Realizing rigid C*-tensor categories as bimodules over C*-algebras

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Overview

Subfactors and their standard invariants

The **standard invariant** of a finite index type II_1 subfactor $N \subset M$, is the lattice of higher relative commutants. For *extremal, irreducible, discrete inclusions*, the standard invariant can be reinterpreted as the triple: **[CJoPe18]**

$$(\mathcal{C},\ \mathbb{F} : \mathcal{C} \hookrightarrow \mathsf{Bim}^{sp}_{bf}(N),\ \mathbf{A}).$$

- $\mathcal{C}$ is a RC*TC,
- $\mathbb{F}$ is a **fully faithful bi-involutive representation**, and
- $\mathbf{A}$ is a (*connected*) W*-*algebra object* in $\mathsf{Vec}(\mathcal{C})$.

From this characterization one can obtain *subfactor reconstruction* results, which we seek to extend to inclusions of C*-algebras.

Representations of RC*TC over II_1-factors

- From such a representation, one recovers the subfactor N by taking *bounded vectors*, and by means of *realizations/crossed products* one recovers the overfactor M.
- Guionnet, Jones, and Shlyakhtenko provided a digrammatic proof of Popa's reconstruction theorem. **[GJS]** The resulting factors are interpolated free group factors in finite depth, and for infinite depth, Hartglass showed that the factors correspond to $L(\mathbb{F}_\infty)$.
- One problem: There is no C*-algebra over which we can universally represent RC*TC's, and by K-theoretical obstructions, there cannot be one.

Representations over C*-algebras

Existence of representations:

Theorem (M. Hartglass, RHP)

*Given $\mathcal{C}$, a countably generated RC*TC, there is a separable, unital, simple C*-algebra, B_0, with unique trace and a fully faithful strong monoidal bi-involutive functor*

$$\mathbb{F} : \mathcal{C}^{op} \hookrightarrow Bim_{fgp}^{sp}(B_0)$$

valued on the finitely generated projective C-bimodules of B_0. Moreover, the K_0 group of B_0 is the free abelian group on the classes of simple objects in $\mathcal{C}$.*

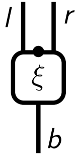
See also **[Yu19]** and **[NaEv19]**.

Construction of the C*-algebra B_0

Let x be a symmetrically self dual object in $\mathcal{C}$, and we consider the full tensor subcategory it generates by $\mathcal{C}_x$. Consider

$$Gr_\infty := \bigoplus_{l,r,b \geq 0} \mathcal{C}(x^{\otimes b} \to x^{\otimes l} \otimes x^{\otimes r})$$

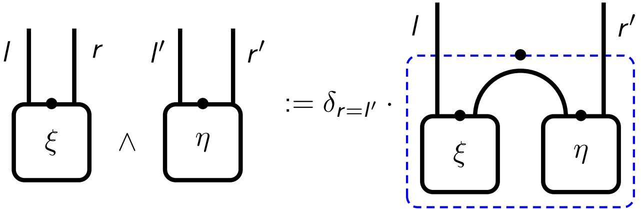
Whose elements can be pictorially visualized as follows: (we are reading from bottom to top)



(1)

Construction of the C*-algebra B_0

By considering diagrams with $b = 0$, we obtain a graded increasing sequence of f.d. C*-algebras, with the multiplication

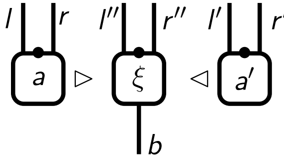


(2)

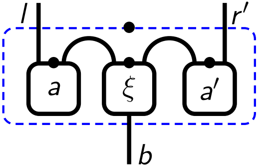
Notice this is the same C*-algebra from the Hom spaces in $\mathcal{C}$ with composition and conjugation as the involution. We denote the **inductive limit C*-algebra** by A_∞.

Construction of the C*-algebra B_0

We now endow Gr_∞ with the structure of a pre-**Hilbert** A_∞**-bimodule**:



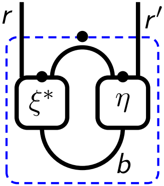
$:= \delta_{r=l''} \cdot \delta_{r''=l'}$



(3)

Where the **right** A_∞**-valued inner** product is given by

$$\langle \xi \mid \eta \rangle_{A_\infty} := \delta_{l=l'} \cdot \delta_{b=b'} \cdot$$



(4)

And we denote its completion by

$$_{A_\infty}(\chi_\infty)_{A_\infty}.$$

Full Fock space and creation operators

Identify χ_∞ with the **full Fock space**

$$\mathcal{F}(\chi_1) = A_\infty \oplus \bigoplus_{n\geq 0} \chi_1^{\otimes_{A_\infty} n} \cong \chi_\infty,$$

allowing nice (diagrammatic) description of the elements of χ_∞ as tensors with a single strand on the bottom.
For $\xi \in \mathcal{F}(\chi_1)$, consider **creation operators**, $L(\xi)$, defined as

$$L(\xi)(a) = \xi \lhd a \qquad \text{for all } a \in A_\infty$$
$$L(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n \qquad \text{for all } \xi_1, \ldots, \xi_n \in \chi_1.$$

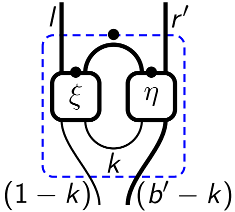
$L(\xi)$ is bounded and **right-adjointable**, and its adjoint is given by

$$L(\xi)^* a = 0 \qquad \text{for all } a \in A_\infty$$
$$L(\xi)^*(\xi_1 \otimes \cdots \otimes \xi_n) = \langle \xi \mid \xi_1 \rangle_{A_\infty} \cdot \xi_2 \otimes \cdots \otimes \xi_n \quad \text{for all } \xi_1, \ldots, \xi_n \in A_\infty$$

Creation operators on full Fock Space

For a self-adjoint diagram, $\xi = \xi^*$, the operator $L(\xi) + L(\xi)^*$ acts on diagrams as multiplication by ξ under the **Walker Product**, $\star$, given by:

$$\xi \star \eta := \delta_{r=l'} \cdot \sum_{k=0}^{1} \tag{5}$$



We now define the C*-algebra

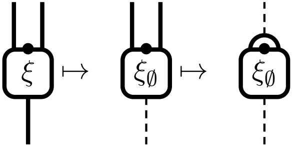
$$B_\infty := \overline{A_\infty \cup \{L(\xi) + L(\xi)^* \mid \xi = \xi^*\}} \subset \mathcal{B}^*((\chi_\infty)_{A_\infty})$$

as a C*-subalgebra of the right-adjointable operators on χ_∞.

Properties of B_∞

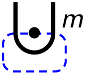
- The composition

$$\text{Tr} : B_\infty \xrightarrow{\mathbb{E}} A_\infty \xrightarrow{\Phi} \mathbb{C}$$



endows B_∞ with a **faithful semifinite tracial weight on** B_∞.The map $\mathbb{E}$ is a faithful conditional expectation onto A_∞, and Φ is a faithful, positive, semifinite, tracial weight on A_∞.

- B_∞ **is simple,** and so is every corner 

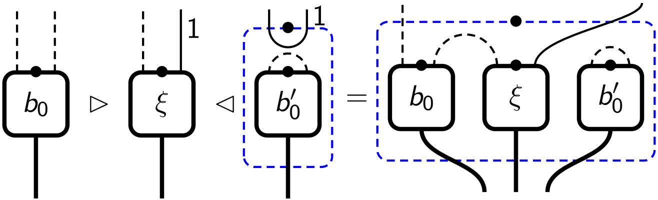
$\wedge B_\infty \wedge$ 

$:= {}_nB_m$, for which $n = m$.

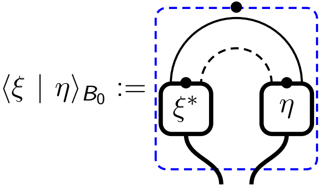
Thus, (B_n, tr_n) **is a tracial, separable, unital simple C*-algebra**, for every $n \geq 0$. We now pay special attention to B_0 and describe some bimodules.

A Hilbert B_0-bimodule

We have the following **commuting actions of B_0 on ${}_0B_1$:**



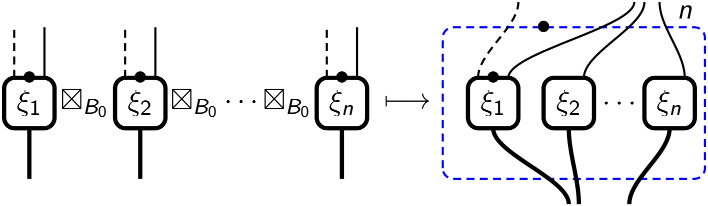
Together with a right B_0-valued inner product:



Bim(B_0): Monoidal structure: Connes' Fusion

Using ${}_0B_1$ to generate the category to support our representation:
There is a unitary B_0-isomorphism from the n-fold **Connes' fusion**

$$({}_0B_1)^{\boxtimes_{B_0} n} \longrightarrow {}_{B_0}({}_0B_n)_{B_0},$$



This isomorphism exists by the simplicity of B_∞, and we will use it to derive the properties of our bimodules.

$\mathrm{Bim}_{fgp}^{sp}(\mathrm{B}_0)$: Rigidity, shpericallity and (bi)finiteness

A few important properties for ${}_0B_n$:

- Existence of **finite left and right** B_0**-bases**: There are collections $\{u_i\}_{i=1}^N$ and $\{v_j\}_{j=1}^M$, so that for each $\xi \in {}_0B_n$,

$$\sum_{j=1}^{M} {}_{B_0}\langle \xi \mid v_j \rangle \triangleright v_j = \xi = \sum_{i=1}^{N} u_i \triangleleft \langle u_i \mid \xi \rangle_{B_0}.$$

- **Normalized and minimal** (these properties are analogous to *sphericality* and *bifiniteness* for bimodules over a II_1-factor.)

We are thus directed to represent $\mathcal{C}$ over the RC*TC of **finitely generated projective** Hilbert B_0-bimodules which are **normalized and minimal** (*a la* **[KaWa00]**).

The representation

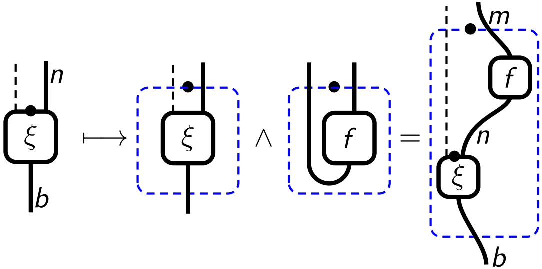
We have a **full**(!) and faithful strong monoidal bi-involutive functor

$$\mathbb{F} : \mathcal{C}_x^{op} \hookrightarrow \mathrm{Bim}_{bf}^{sp}(B_0),$$

$$x^{\otimes n} \mapsto {}_0B_n,$$

$$\mathcal{C}^{op}(x^{\otimes n} \to x^{\otimes m}) \ni f \mapsto \mathbb{F}(f) \in \mathrm{Bim}_{fgp}^{sp}({}_0B_n \to {}_0B_m)$$

whose action on diagrams is given by



The fact that this functor is full is obtained from analogous results regarding W*-categories. **[BrHaPe]**

Applications: Hilbertifying?

Looking for a monoidal functor $- \boxtimes_{B_0} \mathsf{L}^2(B_0)$ transforming C*-bimodules into honest Hilbert spaces so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}^{op} & \xrightarrow{\ \mathbb{F}\ } & \mathrm{Bim}^{sp}_{fgp}(B_0) \\ & \underset{BrHaPe}{\searrow} & \Big\downarrow {\scriptstyle -\boxtimes_{B_0} L^2(B_0)} \\ & & \mathrm{Bim}^{sp}_{bf}(M_0) \end{array}$$

This would recover known results for representations of RC*TC over W*-algebras **[BrHaPe]**.

Applications: (discrete) inclusion reconstruction?

- Using reconstruction techniques on simple C*-algebras towards defining **discrete inclusions for C*-algebras**.(?)
- **Galois correspondences:** Given a discrete countable group Γ acting on B by outer automorphisms, what are all the intermediate C*-algebras

$$B \subset P \subset B \rtimes_r \Gamma?$$

Is every such P of the form $P \cong B \rtimes_r \Lambda$ for some $\Lambda \leq \Gamma$? **[CaSm17]**
Is this correspondence encoded by categorical data? (i.e. *connected* C*-*algebra objects* in $\mathrm{Vec}(\mathcal{C})$. **[CJoPe18]**)

Thanks!

Thanks for listening!

References

- **[BrHaPe]** Brothier, Hartglass and Penneys, *Rigid C^*-tensor categories of bimodules over interpolated free group factors*, arXiv:1208.5505
- **[CaSm17]** Cameron, Smith, *A Galois correspondence for reduced crossed products of unital simple C*-algebras by discrete groups*, 1706.01803
- **[GJS]** A. Guionnet, V.F.R. Jones and D. Shlyakhtenko, *Random matrices, free probability, planar algebras and subfactors*, arXiv: 0712.2904.
- **[HaPe I]** Hartglass and Penneys, *C^*-Algebras from Planar Algebras I: Canonical C^*-algebras Associated to a Planar Algebra*, arXiv: 1401.2485
- **[HaPe II]** Hartglass and Penneys, *C^*-Algebras from Planar Algebras II: The Guionnet-Jones-Shlyakhtenko C^*-algebras*, arXiv:1401.2486.
- **[CJoPe18]** Corey Jones, Penneys, *Realizations of algebra objects and discrete subfactors* arXiv: 1704.02035

References

- **[NaEv19]** NÆs Aaserud and Evans, *Realizing the braided Temperley-Lieb-Jones C*-tensor categories as Hilbert C*-bimodules*, `arXiv: 1908.02674`
- **[JSW]** V. Jones, D. Shlyakhtenko and K. Walker *An orthogonal approach to the subfactor of a planar algebra* `arXiv: 0807.4146`.
- **[KaWa00]** Kajiwara and Watatani, *Jones Index Theory by Hilbert* C^**-bimodules and K-Theory*, Transactions of the AMS, **352**, number 8, Pages 3429-3472, S 0002-9947(00)02392-8. `http://www.numdam.org/item/CTGDC_1988__29_1_9_0/`.
- **[Yu19]** Yuan, *Rigid C*-tensor categories and their realizations as Hilbert* C^**-bimodules*, Proceedings of the Edinburgh Mathematical Society (2019) **62**, 367-393m, `DOI:10.1017/S0013091518000524`.