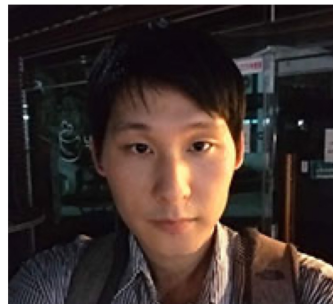


The TASEP on a ring, Schubert polynomials, & evil-avoiding permutations

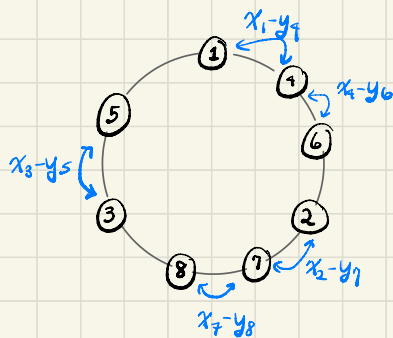
Lauren Williams

joint work w/ Donghyun Kim



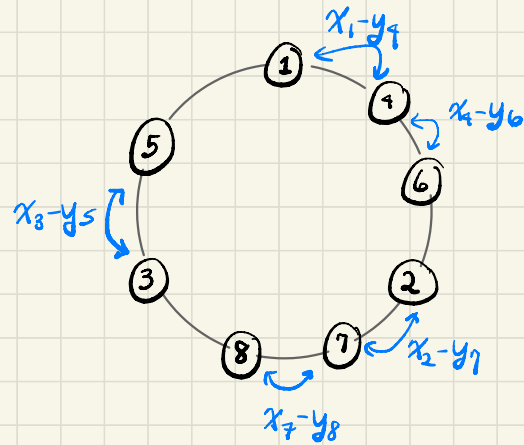
arXiv: 2102.00560 (extended abstract)

Rk: To hear about the "Hypersimplex, amplituhedron, & Eulerian numbers"
see poster of Matteo Parisi



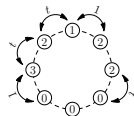
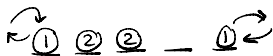
Outline

- Exclusion process on a ring
- Stationary dist & Schubert polynomials
- Results for evil-avoiding permutations

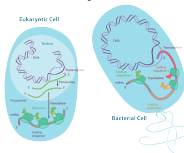
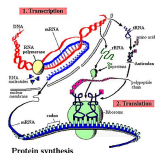


I. The asymmetric simple exclusion process (ASEP)

- Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
- Particles hop on a 1D lattice; at most one particle per site. Particles may have different *weights*, which affect their hopping rate.



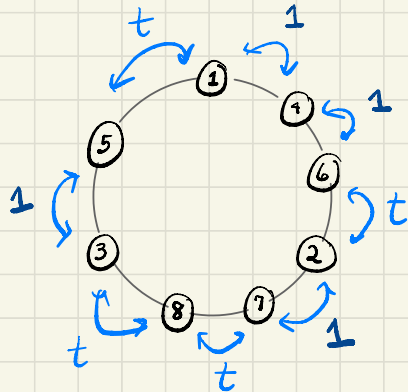
- Lattice could be a line with open boundaries or a *ring*...
- Cited as a model for traffic flow and for translation in protein synthesis



- Over 1000 papers on the exclusion process on the arXiv: Liggett, Derrida, Evans, Hakim, Pasquier, Spohn, Sasamoto, Yau, Borodin, Corwin, Ferrari, Seppalainen, Tracy-Widom, ...

Asymmetric exclusion process on a ring: two versions

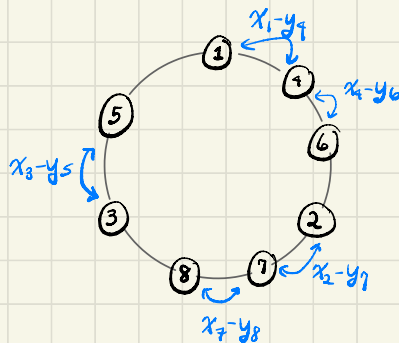
"usual" multispecies ASEP



Connected to Macdonald polynomials

(Cantini-de Gier-Wheeler,
Corteel-Mandelstam-W.)

inhomogeneous, totally asymmetric (TASEP)



connected to Schubert polynomials

(Lam-W., Cantini,
Kim-W.)

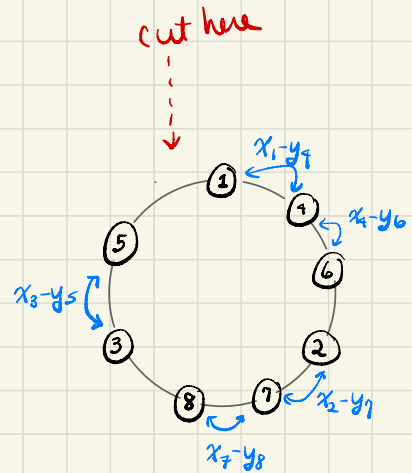
Inhomogeneous TASEP on ring

Fix lattice of n sites on a ring,
filled w/ particles of weights $1, 2, \dots, n$.

Markov chain:

$n!$ states (by cutting open circle, identify states w/ permutation)

Identify state above w/ $(1, 4, 6, 2, 7, 8, 3, 5)$



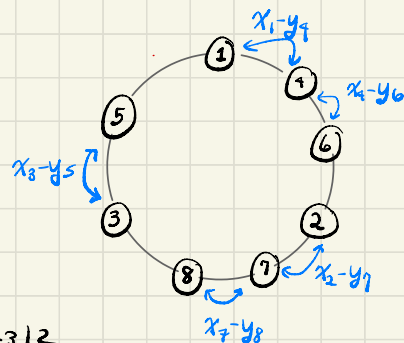
Transitions:

If $i < j$, $\text{prob} \left(\begin{array}{c} \text{state with } i \text{ above } j \end{array} \rightarrow \begin{array}{c} \text{state with } i \text{ below } j \end{array} \right) = x_i - y_j.$

If $i > j$, $\text{prob} (\quad) = 0.$

What are the steady state probabilities?

- For $w \in S_n$, let Ψ_w be (unnormalized) steady state prob of being in state w .
- Circular symmetry in model $\Rightarrow$ e.g. $\Psi_{3124} = \Psi_{1243} = \Psi_{2431} = \Psi_{4312}$
- So can restrict attention to $w \in S_n$ with $w_1 = 1$.



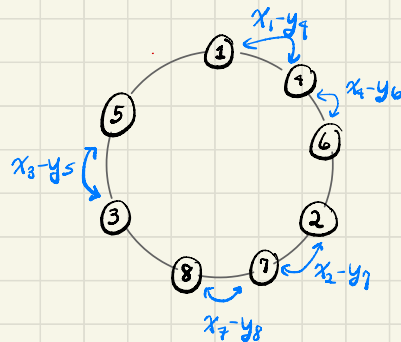
State w	Probability ψ_w (we set $y_i = 0$ for all i)
1234	$x_1^3 x_2$
1324	$x_1 \mathfrak{S}_{1432}$
1342	$x_1 x_2 \mathfrak{S}_{1423}$
1423	$x_1^2 x_2 \mathfrak{S}_{1243}$
1243	$x_1^2 \mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423} \mathfrak{S}_{1342}$

- Here $\mathfrak{S}_w$ is the **Schubert polynomial** associated to permutation w .

- For $w \in S_n$, let ψ_w be (unnormalized) steady state prob of being in state w .

• Circular symmetry in model $\Rightarrow$ e.g. $\psi_{3124} = \psi_{1243} = \psi_{2431} = \psi_{4312}$

- So can restrict attention to $w \in S_n$ with $w_1 = 1$.



Ex: $n = 4$

State w	Probability ψ_w (we set $y_i = 0$ for all i)
1234	$x_1^3 x_2$
1324	$x_1 \mathfrak{S}_{1432}$
1342	$x_1 x_2 \mathfrak{S}_{1423}$
1423	$x_1^2 x_2 \mathfrak{S}_{1243}$
1243	$x_1^2 \mathfrak{S}_{1342}$
1432	$\mathfrak{S}_{1423} \mathfrak{S}_{1342}$

- Here $\mathfrak{S}_w$ is the **Schubert polynomial** associated to permutation w .

(Double) Schubert polynomials

- For $1 \leq i < n$, the *divided difference operator* ∂_i acts on polynomials $P(x_1, \dots, x_n)$ as follows:

$$(\partial_i P)(x_1, \dots, x_n) = \frac{P(\dots, x_i, x_{i+1}, \dots) - P(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

- If $s_{i_1} \dots s_{i_m}$ is a reduced expression for a permutation $w \in S_n$, then $\partial_{i_1} \dots \partial_{i_m}$ depends only on w ; we denote this operator by ∂_w .
- Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two sets of variables;

$$\text{Let } \Delta(x, y) = \prod_{i+j \leq n} (x_i - y_j).$$

- To each $w \in S_n$ we associate the *double Schubert polynomial* (Lascoux-Schutzenberger)

$$\mathfrak{S}_w(x, y) = \partial_{w^{-1}w_0} \Delta(x, y),$$

where the *divided difference operator* acts on the x -variables.

(Double) Schubert polynomials, cont.

- Ordinary Schubert polynomials $\mathfrak{S}_w(x)$ obtained from $\mathfrak{S}_w(x, y)$ when all $y_i = 0$.
- When $w \in S_n$ is a Grassmannian permutation, $\mathfrak{S}_w(x)$ is a Schur polynomial. In particular, both Macdonald polynomials and Schubert polynomials generalize Schur polynomials.
- Geometric significance: Schubert polynomials represent cohomology classes of Schubert varieties in the cohomology ring of the complete flag variety $\mathcal{F}\ell_n$.
- Double Schubert polynomials represent Schubert classes in equivariant cohomology for the Borel group action on $\mathcal{F}\ell_n$.
- Many combinatorial formulas for Schubert polynomials, starting with Billey-Jokusch-Stanley '93, Fomin-Kirillov '96, Kohnert, etc.

Steady State Probabilities when $n=5$

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)} \odot 13452$
12435	$x^{(4,1,0)} \odot 14532$
12453	$x^{(4,1,1)} \odot 14523$
12534	$x^{(5,2,1)} \odot 12453$
12543	$x^{(3,0,0)} \odot 14523 \odot 13452$
13245	$x^{(3,1,1)} \odot 15423$
13254	$x^{(2,0,0)} \odot 15423 \odot 13452$
13425	$x^{(3,2,1)} \odot 15243$
13452	$x^{(3,3,1)} \odot 15234$
13524	$x^{(2,1,0)} (\odot 164325 + \odot 25431)$
13542	$x^{(2,2,0)} \odot 15234 \odot 13452$
14235	$x^{(4,2,0)} \odot 13542$
14253	$x^{(4,2,1)} \odot 12543$
14325	$x^{(1,0,0)} (\odot 1753246 + \odot 265314 + \odot 2743156 + \odot 356214 + \odot 364215 + \odot 365124)$
14352	$x^{(1,1,0)} \odot 15234 \odot 14532$
14523	$x^{(4,3,1)} \odot 12534$
14532	$x^{(1,1,1)} \odot 15234 \odot 14523$
15234	$x^{(5,3,1)} \odot 12354$
15243	$x^{(3,1,0)} (\odot 146325 + \odot 24531)$
15324	$x^{(2,1,1)} (\odot 15432 + \odot 164235)$
15342	$x^{(2,2,1)} \odot 15234 \odot 12453$
15423	$x^{(3,2,0)} \odot 12534 \odot 13452$
15432	$\odot 15234 \odot 14523 \odot 13452$

$$x^{(5,2,0)} = x_1^5 x_2^2 x_3^0$$

Most ψ_w here are
product of monomial
& some Schubert poly's

Probabilities seem to have nice expressions in terms of Schubert polynomials. Often they are products of Schubert polynomials!

Pattern Avoidance

- We say a permutation $w = (w_1, \dots, w_n)$ *avoids the pattern 2413* if there are not four positions $1 \leq i < j < k < l \leq n$ such that w_i, w_j, w_k, w_l are in relative order 2413.
- Example: $w = (3, 4, 6, 1, 2, 5)$ contains the pattern 2413 while $w = (1, 2, 6, 4, 3, 5)$ avoids the pattern 2413.

Def: Say permutation w is **evil-avoiding** if it avoids patterns $\begin{matrix} 2 & 4 & 1 & 3 \\ \parallel & \parallel & \parallel & \parallel \\ \text{e} & \text{v} & \text{i} & \text{l} \end{matrix}$ and also patterns coming from

these anagrams of evil: **vile** = 4132

veil = 4213

leiv = 3214

Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP. WLOG $w_1 = 1$. Then the (unnormalized) probability Ψ_w is proportional to product of k (double) Schubert poly's, where $k = \# \text{ descents of } w^{-1}$.

evil = 2413

vile = 4132

veil = 1213

leiv = 3214

pattern
2413

3214

4132

4213

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)} \mathfrak{S}_{13452}$
12435	$x^{(4,1,0)} \mathfrak{S}_{14532}$
12453	$x^{(4,1,1)} \mathfrak{S}_{14523}$
12534	$x^{(5,2,1)} \mathfrak{S}_{12453}$
12543	$x^{(3,0,0)} \mathfrak{S}_{14523} \mathfrak{S}_{13452}$
13245	$x^{(3,1,1)} \mathfrak{S}_{15423}$
13254	$x^{(2,0,0)} \mathfrak{S}_{15423} \mathfrak{S}_{13452}$
13425	$x^{(3,2,1)} \mathfrak{S}_{15243}$
13452	$x^{(3,3,1)} \mathfrak{S}_{15234}$
13524	$x^{(2,1,0)} (\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)} \mathfrak{S}_{15234} \mathfrak{S}_{13452}$
14235	$x^{(4,2,0)} \mathfrak{S}_{13542}$
14253	$x^{(4,2,1)} \mathfrak{S}_{12543}$
14325	$x^{(1,0,0)} (\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)} \mathfrak{S}_{15234} \mathfrak{S}_{14532}$
14523	$x^{(4,3,1)} \mathfrak{S}_{12534}$
14532	$x^{(1,1,1)} \mathfrak{S}_{15234} \mathfrak{S}_{14523}$
15234	$x^{(5,3,1)} \mathfrak{S}_{12354}$
15243	$x^{(3,1,0)} (\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)} (\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)} \mathfrak{S}_{15234} \mathfrak{S}_{12453}$
15423	$x^{(3,2,0)} \mathfrak{S}_{12534} \mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234} \mathfrak{S}_{14523} \mathfrak{S}_{13452}$

← inverse has 3 descents

Combinatorics of evil-avoiding permutations

- Let $e(n) = \#$ evil-avoiding perms in S_n .

$$\text{Then } e(n) = \frac{(2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1}}{2}$$

(Sloane Encyclopedia A006012)

$$\text{evil} = 2413$$

$$\text{vile} = 4132$$

$$\text{veil} = 4213$$

$$\text{leiv} = 3214$$

- Apparently $e(n)$ also counts

- rectangular perms on $[n]$, those avoiding 2413, 2431, 4213, 4231
- (several other objects)

Bijective proof?

Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP w/ $\text{des}(w^{-1}) = k$. WLOG $w_1 = 1$. Then there are partitions $\lambda^1 \dots \lambda^k$ and $\mu := \left(\binom{n-1}{2}, \binom{n-1}{2}, \dots, \binom{2}{2} \right) - \sum_{i=1}^k \lambda^i$

such that

$$\Psi_w = x^\mu \prod_{i=1}^k \underbrace{S_{\lambda^i}(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \dots)}_{\text{flagged Schur poly of shape } \lambda^i \text{ w/ entries in row } j \text{ bounded above by } n-\lambda^i_j}$$

To construct $\lambda^1, \dots, \lambda^k$:

Let $C = (c_1, \dots, c_n)$ be Lehmer code of w^{-1} .

Let $a_0 = 0$, and $a_1 \dots a_k$ be descent positions of $c_1 \dots c_n$.

Define $\lambda^i = (n - a_i)^{a_i} - \underbrace{(0, \dots, 0)}_{a_{i-1}}, c_{a_{i-1}+1}, c_{a_{i-1}+2}, \dots, c_{a_i}$

Ex: If $w = 14253$, $\text{code}(w^{-1}) = (0, \underline{1}, 2, 0, 0)$, $a_0 = 0, a_1 = 3$,
 $\lambda^1 = (5 - 3)^3 - (0, \underline{1}, 2) = (2, 2, 2) - (0, 1, 2) = (2, 1, 0)$.

Rem: I stated theorem in case all y_i 's = 0, but we have version when y_i 's $\neq 0$ using double Schub poly's.

descents
of w^{-1}

k	$w \in \text{St}(5, k)$	$\Psi(w)$	probability ψ_w	$s(w)$
0	12345	$\emptyset$	$\mathbf{x}^{(6,3,1)}$	(0)
1	12354	(1, 1, 1)	$\mathbf{x}^{(5,2,0)} \mathfrak{S}_{13452}$	(0)
1	12435	(2, 2, 1)	$\mathbf{x}^{(4,1,0)} \mathfrak{S}_{14532}$	(0)
1	12453	(2, 2)	$\mathbf{x}^{(4,1,1)} \mathfrak{S}_{14523}$	(0)
1	12534	(1, 1)	$\mathbf{x}^{(5,2,1)} \mathfrak{S}_{12453}$	(0)
1	13245	(3, 2)	$\mathbf{x}^{(3,1,1)} \mathfrak{S}_{15423}$	(0)
1	13425	(3, 1)	$\mathbf{x}^{(3,2,1)} \mathfrak{S}_{15243}$	(0)
1	13452	(3)	$\mathbf{x}^{(3,3,1)} \mathfrak{S}_{15234}$	(0)
1	14235	(2, 1, 1)	$\mathbf{x}^{(4,2,0)} \mathfrak{S}_{13542}$	(0)
1	14253	(2, 1)	$\mathbf{x}^{(4,2,1)} \mathfrak{S}_{12543}$	(0)
1	14523	(2)	$\mathbf{x}^{(4,3,1)} \mathfrak{S}_{12534}$	(0)
1	15234	(1)	$\mathbf{x}^{(5,3,1)} \mathfrak{S}_{12354}$	(0)
2	12543	(2, 2), (1, 1, 1)	$\mathbf{x}^{(3,0,0)} \mathfrak{S}_{14523} \mathfrak{S}_{13452}$	(0, -1)
2	13254	(3, 2), (1, 1, 1)	$\mathbf{x}^{(2,0,0)} \mathfrak{S}_{15423} \mathfrak{S}_{13452}$	(0, 0)
2	13542	(3), (1, 1, 1)	$\mathbf{x}^{(2,2,0)} \mathfrak{S}_{15234} \mathfrak{S}_{13452}$	(0, -1)
2	14352	(3), (2, 2, 1)	$\mathbf{x}^{(1,1,0)} \mathfrak{S}_{15234} \mathfrak{S}_{14532}$	(0, -1)
2	14532	(3), (2, 2)	$\mathbf{x}^{(1,1,1)} \mathfrak{S}_{15234} \mathfrak{S}_{14523}$	(0, -1)
2	15342	(3), (1, 1)	$\mathbf{x}^{(2,2,1)} \mathfrak{S}_{15234} \mathfrak{S}_{12453}$	(0, -1)
2	15423	(2), (1, 1, 1)	$\mathbf{x}^{(3,2,0)} \mathfrak{S}_{12534} \mathfrak{S}_{13452}$	(0, -2)
3	15432	(3), (2, 2), (1, 1, 1)	$\mathfrak{S}_{15234} \mathfrak{S}_{14523} \mathfrak{S}_{13452}$	(0, -1, -2)

sum of these
vectors is
constant
 $(2, 2) + (1, 1, 1) + (3, 0, 0)$
 $= (6, 3, 1)$

sequence
of partitions
 $\gamma^1 \dots \gamma^k$

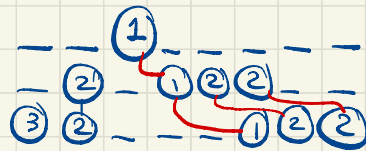
Theorem (Kim-W.) Let $w \in S_n$ be an evil-avoiding permutation representing state of TASEP w/ $\text{des}(w^{-1}) = k$. WLOG $w_1 = 1$. Then there are partitions $\lambda^1 \dots \lambda^k$ and $\mu := \left(\binom{n-1}{2}, \binom{n-1}{2}, \dots, \binom{2}{2} \right) - \sum_{i=1}^k \lambda^i$

such that

$$\Psi_w = x^\mu \prod_{i=1}^k S_{\lambda^i}(X_{n-\lambda^i}, X_{n-\lambda^i}, \dots)$$

Note: If y_i 's = 0, there is combinatorial formula for stationary dist in terms of multiline queues (Ayyer-Linusson and Arita-Mallick). In this case & when $k=1$, can use it to prove Thm.

But to prove theorem for general y_i 's & k , need another technique...



To prove our main result connecting steady state prob's $\Psi_w(x_1, \dots, x_n, y_1, \dots, y_n)$ to double Schubert poly's, use theorem of Cantini that adds auxiliary parameters $z_1, \dots, z_n$ into the picture.

Gives recursion for the Ψ_w .

Theorem (Cantini)

For each state $w \in S_n$ we define the quantity $\psi_w(z) = \psi_w(z_1, \dots, z_n)$ via:

$$\psi_{(1, 2, \dots, n)}(z) = \prod_{1 \leq i < j \leq n} (x_i - y_j)^{j-i-1} \prod_{i=1}^n \left(\prod_{j=1}^{i-1} (z_i - x_j) \prod_{j=i+1}^n (z_i - y_j) \right),$$

$$\psi_{s_\ell w}(z) = \pi_\ell(w_\ell, w_{\ell+1}; n) \psi_w(z) \quad \text{if } w_\ell > w_{\ell+1},$$

where $\pi_\ell(\beta, \alpha; n)$ is the *isobaric divided difference operator* defined by

$$\pi_\ell(\beta, \alpha; n) G(z) = \frac{(z_\ell - y_\beta)(z_{\ell+1} - x_\alpha)}{x_\alpha - y_\beta} \frac{G(z) - s_\ell G(z)}{z_\ell - z_{\ell+1}}.$$

Then the leading coefficient $LC_z(\psi_w(z))$ w/ respect to z equals the steady state prob ψ_w .

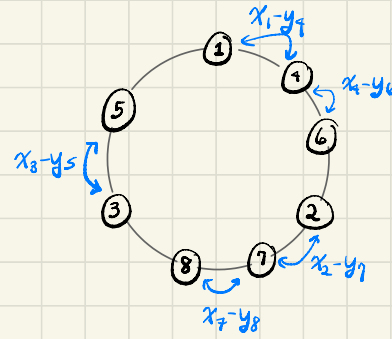
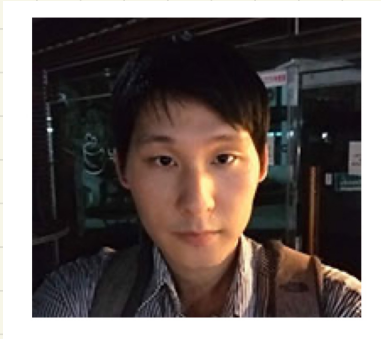
Then need to show that for evil-avoiding w ,

$$LC_z(\Psi_w(z)) = \text{product of Schuberts.}$$

Conj: If w not evil-avoiding, can write Ψ_w as
monomial $\cdot$ (pos sum of Schubert poly's)

State w	Probability ψ_w (we set $y_i = 0$ for all i)
12345	$x^{(6,3,1)}$
12354	$x^{(5,2,0)} \mathfrak{S}_{13452}$
12435	$x^{(4,1,0)} \mathfrak{S}_{14532}$
12453	$x^{(4,1,1)} \mathfrak{S}_{14523}$
12534	$x^{(5,2,1)} \mathfrak{S}_{12453}$
12543	$x^{(3,0,0)} \mathfrak{S}_{14523} \mathfrak{S}_{13452}$
13245	$x^{(3,1,1)} \mathfrak{S}_{15423}$
13254	$x^{(2,0,0)} \mathfrak{S}_{15423} \mathfrak{S}_{13452}$
13425	$x^{(3,2,1)} \mathfrak{S}_{15243}$
13452	$x^{(3,3,1)} \mathfrak{S}_{15234}$
13524	$x^{(2,1,0)} (\mathfrak{S}_{164325} + \mathfrak{S}_{25431})$
13542	$x^{(2,2,0)} \mathfrak{S}_{15234} \mathfrak{S}_{13452}$
14235	$x^{(4,2,0)} \mathfrak{S}_{13542}$
14253	$x^{(4,2,1)} \mathfrak{S}_{12543}$
14325	$x^{(1,0,0)} (\mathfrak{S}_{1753246} + \mathfrak{S}_{265314} + \mathfrak{S}_{2743156} + \mathfrak{S}_{356214} + \mathfrak{S}_{364215} + \mathfrak{S}_{365124})$
14352	$x^{(1,1,0)} \mathfrak{S}_{15234} \mathfrak{S}_{14532}$
14523	$x^{(4,3,1)} \mathfrak{S}_{12534}$
14532	$x^{(1,1,1)} \mathfrak{S}_{15234} \mathfrak{S}_{14523}$
15234	$x^{(5,3,1)} \mathfrak{S}_{12354}$
15243	$x^{(3,1,0)} (\mathfrak{S}_{146325} + \mathfrak{S}_{24531})$
15324	$x^{(2,1,1)} (\mathfrak{S}_{15432} + \mathfrak{S}_{164235})$
15342	$x^{(2,2,1)} \mathfrak{S}_{15234} \mathfrak{S}_{12453}$
15423	$x^{(3,2,0)} \mathfrak{S}_{12534} \mathfrak{S}_{13452}$
15432	$\mathfrak{S}_{15234} \mathfrak{S}_{14523} \mathfrak{S}_{13452}$

Thank you!



Donghyun Kim — W. "Schubert polynomials & the inhomogeneous TASEP on a ring"
arXiv: 2102.00560 (extended abstract)