Classifying Module Categories for Generalized TLJ $*$-2-Categories

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Weighted Bidirected Graphs

A **weighted bidirected graph** is a triple $(\Gamma, \delta, \overline{\cdot})$ where

- the directed graph Γ is countable and locally finite,
- the map $\delta : E(\Gamma) \to (0, \infty)$ is a weighting of the edges, and
- the function $\overline{\cdot} : E(\Gamma) \to E(\Gamma)$ is a direction-reversing involution.

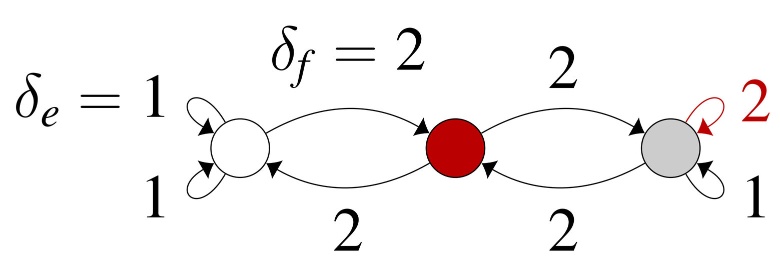


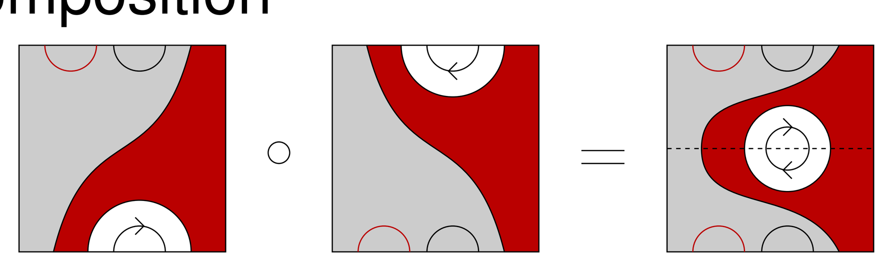
Figure: A weighted bidirected graph $\Gamma = \Gamma_0$

Description of the $*$-2-Category TLJ(Γ) [Mw10]

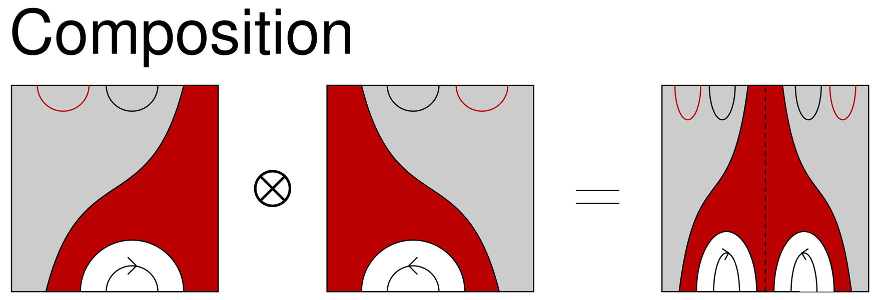
- **Objects** in **TLJ**(Γ) are vertices of Γ.
- **1-morphisms** are the paths in Γ, and composition is by concatenation.
- **2-morphisms** from path a to path b are $\mathbb{C}$-linear combinations of Kauffman diagrams.

We can perform the following operations on 2-morphisms:

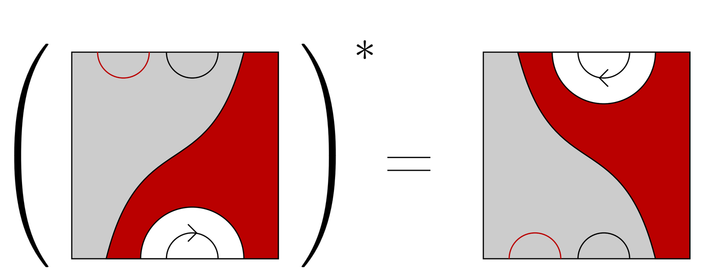
- Vertical composition



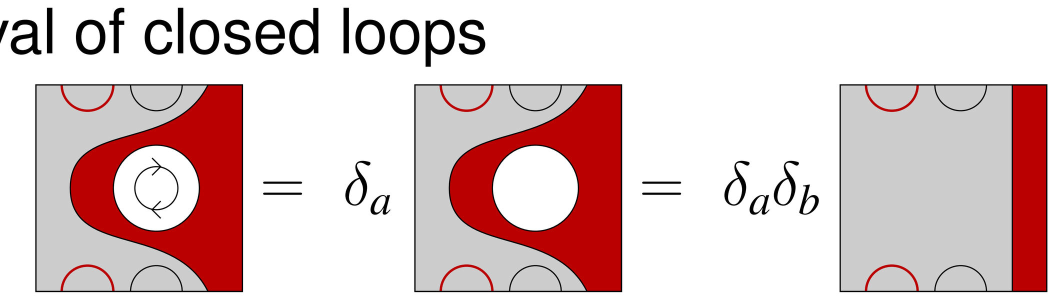
- Horizontal Composition



- Involution reflects horizontally and reverses orientations



- Removal of closed loops



Unitary Modules for TLJ(Γ): Motivation and Definitions

Fiber functors relate the representation theory of quantum groups with TLJ categories. These are strong monoidal functors $\mathbb{F}$: TLJ $\to$ Vec (turning Vec into a TLJ-module), and were classified in **[Yam04]** using spectral theory.
General TLJ(δ)-modules were classified as fiber functors into the category of bi-graded Vector spaces in terms of graphs in **[EO04]**.
Module C^*-categories for $SU(2)$ where classified in **[DcY13a]** as fiber functors into bi-graded Hilbert spaces, $\mathbb{F}$: Rep$(SU_q(2)) \to$ BigHilb in terms of weighted graphs. (An appropriate choice of δ makes TLJ(δ) unitarily equivalent to Rep$(SU_q(2))$.) We thus define a unitary **TLJ**(Γ)-module as a $*$-pseudofunctor

$$\mathcal{F} : \textbf{TLJ}(\Gamma) \Rightarrow \textbf{UCat}.$$

UCat is a strict model for **BigHilb**, the $*$-2-category of countably bigraded row and column finite Hilbert spaces.

Γ-fundamental solutions

Strict $*$-pseudofunctors

$$\mathcal{F} : \textbf{TLJ}(\Gamma) \Rightarrow \textbf{UCat}.$$

are determined by the image of each **cup element**, satisfying the **balancing equations**. To specify $\mathcal{F}$ it is then sufficient to describe a Γ-**fundamental solution**:

$$(\{\mathcal{F}(a)\}_{a \in V(\Gamma)}, \{\mathcal{F}(e)\}_{e \in E(\Gamma)}, \{\mathcal{F}(\text{coev}_e)_{e \in E(\Gamma)}\}).$$

References

[DcY13b], [DcY13a] De Commer and Yamashita, *Tannaka-Krein duality for compact quantum homogeneous spaces. I, II.* arXiv:1211.6552v3, 1212.3413v2
[EO04] Etingof and Ostrik, *Module categories over representations of $SL_q(2)$ and graphs*, arXiv:math/0302130v1
[FeHe19] Ferrer, Hernandez, *Classifying Module Categories for Generalized TLJ $*$-2-Categories*, arXiv:1905.00471
[MW10] Morrison and Walker, *The graph planar algebra embedding*, 2010,http://tqft.net/gpa.
[Yam04] Yamagami, *Fiber functors on Temperley-Lieb categories*, 2004, arXiv:math/0405517v2.

A Combinatorial Description for Γ-Fundamental Sol's

A **balanced** Γ-**fair** graph **[FeHe19]** is a triple (Λ, w, π), where $\pi : \Lambda \to \Gamma$ is a surjective bidirected graph homomorphism, and $w : E(\Lambda) \to (0, \infty)$ is a weight function such that

$$\sum_{\substack{\{\epsilon \mid \text{source}(\epsilon)=\alpha \\ \text{and } \pi(\epsilon)=e\}}} w(\epsilon) = \delta_e.$$

Together with the existence of an involution $\overline{\cdot}$ on $E(\Lambda)$ with

$$w(\epsilon)w(\overline{\epsilon}) = 1 \text{ and } \pi(\overline{\epsilon}) = \overline{\pi(\epsilon)}.$$

These graphs fully encode Γ-fundamental solutions, since one can produce a balanced Γ-fair graph from a Γ-fundamental solution and conversely. This equivalence is made explicit by certain conjugate-linear maps associated to balanced Γ-fair graph or either to a Γ-fundamental solution.

A Balanced Γ_0-Fair Graph Λ_0

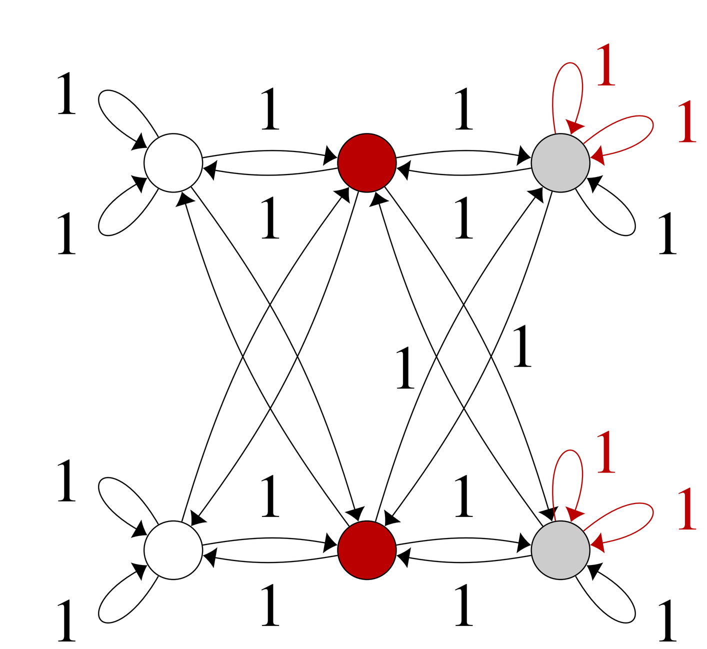


Figure: Description of the triple (Λ_0, w, π)

Classification of Unitary TLJ(Γ)-Modules

Theorem: [FeHe19]
Every balanced Γ-fair graph arises from a Γ-fundamental solution.
There is an equivalence of isomorphism classes of balanced Γ-fair graphs and unitary isomorphism classes of strong $*$-pseudofunctors $\mathcal{F}$: **TLJ**$(\Gamma) \Rightarrow$ **BigHilb**.
Corollary: We recover Proposition 2.3 in **[DCY13a]** for Rep$(SU_q(2))$ for $q < 0$, by taking Γ to be a single vertex and self-dual loop, which recovers unshaded unoriented TLJ(δ).

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