Realizations of Rigid C*-Tensor Categories as bimodules over GJS C*-algebras

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Overview

Subfactors and their standard invariants

The **standard invariant** of a finite index type II_1 subfactor $N \subset M$, is the lattice of higher relative commutants. We can view it as the RC*TC generated by the Hilbert bimodule ${}_N\mathrm{L}^2(M)_N$ and whose morphisms are bounded N-N bilinear maps. For *extremal, irreducible, discrete inclusions*, the standard invariant can be reinterpreted as the triple:

$$(\mathsf{C},\ \mathbb{F} : \mathsf{C} \hookrightarrow \mathrm{Bim}^{\mathrm{sp}}_{\mathrm{bf}}(N),\ A),$$

where

- C is a RC*TC,
- $\mathbb{F}$ is an embedding into *spherical bi-finite* N-bimodules, and
- is a *unitary Frobenius algebra* generating C. **[JP18]**

Note: This formalism allows M to be type III.

Representations over II_1-factors

♣ This setting allows for subfactor reconstruction: one recovers N by taking *bounded vectors* in $\mathbb{F}(A)$, and by means of *realizations/crossed products* $N \rtimes_{\mathbb{F}} A$, one recovers M.

♣♣ **[GJS10]** provided a diagrammatic proof of Popa's Reconstruction Theorem, obtaining interpolated free group factors in finite depth. For infinite depth Hartglass showed the factors correspond to $\mathrm{L}(\mathbb{F}_\infty)$.

♣♣♣ **One problem:** by K-theoretical considerations, there is no separable C*-algebra over which we can universally represent every RC*TC.

Representations over C*-algebras

Existence of representations:

Theorem (HaHe19)

*For every countably generated RC*TC C, there is a separable unital simple exact C*-algebra B_0 with unique trace, and a full and faithful bi-involutive strong monoidal functor*

$$\mathbb{F} : \mathsf{C} \hookrightarrow Bim^{\mathrm{tr}}_{\mathrm{fgp}}(B_0)$$

valued on the finitely generated projective C-bimodules of B_0. Moreover, the K_0 group of B_0 is the free abelian group on the classes of simple objects in C.*

See also **[Yua19]** and **[NE19]**.

Tensor categories

A **tensor category** is a tuple $(\mathsf{C}, \otimes, 1, \alpha, \lambda, \rho)$ consisting of:

- a semisimple category C enriched over complex vector spaces; i.e., for objects $a, b \in \mathsf{C}$, the morphisms $\mathsf{C}(a \to b)$ form a complex vector space,
- a *tensor product* bilinear functor $- \otimes - : \mathsf{C} \otimes \mathsf{C} \longrightarrow \mathsf{C}$, together with the *tensor unit* 1, which we assume to be simple; i.e., $\mathsf{C}(1) = \mathrm{End}(1) \cong \mathbb{C}$,
- an *associator* α, and *left and right unitors* λ and ρ, satisfying *triangle and pentagon axioms* which we will omit.

For all our categories, we assume the isomorphism classes of objects form a set.
Useful references are **[EGNO]**, **[Rie14]** and **[Mac71]**.

Dagger categories

We say that C is a **dagger category** iff for each $a, b \in \mathsf{C}$ there is an anti-linear map

$$\dagger : \mathsf{C}(a \to b) \longrightarrow \mathsf{C}(b \to a)$$

with the following properties:

- The map $\dagger$ is an involution; i.e. for each $a, b \in \mathsf{C}$ and every $f \in \mathsf{C}(a \to b)$ we have $(f^\dagger)^\dagger = f$.
- For composable morphisms f and g in C, we have $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$.
- Moreover, we have the identity $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$.

In the graphical calculus of C, this corresponds to reflection with respect to a horizontal axis.

C*-categories

A dagger category is a **C*-category** if and only if for each $a, b \in \mathrm{C}$:

- For every morphism $f \in \mathrm{C}(a \to b)$ there exists an endomorphism $g \in \mathrm{C}(a)$ such that $f^\dagger \circ f = g^\dagger \circ g$.
- The map $||\cdot|| : \mathrm{C}(a \to b) \longrightarrow [0, \infty]$ defined by

$$||g||^2 := \sup\{|\lambda| \geq 0 \mid (g^\dagger \circ g - \lambda \cdot \mathrm{id}_a) \notin \mathrm{GL}(\mathrm{C}(a))\}$$

  defines a C*-norm.

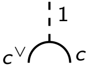
Note: being C* is a property of a dagger category and not extra structure.

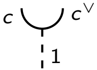
Rigid categories I

We say C is a **rigid category** if each object $c \in$ C has a *dual object*, $c^\vee \in$ C together with *evaluation and coevaluation maps*

$$\mathrm{ev}_c : c^\vee \otimes c \longrightarrow 1 \text{ and } \mathrm{coev}_c : 1 \longrightarrow c \otimes c^\vee.$$

And in *the graphical calculus*:

$\mathrm{ev}_c =$ 

and $\mathrm{coev}_c =$ 

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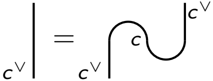
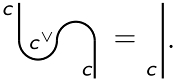
These maps satisfy *the Zig-Zag equations* **[LR97]**:



and


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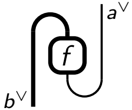
We moreover require that there exists a *predual object* to c, denoted by $c_\vee$, such that $(c_\vee)^\vee \cong c$.

Rigid categories II: dual functors

The *dual of a map* $f \in \mathsf{C}(a \to b)$ is computed graphically as:

$$f^\vee \;=$$



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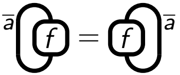
where now $f^\vee \in \mathsf{C}(b^\vee \to a^\vee)$. These choices of dual objects can be arranged into a strong-monoidal **dual functor**

$$(\bullet)^\vee : (\mathsf{C}, \circ, \otimes) \longrightarrow (\mathsf{C}, \circ_{\mathrm{op}}, \otimes_{\mathrm{op}})$$
$$c \mapsto c^\vee \text{ and } f \mapsto f^\vee.$$

♢ In a rigid category every morphism space is finite dimensional.

RC*TC: unitary dual and bi-involutive structures

In any RC*TC there is a **canonical unitary dual functor** satisfying the Zig-Zag equations and *the balancing condition*: on arbitrary endomorphisms its *left and right traces* match; i.e

$\mathrm{ev}_a \circ (\mathrm{id}_{\overline{a}} \otimes f) \circ \mathrm{ev}_a{}^\dagger =$  $= \mathrm{coev}_a{}^\dagger \circ (f \otimes \mathrm{id}_{\overline{a}}) \circ \mathrm{coev}_a \in \mathbb{C}.$

For each morphism f we obtain $f^{\dagger\vee} = f^{\vee\dagger}$. This choice of dual functor is unique up to a unique natural isomorphism.
There is a canonical **bi-involutive structure** on a rigid dagger category, given by

$$\overline{\cdot} : (\mathsf{C}, \circ, \otimes) \longrightarrow (\mathsf{C}, \circ, \otimes_{\mathrm{op}})$$
$$c \mapsto \overline{c} := c^\vee \text{ and } f \mapsto \overline{f} = (f^\dagger)^\vee,$$

which graphically can be viewed as reflection w.r.t. a vertical axis.
[HP17b], [Pen18], [Yam14]

$\mathrm{Bim}^{\mathrm{sp}}_{\mathrm{bf}}(N)$: bi-finite spherical bimodules over a II_1-factor

The tensor product is the *Connes fusion*, denoted $-\boxtimes_N -$. For $H \in \mathrm{Bim}^{\mathrm{sp}}_{\mathrm{bf}}(N)$ there is an *N-valued inner product* $\langle \eta \mid \xi \rangle_N := L_\eta^* L_\xi$. Its **unitary dual** is $\overline{H}$, the *conjugate Hilbert space* with N-actions:

$$n \triangleright \overline{\xi} \triangleleft m := \overline{m^* \triangleright \xi \triangleleft n^*},$$

and for $\overline{\xi}, \overline{\eta} \in \overline{H}$ their inner product is $\langle \overline{\xi} \mid \overline{\eta} \rangle_N := \langle \eta \mid \xi \rangle_N$. H has a *finite (right) N-basis* $\{\beta\}$, which induces the evaluation and coevaluation maps (given on *bounded vectors* by):

$$\mathrm{ev}_H : \overline{H} \underset{N}{\boxtimes} H \to \mathrm{L}^2(N) \qquad\qquad \mathrm{coev}_H : \mathrm{L}^2(N) \longrightarrow H \underset{N}{\boxtimes} \overline{H}$$

$$\overline{\eta} \boxtimes \xi \mapsto \langle \eta \mid \xi \rangle_N \qquad\qquad n\Omega \mapsto \sum (n \triangleright \beta) \boxtimes \overline{\beta},$$

which is independent on the choice of basis. [Bis97],[Pen18]

Hilbert C*-bimodules

A C-D **Hilbert C*-bimodule** $\mathcal{Y}$ is a vector space with commuting left C and right D-actions, $- \triangleright -$ and $- \triangleleft -$, endowed with:

- a C-valued, positive-definite form ${}_C\langle -, \cdot \rangle$ which is C-linear on the left and conjugate linear on the right,
- a D-valued, positive-definite form $\langle \cdot | - \rangle_D$ which is D-linear on the right and conjugate linear on the left, and
- Compatibility requirement: ${}_C\langle y_1, y_2 \rangle = ({}_C\langle y_2, y_1 \rangle)^*$, $\langle y_1 \mid y_2 \rangle_D = (\langle y_2 \mid y_1 \rangle_D)^*$, ${}_C\langle y_1 \triangleleft d, y_2 \rangle = {}_C\langle y_1, y_2 \triangleleft d^* \rangle$, and $\langle c \triangleright y_1 \mid y_2 \rangle_D = \langle y_1 \mid c^* \triangleright y_2 \rangle_D$, for $c \in C$ and $d \in D$.

These forms induce complete and equivalent norms. We only consider right or *bi-adjointable* C-D morphisms when studying C*-bimodules.

$\mathrm{Bim}_{\mathrm{fgp}}(B)$: fgp bimodules over a unital simple C*-algebra:

$\mathrm{Bim}_{\mathrm{fgp}}(B)$ is a RC*TC whose objects are **finitely generated projective** Hilbert B-bimodules $\mathcal{Y}$. This is:

there are finite *left and right B-basis* $\{v_j\}$ and $\{u_i\} \subset \mathcal{Y}$; i.e.,

$$\sum_j {}_B\langle y, v_j\rangle \triangleright v_j = y = \sum_i u_i \triangleleft \langle u_i |\, y\rangle_B .$$

The **tensor product** $- \boxtimes_B -$ is fusion relative to B, and existence of finite basis allows to construct a **dual functor** as was done for $\mathrm{Bim}^{\mathrm{sp}}_{\mathrm{bf}}(B)$. It is possible to impose two algebraic conditions:

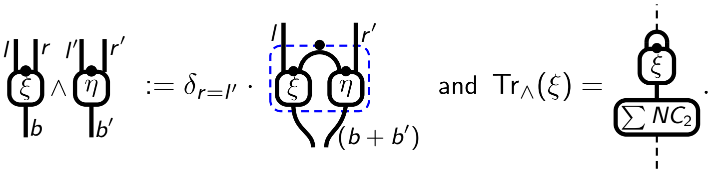
minimality: for any bi-adjointable $T \in \mathrm{End}({}_B\mathcal{Y}_B)$, $\sum_i {}_B\langle Tu_i, u_i\rangle = \sum_j \langle v_j |\, Tv_j\rangle_B$, and
to be **normalized**: $\sum_i {}_B\langle u_i, u_i\rangle = \sum_j \langle v_j | v_j\rangle_B$. The square of these common values is *the index*, denoted $\mathrm{Ind}(\mathcal{Y})$.

The graded tracial $*$-algebra Gr_∞

Fix a symmetrically self-dual object $x \in \mathcal{C}$ with dimension $\delta_x > 1$. (So there is no need to orient strings labeled by x.) We construct a graded tracial $*$-algebra Gr_∞ from the morphisms in $\mathcal{C}$:

$$\mathrm{Gr}_\infty := \bigoplus_{b,l,r \geq 0} \mathcal{C}(x^{\otimes b} \to x^{\otimes l} \otimes x^{\otimes r}),$$

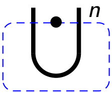
whose elements and operations can be visualized as follows:



The bi-involutive structure induces the involution: $(\xi^\vee)^\dagger = \xi^* = (\xi^\dagger)^\vee$.

The ambient C*-algebra B_∞ and its corners B_n.

♠ Define **the ambient C*-algebra** B_∞ as the completion of Gr_∞ in the GNS representation $\mathrm{L}^2(\mathrm{Gr}_\infty, \mathrm{Tr}_\wedge)$. [JSW10]
♡ The trace on Gr_∞ extends to a faithful, semifinite tracial weight on B_∞, denoted by Tr . [HP17a]

For each $n \geq 0$ let $p_n :=$  $\in \mathrm{Gr}_\infty$. Notice that p_0 corresponds to an empty diagram.
♣ Define **the corners of** B_∞ as ${}_nB_m := p_n \wedge B_\infty \wedge p_m$, and notice that $B_n := {}_nB_n$ are unital C*-algebras. The corner B_0 containing all diagrams with no strings on the top will be of special interest to us.
♢ B_∞ is a simple C*-algebra and thus, so is each B_n.

Properties of B_∞ and its corners

For each $n \geq 0$ we have the following:

- The tracial weight Tr descends to a positive, faithful linear tracial functional $\mathrm{Tr}_n(\bullet) = \mathrm{Tr}(p_n \bullet p_n)$ on the unital C*-algebra B_n. This is in fact the **unique trace** on B_n.
- The von Neumann algebra defined by

$$M_n := B_n'' \subseteq \mathcal{B}(\mathrm{L}^2(B_n), \mathrm{Tr}_n)$$

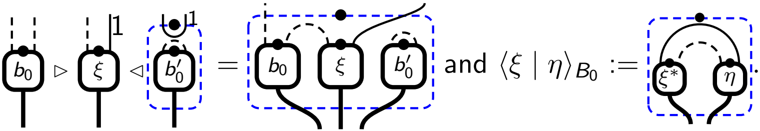
is an interpolated free group factor $L(\mathbb{F}_t)$, for $t \in (1, \infty]$.

- For Irr(C) any fixed set of isomorphism classes of simple objects in C, we have $K_0(B_0) = \mathbb{Z}[\mathrm{Irr}(\mathrm{C})]$.
- The inclusion $B_0 \subset B_n$ has finite Watatani Index δ_x^{2n}.

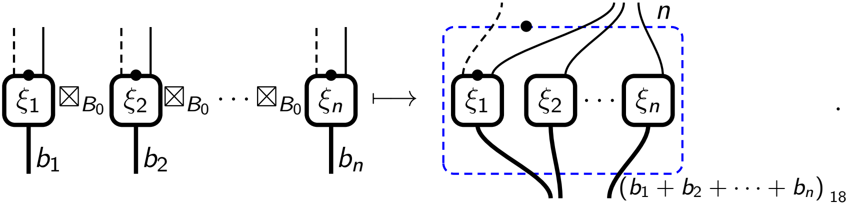
(**[HP14]**, **[GJS10]**, **[Har13]**, **[BHP12]**.)

The Hilbert B_0-bimodule ${}_0B_1$

The left and right B_0-actions together with the right B_0-valued inner product:



Fusion behaves as follows: ${}_{B_0}({}_0B_1)_{B_0}^{\boxtimes_{B_0} n} \cong {}_{B_0}({}_0B_n)_{B_0}$ via

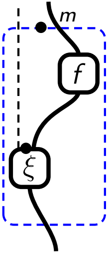


Representing the category C as Hilbert C*-bimodules

Consider the correspondence

$$\mathbb{F} : \mathrm{C} \hookrightarrow \mathrm{Bim}_{\mathrm{fgp}}(B_0)$$
$$x^{\otimes n} \mapsto {}_0B_n$$

$\mathrm{C}(x^{\otimes n} \to x^{\otimes m}) \ni f \longmapsto \mathbb{F}(f) :$ 

$\longmapsto$ 

$\in \mathrm{Bim}({}_0B_n \to {}_0B_m).$

This functor is easily seen to be faithful bi-involutive and strong monoidal. Proving it is also full requires deeper insights.

Constructing Hilbert spaces and extending actions

Consider the following functor which turns Hilbert C*-bimodules into Hilbert spaces:

$$
{}_{M_0}(- \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0))_{M_0} : \mathrm{Bim}^{\mathrm{tr}}_{\mathrm{fgp}}(B_0) \longrightarrow \mathrm{Bim}^{\mathrm{sp}}_{\mathrm{bf}}(M_0)
$$

$$
\mathcal{Y} \longmapsto {}_{M_0}(\mathcal{Y} \underset{B_0}{\boxtimes} \mathrm{L}^2(B_0))_{M_0}
$$

$$
\mathrm{Bim}^{\mathrm{tr}}_{\mathrm{fgp}}(B_0)(\mathcal{Y} \to \mathcal{Z}) \ni f \longmapsto f \boxtimes \mathrm{id}_{\mathrm{L}^2(B_0)} .
$$

Here, $\mathrm{Bim}^{\mathrm{tr}}_{\mathrm{fgp}}(B_0) \subset \mathrm{Bim}_{\mathrm{fgp}}(B_0)$ is the RC*TC consisting of those bimodules $\mathcal{Y}$ which are **compatible with the trace** **[KW00]** :

$$
\mathrm{tr}_{B_0}\left(\langle \eta \mid \xi \rangle_{B_0}\right) = \mathrm{tr}_{B_0}\left({}_{B_0}\langle \xi, \eta \rangle\right), \text{on } \mathcal{Y} \times \mathcal{Y}.
$$

A comment on Hilbertification

Defining this functor requires one to extend the B_0-actions to normal M_0-actions. The distinctive elements that make this extension work are **the existence of a a positive trace** and a **faithful, trace-preserving conditional expectation**, given by:

$$\mathrm{Tr} : \mathcal{B}^*(\mathcal{Y}) \longrightarrow \mathbb{C} \qquad \text{and} \qquad \mathcal{E} : \mathcal{B}^*(\mathcal{Y}) \longrightarrow B_0,$$

$$\xi\rangle\langle\eta| \ \mapsto \frac{1}{\mathrm{Ind}(\mathcal{Y})} \cdot \mathrm{Tr}_{B_0}[\ \langle\eta \mid \xi\rangle_{B_0}] \qquad |\xi\rangle\langle\eta| \ \mapsto \frac{1}{\sqrt{\mathrm{Ind}(\mathcal{Y})}} \cdot {}_{B_0}\langle\xi,\eta\rangle.$$

Implying that the correspondence $\mathcal{Y} \longmapsto \ {}_{M_0}(\mathcal{Y} \boxtimes_{B_0} \mathrm{L}^2(B_0))_{M_0}$ defines a faithful bi-involutive strong monoidal functor.

A commutative 2-cell

There is a strong monoidal unitary natural isomorphism filling the following 2-cell: (*the following diagram commutes*)

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{\mathbb{F}} & \mathsf{Bim}^{\mathrm{tr}}_{\mathrm{fgp}}(B_0) \\ & \searrow^{\mathbb{G}} & \Big\downarrow {\scriptstyle {}_{M_0}(- \boxtimes_{B_0} \mathrm{L}^2(B_0))_{M_0}} \\ & & \mathsf{Bim}^{\mathrm{sp}}_{\mathrm{bf}}(M_0) \end{array}$$

Since the functor $\mathbb{G}$ constructed in **[BHP12]** is full and faithful, it then follows that $\mathbb{F}$ **is full**.

A summary

Theorem (HaHe19)

*Given a countably generated RC*TC* C*, there exists a unital simple separable exact C*-algebra* B_0 *with unique trace and a fully-faithful bi-involutive strong monoidal functor*

$$\mathbb{F} : \mathsf{C} \hookrightarrow \mathsf{Bim}^{\mathrm{tr}}_{\mathrm{fgp}}(B_0).$$

Despite K-theoretical obstructions, there exists a separable C*-algebra B whose category of bimodules contains an image of every fusion category.

Corollary

There exists a unital simple exact separable C-algebra* B *with unique trace over which we can full and faithfully realize every unitary fusion category.*

Some future questions

♠ Using reconstruction techniques on simple C*-algebras towards defining **discrete inclusions for C*-algebras**.(?)

♠♠ **Galois correspondences:** Given a discrete countable group Γ acting on B by outer automorphisms, what are all the intermediate C*-algebras

$$B \subset P \subset B \rtimes_r \Gamma?$$

Is every such P of the form $P \cong B \rtimes_r \Lambda$ for some $\Lambda \leq \Gamma$? **[CS17]**

♠♠♠ Is this correspondence encoded by categorical data? (i.e. *connected* C*-*algebra objects* in Vec(C).

♠♠♠♠...

Thank you!

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