}$.

If $x < \kappa$, then $B(x) = \{x\}$; so $\{x\}$ is an open base element of $\mathbb{R}$. So $(\mathbb{R}, \tau_{\mathscr{U}})$ is a discrete topological space.

The following technical result will be a useful tool in proofs to come.

Example 8. Suppose $(S, \mathscr{U})$ is uniform space and $U, W \in \mathscr{U}$ where W is known to be symmetric. Show that

$$W{\circ}U{\circ}W = \bigcup_{(x,y)\in U} \{W(x) \times W(y)\}$$

Solution: Consider the composition $W{\circ}U{\circ}W$ where $U \in \mathscr{U}$ and W is symmetric in $\mathscr{U}$.

See that $(a, b) \in W{\circ}U{\circ}W$ if and only if $(a, x) \in W$ and $(y, b) \in W$ for some $(x, y) \in U$.

Then, $W{\circ}U{\circ}W \subseteq W(x) \times W(y)$, for some $(x, y) \in U$.

Conversely, suppose $(a, b) \in W(x) \times W(y)$, for some $(x, y) \in U$.

Then $a \in W(x)$ and $b \in W(y)$. So $(x, a) \in W$ and $(y, b) \in W$. Since W is symmetric, $(a, x) \in W$. Since $(x, y) \in U$, then $(a, b) \in W{\circ}U{\circ}W$, so $W(x) \times W(y) \subseteq W{\circ}U{\circ}W$.

We conclude that, whenever W is symmetric, there is $(x, y) \in U$ such that

$$W{\circ}U{\circ}W = W(x) \times W(y)$$

Then

$$W{\circ}U{\circ}W = \bigcup_{(x,y)\in U} \{W(x) \times W(y)\}$$

29.5 On the product topology derived from $(S, \tau_{\mathscr{U}}) \times (S, \tau_{\mathscr{U}})$.

Since a uniformity, $\mathscr{U}$, is a subfamily of $\mathscr{P}(S \times S)$, then every element, M, in $\mathscr{U}$ is subset of the Cartesian product $S \times S$ which contains the diagonal, $\Delta(S)$. The structure of $(S, \mathscr{U})$ is determined by the five uniformity axioms. Since $\Delta(S) \subseteq U$ for each $U \in \mathscr{U}$ one might wonder whether U is a "neighborhood" of $\Delta(S)$. But with respect to what topology? Suppose one decides to equip $S \times S$ with the product topology. But to do this we must have a topology which is defined on S. If S is equipped with the topology, $\tau_{\mathscr{U}}$, induced by $(S, \mathscr{U})$,then this then becomes a valid question. One which we will address now.

Theorem 29.14 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. Suppose S is equipped with the uniform topology, $\tau_{\mathscr{U}}$ induced on S by $\mathscr{U}$. If $M \in \mathscr{U}$ then $\text{int}_{S\times S} M \in \mathscr{U}$ (with respect to the product topology on $S \times S$).

Proof : We are given a uniformity $\mathscr{U}$ on a set S and the uniform topology, $\tau_{\mathscr{U}}$ induced on S by $\mathscr{U}$. A set U belongs to $\tau_{\mathscr{U}}$ provided, for every $x \in U$, there exists $B \in \mathscr{B}$ such that $B(x) \subseteq U$.

If $M \in \mathscr{U}$,

$$\text{int}_{S\times S} M = \{(x, y) : \exists \;\; U, V \in \tau_{\mathscr{U}} \text{ s.t. } (x, y) \in U \times V \subseteq M\}$$

$$\text{int}_{S\times S} M = \{(x, y) : \exists \;\; U, V \in \mathscr{U} \text{ s.t. } (x, y) \in U(x) \times V(x) \subseteq M\}$$

To prove the theorem we first establish the following three facts.

Fact #1. For any $M \in \mathscr{U}$,

$$\text{int}_{S\times S} M = \{(x, y) : \exists \, W \in \mathscr{B} \text{ s.t. } (x, y) \in W(x) \times W(y) \subseteq M\}$$

Proof of fact #1 : We are given,

$$\text{int}_{S\times S} M = \{(x, y) : \exists \;\; U, V \in \tau_{\mathscr{U}} \text{ s.t. } (x, y) \in U \times V \subseteq M\}$$

$$\text{int}_{S\times S} M = \{(x, y) : \exists \;\; B_1, B_2 \in \mathscr{B} \text{ s.t. } (x, y) \in B_1(x) \times B_2(y) \subseteq U \times V \subseteq M\}$$

By U2* there exists $W \in \mathscr{B}$ such that $W \subseteq B_1 \cap B_2$.

Then $x \in W(x) \subseteq (B_1 \cap B_2)(x) \subseteq B_1(x) \subseteq U$ and $y \in W(y) \subseteq (B_1 \cap B_2)(y) \subseteq B_2(y) \subseteq V$. Then

$$\text{int}_{S\times S} M = \{(x, y) : \exists \, W \in \mathscr{B} \text{ s.t. } (x, y) \in W(x) \times W(y) \subseteq M\}$$

This proves fact #1.

Fact #2. For any $M \in \mathscr{U}$, there exists $W \in \mathscr{B}_{\text{sym}}$ such that $W \circ W \circ W \subseteq M$.

This fact is proven in proposition 29.13.

Fact #3. For any $U, W \in \mathscr{U}$ where W is known to be symmetric,

$$W \circ U \circ W = \bigcup_{(x,y)\in U} \{W(x) \times W(y)\}$$

Proof of fact #3 is given in Example 8.

We now combine these three facts, to conclude the proof. Let $M \in \mathscr{U}$.

If we prove that $W \circ W \circ W \subseteq \text{int}_{S\times S}M$, then by applying U5, since $W \circ W \circ W \in \mathscr{U}$, $\text{int}_{S\times S}M \in \mathscr{U}$ and so we will be done.

By fact #2, there exists $W \in \mathscr{B}_{\text{sym}}$ such that $W \circ W \circ W \subseteq M$.

By fact #3, for a symmetric $W \in \mathscr{U}$,

$$W \circ W \circ W = \bigcup_{(x,y)\in W} \{W(x) \times W(y)\} \subseteq M$$

It suffices to show that, for $M \in \mathscr{U}$, there is some element A of $\mathscr{U}$ such that $(x, y) \in A \subseteq \text{int}_{S\times S}M$. That is, that $(x, y) \in A \subseteq U(x) \times V(y) \subseteq M$ for some $U, V \in \mathscr{U}$.
By fact #2, given $M \in \mathscr{U}$, there exists $W \in \mathscr{B}_{\text{sym}}$ such that $W \circ W \circ W \subseteq M$. Let $(x, y) \in W$. Then $y \in W(x)$ and $x \in W(y)$. So

$$\begin{aligned} (y, x) &\in W(x) \times W(y) \\ &\subseteq \cup_{(x,y)\in W}\{W(x) \times W(y)\} \\ &= W \circ W \circ W \\ &\subseteq M \end{aligned}$$

Since $W \circ W \circ W \in \mathscr{B}$, by U5, $\text{int}_{S\times S}M \in \mathscr{U}$. We are done.

We know that $\mathscr{B}_{\text{sym}} = \{B \in \mathscr{U} : B \text{ is symmetric }\}$ forms a base for $\mathscr{U}$. This means that, for every $U \in \mathscr{U}$, there is a $B \in \mathscr{B}$ such that $B \subseteq U$. By the above theorem, since $\mathscr{B}_{\text{sym}} \subseteq \mathscr{U}$, if $B \in \mathscr{B}_{\text{sym}}$, $\text{int}_{S\times S}B \in \mathscr{U}$.

We then have the following corollary.

Corollary 29.15 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. Suppose S is equipped with the uniform topology, $\tau_{\mathscr{U}}$ induced on S by $\mathscr{U}$.

Let

$$\mathscr{B}^{\circ} = \{\text{int}_{S\times S}B : B \in \mathscr{B}_{\text{sym}}\}$$

Then the family, $\mathscr{B}^{\circ}$ forms a base for $\mathscr{U}$. Furthermore, for each $x \in S$, $\mathscr{B}^{\circ}(x) = \{B(x) : B \in \mathscr{B}^{\circ}\}$ is an open neighborhood base at x in $\tau_{\mathscr{U}}$.

Proof: Since $\mathscr{B} = \{B : B \in \mathscr{B}_{\text{sym}}\}$ forms a base for in $\mathscr{U}$ and $\text{int}_{S\times S}B \in \mathscr{U}$ for each $B \in \mathscr{B}_{\text{sym}}$ then $\mathscr{B}^{\circ}$ forms a base for in $\mathscr{U}$.

Suppose $x \in S$. If $B \in \mathscr{B}^{\circ}$, $x \in B(x)$. Since $\text{int}_{S\times S}B$ is open in $S \times S$, then $(\{x\} \times S) \cap \text{int}_{S\times S}B$ is open in $\{x\} \times S$.
We know that,

$$\pi_2|_{\{x\}\,\times\,S} : \{x\} \times S \to S$$

is a projection continuous function mapping $\{x\} \times S$ one-to-one onto S. Then,

$$(\text{int}_{S\times S}B)(x) \;=\; \pi_2|_{\{x\}\,\times\,S}[(\{x\} \times S) \cap \text{int}_{S\times S}B]$$

is an open neighborhood of x with respect to $\tau_{\mathscr{U}}$. Since $\{B(x) : B \in \mathscr{B}_{\text{sym}}\}$ has been shown to be a neighborhood base at x,

$$\mathscr{B}^{\circ}(x) = \{(\text{int}_{S\times S}B)(x) : B \in \mathscr{B}_{\text{sym}}\}$$

is an open neighborhood base at x with respect to $\tau_{\mathscr{U}}$.

We briefly summarize the main ideas behind what we have accomplished up to this point.

- We first give ourselves a set S equipped with a uniformity $\mathscr{U}$. A uniformity is a collection of subsets of $S \times S$ (each containing the diagonal) which satisfies five uniformity axioms.
- The uniformity allows us to define a topology, $\tau_{\mathscr{U}}$, (called the uniform topology induced by $\mathscr{U}$) on S.
- This topology on S allows us to define the product topology $(S, \tau_{\mathscr{U}}) \times (S, \tau_{\mathscr{U}})$ which, in turn, allows us to describe the elements of $\mathscr{U}$ in terms of the open sets in the product space.
- For each set $U \in \mathscr{U}$ we can associate the set, $\text{int}_{S\times S}U$, proven to belong to $\mathscr{U}$.
- Then $\{\text{int}_{S\times S}B : B \in \mathscr{B}_{\text{sym}}\}$ forms a base for $\mathscr{U}$.
- Then $\{(\text{int}_{S\times S}B)(x) : B \in \mathscr{B}_{\text{sym}}\}$ forms an open base at x with respect to $\tau_{\mathscr{U}}$.

We now describe a method for representing the closure (with respect to the product topology on $(S, \tau_{\mathscr{U}}) \times (S, \tau_{\mathscr{U}})$) of the elements of $\mathscr{U}$. When we write "$\text{cl}_{S\times S}$" we mean the closure in $S \times S$ equipped with the product topology $(S, \tau_{\mathscr{U}}) \times (S, \tau_{\mathscr{U}})$. This will eventually show that $\{\text{cl}_{S\times S} B : B \in \mathscr{B}_{\text{sym}}\}$ forms a base for $\mathscr{U}$.

Theorem 29.16 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. If $M \in \mathscr{U}$ then

$$\text{cl}_{S\times S} M = \bigcap_{B \in \mathscr{B}_{\text{sym}}} \{B \circ M \circ B\}$$

Furthermore, if $M \in \mathscr{B}_{\text{sym}}$ then $\text{cl}_{S\times S} M \in \mathscr{B}_{\text{sym}}$.

Proof : We are given a set S, a uniformity $\mathscr{U}$ on S and the base, $\mathscr{B}_{\text{sym}}$, of all symmetric elements of $\mathscr{U}$. To show that $\text{cl}_{S\times S} M = \bigcap_{B\in\mathscr{B}_{\text{sym}}} \{B \circ M \circ B\}$ it suffices to show that

$$\{(x, y) \in S \times S : \text{ any nbd of } (x, y) \text{ intersects } M\} = \bigcap_{B \in \mathscr{B}_{\text{sym}}} \{B \circ M \circ B\}$$

The nbh $B(x) \times B(y)$ of (x, y) (where $B \in \mathscr{B}_{\text{sym}}$) intersects M

$$\Leftrightarrow$$

There exists $(u, v) \in M$ such that $(u, v) \in B(x) \times B(y)$

$$\Leftrightarrow$$

There exists $(u, v) \in M$ such that $(x, u) \in B$ and $(y, v) \in B$

$$\Leftrightarrow$$

There exists $(u, v) \in M$ such that $(u, x) \in B$ and $(v, y) \in B$

$$\Leftrightarrow$$

There exists $(u, v) \in M$ such that $(x, y) \in B(u) \times B(v)$

$$\Leftrightarrow$$

$$(x, y) \in \bigcup\nolimits_{(u,v)\in M} B(u) \times B(v)$$

$$\Leftrightarrow$$

$$(x, y) \in B \circ M \circ B \qquad \text{(See Example 8.)}$$

Then the neighborhood, $B(x) \times B(y)$ of (x, y)) intersects M, for each $B \in \mathscr{B}_{\text{sym}}$, if and only if $(x, y) \in B \circ M \circ B$, for each $B \in \mathscr{B}_{\text{sym}}$, if and only if $(x, y) \in \bigcap_{B\in\mathscr{B}_{\text{sym}}} B \circ M \circ B$.

We conclude that $\mathrm{cl}_{S\times S}M = \bigcap_{B\in\mathscr{B}_{\mathrm{sym}}}\{B\circ M\circ B\}$.
Suppose $M \in \mathscr{B}_{\mathrm{sym}}$. Then, by U5, $\mathrm{cl}_{S\times S}M \in \mathscr{U}$.

We verify that $\mathrm{cl}_{s\times S}M$, is symmetric.

Suppose $(a,b) \in \mathrm{cl}_{S\times S}M = \cap_{B\in\mathscr{B}_{\mathrm{sym}}}\{B\circ M\circ B\}$. Then $(a,b) \in B\circ M\circ B$, for all $B \in \mathscr{B}_{\mathrm{sym}}$.

Then, $(a,x) \in B$, $(y,b) \in B$ some $(x,y) \in M$, for all B.

Then $(x,a) \in B$, $(y,x) \in M$ some $(b,y) \in B$, for all B. So $(b,a) \in B\circ M\circ B$, for all B. So $(b,a) \in \mathrm{cl}_{s\times S}M$.

So $\mathrm{cl}_{s\times S}M$ is symmetric.

Corollary 29.17 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. The family

$$\mathscr{C} = \{C : C \text{ is closed in } S\times S \text{ and symmetric}\}$$

is a base for $\mathscr{U}$.

Proof: Let $U \in \mathscr{U}$. By Proposition 29.13, there exists $W \in \mathscr{B}_{\mathrm{sym}}$ such that $W\circ W\circ W \subseteq U$. By the previous theorem,

$$\mathrm{cl}_{S\times S}W = \cap_{B\in\mathscr{B}_{\mathrm{sym}}}\{B\circ W\circ B\}$$

So $\mathrm{cl}_{S\times S}W = \cap_{B\in\mathscr{B}_{\mathrm{sym}}}\{W\circ W\circ W\} \subseteq W\circ W\circ W \subseteq U$. Since W is symmetric then $\mathrm{cl}_{S\times S}W$ is symmetric.
Then every element $U \in \mathscr{U}$ contains an element of $\mathscr{C}$. So $\mathscr{C}$ is a base for $\mathscr{U}$. Then the collection $\{\mathrm{cl}_{S\times S}W : W \in \mathscr{B}_{\mathrm{sym}}\}$ is a base for $\mathscr{U}$.

Summarizing: We previously described the properties of a base $\mathscr{B}$ for $\mathscr{U}$. We also previously showed that

- $\mathscr{B}_{\mathrm{sym}} = \{B \in \mathscr{B} : B \text{ is symmetric}\}$ is a base for $\mathscr{U}$.
- $\mathscr{B}(x) = \{B(x) : B \in \mathscr{B}\}$ forms a base for the neighborhood system of x, with respect to $\tau_{\mathscr{U}}$. (Theorem 29.11)
- $\mathscr{B}^\circ = \{\mathrm{int}_{S\times S}B : B \in \mathscr{B}_{\mathrm{sym}}\}$ is a neighborhood base of $\mathscr{U}$. (Corollary 29.15)

- $\mathscr{B}^{\circ}(x) = \{B(x) : B \in \mathscr{B}^{\circ}\}$ is an open neighborhood base at x in $\tau_{\mathscr{U}}$. (Corollary 29.15)
- $\mathscr{B}_{\text{cl-sym}} = \{\text{cl}_{S\times S}W : W \in \mathscr{B}_{\text{sym}}\}$ is a neighborhood base of $\mathscr{U}$ where $\text{cl}_{S\times S}W \in \mathscr{B}_{\text{sym}}$. (Corollary 29.17)
- $(\mathscr{B}_{\text{cl-sym}})(x) = \{B(x) : B \in \mathscr{B}_{\text{cl-sym}}\}$ is a neighborhood base for the neighborhood system of x, with respect to $\tau_{\mathscr{U}}$.

29.6 The uniform space $(S, \tau_{\mathscr{U}})$ is T_3.

The uniform space, $(S, \tau_{\mathscr{U}})$, inherits from the uniformity $\mathscr{U}$ on S some separation properties. We can show immediately that the uniformity $\mathscr{U}$ on S guarantees that $(S, \tau_{\mathscr{U}})$ is T_3.

In what follows we will represent the base of all closed symmetric elements of $\mathscr{U}$ by, $\mathscr{B}_{\text{cl-sym}}$

Theorem 29.18 Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S such that $\bigcap_{U\in\mathscr{U}}\{U\} = \Delta(S)$. Let $\tau_{\mathscr{U}}$ be the topology on S generated by $\mathscr{U}$. Then the space, $(S, \tau_{\mathscr{U}})$, is regular.

Proof: We are given a set S, a uniformity $\mathscr{U}$ on S such that $\bigcap_{U\in\mathscr{U}}\{U\} = \Delta(S)$ and the derived topological space, $(S, \tau_{\mathscr{U}})$, equipped with the uniform topology.

Recall that $\mathscr{B}^{\circ} = \{\text{int}_{S\times S}B : B \in \mathscr{B}_{\text{sym}}\}$ forms a base for $\mathscr{U}$ and that $(\mathscr{B}^{\circ})(x) = \{(\text{int}_{S\times S}B)(x) : B \in \mathscr{B}_{\text{sym}}\}$ forms an open base at x with respect to $\tau_{\mathscr{U}}$.

Let $(x, y) \in S \times S$ such that $x \neq y$. Since $\bigcap_{U\in\mathscr{U}}\{U\} = \Delta(S)$ there exists $B \in \mathscr{B}^{\circ}$ such that neither (x, y) nor (y, x) belong to B. Then $x \in B(x)$ and $y \notin B(x)$ and $y \in B(y)$ and $x \notin B(y)$. Since $B(x)$ and $B(y)$ are both basic open neighborhoods with respect to $\tau_{\mathscr{U}}$, then by definition, $(S, \tau_{\mathscr{U}})$ is T_1.

We now show that $(S, \tau_{\mathscr{U}})$ is T_3. Let F be a closed subset in S (with respect to $\tau_{\mathscr{U}}$) and $x \in S\backslash F$. Then $S\backslash F$ is $\tau_{\mathscr{U}}$-open in S.

To show that $(S, \tau_{\mathscr{U}})$ is T_3, it suffices to show that x has a closed neighborhood which misses F.

Since $(\mathscr{B}_{\text{cl-sym}})(x) = \{B(x) : B \in \mathscr{B}_{\text{cl-sym}}\}$ is a neighborhood base for the neighborhood system of x, with respect to $\tau_{\mathscr{U}}$, then $x \in B(x) \subseteq S\backslash F$

for some $B \in \mathscr{B}_{\text{cl-sym}} = \{\text{cl}_{S \times S} W : W \in \mathscr{B}_{\text{sym}}\}$.

We know that,

$$\pi_2|_{\{x\} \times S} : \{x\} \times S \to S$$

is an open projection continuous function mapping $\{x\} \times S$ one-to-one onto S.

Then if $B \in \mathscr{B}_{\text{cl-sym}}$,

$$B(x) = \pi_2|_{\{x\} \times S}[B]$$

is a closed neighborhood of x. So $x \in B(x)$, a closed neighborhood which misses F. So $(S, \tau_{\mathscr{U}})$ is T_3.

29.7 Uniform continuity on a uniform space $(S, \mathscr{U})$.

The uniformity $\mathscr{U}$ defined on a set S will allow us to introduce the notion of "uniform continuity" on the resulting uniform space, $(S, \mathscr{U})$.

Definition 29.19 Let $(S, \mathscr{U})$ and $(T, \mathscr{V})$ be two sets equipped with the uniformities, $\mathscr{U}$ and $\mathscr{V}$, respectively. Let $f : S \to T$ be a function mapping S into T. Let $g_f : S \times S \to T \times T$ be the function defined as

$$g_f(x, y) = (\, f(x), f(y)\,)$$

and

$$g_f^{\leftarrow}[V] = \{(x, y) \in S \times S : (f(x), f(y)) \in V\}$$

We will say that the function

$$f : (S, \mathscr{U}) \to (T, \mathscr{V}) \text{ is "uniformly continuous" relative to } \mathscr{U} \text{ and } \mathscr{V}$$

if and only if

$$V \in \mathscr{V} \quad \Rightarrow \quad g_f^{\leftarrow}[V] \in \mathscr{U}$$

Equivalently, $f : S \to T$ is uniformly continuous if and only if,

$$\text{for any } V \in \mathscr{V} \text{ there exists } U \in \mathscr{U} \text{ such that } g_f[U] \subseteq V$$

Example 9. For $\varepsilon > 0$, let $B_\varepsilon = \{(a,b) \in \mathbb{R} \times \mathbb{R} : |a-b| < \varepsilon\}$. Recall that the *usual uniformity* on $\mathbb{R}$ is the uniformity $\mathscr{V}$ with base $\mathscr{B} = \{B_\varepsilon : \varepsilon > 0\}$.

Let $(S, \mathscr{U})$ be a uniform space and $f : S \to \mathbb{R}$ be a function mapping S into $\mathbb{R}$, where $(\mathbb{R}, \mathscr{V})$ is equipped with the usual uniformity, $\mathscr{V}$. By definition, $f : S \to \mathbb{R}$ is uniformly continuous on S if and only if, for any $\varepsilon > 0$, there exists $U \in \mathscr{U}$ such that $g_f[U] \subseteq B_\varepsilon$. That is, for any pair, $(x,y) \in U$, $|f(x) - f(y)| < \varepsilon$. Note that the choice of U is independent of the location of (x,y) in $S \times S$.

In the following more familiar example, we examine what uniform continuity means for a function $f : \mathbb{R} \to \mathbb{R}$.

Example 10. Let $(\mathbb{R}, \mathscr{U})$ be the uniform space for $\mathbb{R}$ equipped with the usual uniformity with the base $\mathscr{B} = \{B_\varepsilon : \varepsilon > 0\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function mapping $\mathbb{R}$ into $\mathbb{R}$. By definition, f is uniformly continuous on $\mathbb{R}$ if and only if for any $B_\varepsilon \in \mathscr{U}$, there is a $B_\delta \in \mathscr{U}$ such that $g_f[B_\delta] \subseteq B_\varepsilon$. Equivalently, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Note that the composition of two uniformly continuous functions is easily verified to be uniformly continuous.

We will immediately formally define the following three concepts.

Definition 29.20 If, for a given one-to-one function, $f : (S, \mathscr{U}) \to (T, \mathscr{V})$, both f and $f^{\leftarrow}$ are known to be uniformly continuous, then we say that the function $f : S \to T$ is a *uniform isomorphism* from between S and T.

If $f : S \to T$ is a uniform isomorphism, then we will say that $(S, \mathscr{U})$ and $(T, \mathscr{V})$ are *uniformly isomorphic*.

A property on $(S, \mathscr{U})$ which is shared by all uniform spaces which are uniformly isomorphic to $(S, \mathscr{U})$ is said to be a *uniform invariant*.

29.8 Uniform continuity implies continuity.

Continuity is always defined with respect to some topology. If we are given a function $f : (S, \mathscr{U}) \to (T, \mathscr{V})$ which is known to be uniformly continuous with respect to the uniformities $\mathscr{U}$ and $\mathscr{V}$, respectively, what can we say about the same function $f : S \to T$ mapping S to T

with respect to the corresponding topologies $\tau_{\mathscr{U}}$ and $\tau_{\mathscr{V}}$. We answer this question with the following theorem.

Theorem 29.21 Let $(S, \mathscr{U})$ and $(T, \mathscr{V})$ be sets with corresponding uniformities, $\mathscr{U}$ and $\mathscr{V}$, respectively. If $f : (S, \mathscr{U}) \to (T, \mathscr{V})$ is a uniformly continuous function with respect to the uniformities $\mathscr{U}$ and $\mathscr{V}$, then this same function is a continuous function with respect to the uniform topologies, $\tau_{\mathscr{U}}$ and $\tau_{\mathscr{V}}$.

Proof: We are given sets $(S, \mathscr{U})$ and $(T, \mathscr{V})$ with respective uniformities $\mathscr{U}$ and $\mathscr{V}$. Suppose $f : (S, \mathscr{U}) \to (T, \mathscr{V})$ is a uniformly continuous function.

To prove continuity of $f : S \to T$, it suffices to show f is continuous at each point $x \in S$.

Let $x \in S$ and let V be a base element of $\mathscr{V}$ such that $f(x) \in V(f(x))$, where $V(f(x))$ is a neighborhood of $f(x)$ in T. It suffices to produce a neighborhood W, of x such that $f[W] \subseteq V(f(x))$. (See 6.2 and 6.3.) By the definition of "$f : S \to T$ is uniformly continuous", $g_f^{\leftarrow}[V]$ is an element of $\mathscr{U}$ (with $\Delta(S)$ in its $(S \times S)$-interior).

See that,

$$\begin{aligned} f^{\leftarrow}[V(f(x))] &= \{\, y \in S : f(y) \in V(f(x)) \,\} \\ &= \{y \in S : (f(x), f(y)) \in V\} \\ &= \{y \in S : g_f(x,y) \in V\} \\ &= \left[\, g_f^{\leftarrow}[V] \,\right](x) \quad \text{{\scriptsize (Where } g_f^{\leftarrow}[V] \in \mathscr{U}\text{)}} \end{aligned}$$

Since $\left[\, g_f^{\leftarrow}[V] \,\right](x)$ is a neighborhood of x, then $f^{\leftarrow}[V(f(x))]$ is a neighborhood of x.

Since $f[f^{\leftarrow}[V(f(x))]] = V(f(x))$, we have produced a neighborhood $W = f^{\leftarrow}[V(f(x))]$ of x such that $f[W] \subseteq V(f(x))$, so f is continuous at x. Since x was arbitrarily chosen in S, f is continuous on all of S.

It follows from the theorem that a function $f : S \to T$ which is a uniform isomorphism can also be seen as a homeomorphism mapping S onto T.

Uniform continuity on a uniform space generalizes the notion of uniform continuity from one metric space to another. In the case of a metric space, it is more often discussed without explicitly introducing the concept of uniformities on a set. To see this we present the following example.

Example 11. Suppose (M_1, ρ_1) and (M_2, ρ_2) are two metric spaces. Suppose $f : M_1 \to M_2$ is a function satisfying the condition: "*For every* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that* $f[B_\delta(x)] \subseteq B_\varepsilon(f(x))$, *for all* $x \in S$." In this case, the condition imposed on the existence of δ is stronger than the one required in the definition of continuity at x (where the value of δ depends on both the value of ε and the point, x). It easily seen here that, if f satisfies the given condition, then f is continuous at each point.

We compare this to the same function $f : M_1 \to M_2$ where the sets M_1 and M_2 are equipped with the metric uniformities, $\mathscr{U}$ and $\mathscr{V}$, respectively. A metric uniformity has as base, $\mathscr{B} = \{B_\kappa : \kappa > 0\}$ where $B_\kappa = \{(x, y) : \rho(x, y) < \kappa\}$. By using the language and notation for uniformities, we can say that f is uniformly continuous if,

> ... for any $\varepsilon > 0$, there is $\delta > 0$ such that, if $(x, y) \in B_\delta \in \mathscr{U}$, then $(f(x), f(y)) \in B_\varepsilon \in \mathscr{V}$. Equivalently, $f[B_\delta(x)] \subseteq B_\varepsilon(f(x))$, for all $x \in S$.