Schur functions in noncommuting variables

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Integer partitions

An integer partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of n is a list of positive integers whose sum is n: $3221 \vdash 8$.

Let $\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell = 1^{r_1} 2^{r_2} \cdots n^{r_n}$. Then

$$\lambda! = \lambda_1! \lambda_2! \cdots \lambda_\ell!$$

and

$$\lambda^! = r_1! r_2! \cdots r_n!$$

Example

If $\lambda = 3221 = 1^1 2^2 3^1 4^0 5^0 6^0 7^0 8^0 \vdash 8$, then

$$\lambda! = 3!2!2!1! = 6 \times 2 \times 2 \times 1 = 24$$

$$\lambda^! = 1!2!1!0!0!0!0!0! = 2.$$

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Set partitions

A set partition π of $[n] = \{1, 2, \ldots, n\}$ is a partitioning of $[n]$ into disjoint sets $B_1, B_2, \ldots, B_\ell$ called blocks so

- $B_i \neq \emptyset$
- $B_1 \cup B_2 \cup \cdots \cup B_\ell = [n]$.

$$\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$$

Example

$$\{1, 3, 4\}, \{2, 5\}, \{6\}, \{7, 8\}$$

is a set partition of $[8]$, or

$$\pi = 134/25/6/78 \vdash [8].$$

Partitions and permutations

If $\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$, then

$$\lambda(\pi) = |B_1||B_2|\cdots|B_\ell|$$

with sizes weakly decreasing. If $\delta \in \mathfrak{S}_n$, then

$$\delta(\pi) = \delta(B_1)/\delta(B_2)/\cdots/\delta(B_\ell).$$

Example

If $\pi = 134/25/6/78 \vdash [8]$, then

$$\lambda(134/25/6/78) = 3221$$

and $\delta = 14325876 \in \mathfrak{S}_8$

$$\begin{aligned}\delta(134/25/6/78) &= \delta(1)\delta(3)\delta(4)/\delta(2)\delta(5)/\delta(6)/\delta(7)\delta(8)\\ &= 132/45/8/76 = 123/45/67/8.\end{aligned}$$

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Slash product

If

$$S + n = \{s + n : s \in S\}$$

then for $\pi \vdash [n]$ and $\sigma = B_1/B_2/\cdots/B_\ell \vdash [m]$ the slash product is

$$\pi \mid \sigma = \pi/(B_1 + n)/(B_2 + n)/\cdots/(B_\ell + n) \vdash [n + m].$$

Example

If $\pi = 134/25 \vdash [5]$ and $\sigma = 1/23 \vdash [3]$ then

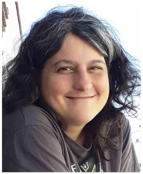
$$\pi \mid \sigma = 134/25/6/78 \vdash [8].$$

NCSym (Rosas-Sagan 2004)

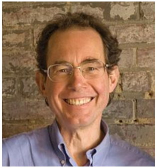
NCSym is the algebra of symmetric functions in noncommuting variables $x_1, x_2, x_3, \ldots$

$$\mathrm{NCSym} = \mathrm{NCSym}^0 \oplus \mathrm{NCSym}^1 \oplus \cdots \subset \mathbb{Q} \ll x_1, x_2, x_3, \ldots \gg$$

where $\mathrm{NCSym}^0 = \mathrm{span}\{1\}$ and for $n > 0$



$$\begin{aligned} \mathrm{NCSym}^n &= \mathrm{span}\{m_\pi \ : \ \pi \vdash [n]\} \\ &= \mathrm{span}\{p_\pi \ : \ \pi \vdash [n]\} \\ &= \mathrm{span}\{e_\pi \ : \ \pi \vdash [n]\} \\ &= \mathrm{span}\{h_\pi \ : \ \pi \vdash [n]\}. \end{aligned}$$



Note: The e_π defined by Wolf in 1936.

Monomial function in NCSym

The monomial symmetric function in NCSym for $\pi \vdash [n]$ is

$$m_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if and only if j and k are in the same block of π.

Example

$m_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + x_1 x_3 x_1 + x_2 x_3 x_2 + \cdots$

Power sum function in NCSym

The power sum symmetric function in NCSym for $\pi \vdash [n]$ is

$$p_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j = i_k$$

if j and k are in the same block of π.

Example

$p_{13/2} = x_1 x_2 x_1 + x_2 x_1 x_2 + \cdots + x_1^3 + x_2^3 + \cdots$

Elementary function in NCSym

The elementary symmetric function in NCSym for $\pi \vdash [n]$ is

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j \neq i_k$$

if j and k are in the same block of π.

Example

$e_{13/2} = x_1 x_1 x_2 + x_1 x_2 x_2 + x_2 x_2 x_1 + x_2 x_1 x_1 + \cdots + x_1 x_2 x_3 + \cdots$

Complete homogeneous function in NCSym

The complete homogeneous symmetric function in NCSym for $\pi \vdash [n]$ is

$$h_\pi = \sum_\sigma \lambda(\sigma \wedge \pi)! m_\sigma$$

where $\sigma \wedge \pi$ is the maximal set partition where every block is a subblock of σ and π.

Example

$$h_{13/2} = 2m_{123} + m_{12/3} + m_{1/23} + 2m_{13/2} + m_{1/2/3}$$

$$123 \wedge 13/2 = 13/2 \wedge 13/2 = 13/2 \quad \lambda(13/2)! = 2$$

$$12/3 \wedge 13/2 = 1/23 \wedge 13/2 = 1/2/3 \wedge 13/2 = 1/2/3 \quad \lambda(1/2/3)! = 1$$

Permutations and products

Fact: If $\pi \vdash [n]$, $\delta \in \mathfrak{S}_n$ and $\delta \circ m_\pi = m_{\delta(\pi)}$, then for $b = p, e, h$

$$\delta \circ b_\pi = b_{\delta(\pi)}.$$

Fact: If π, σ are set partitions, then for $b = p, e, h$

$$b_\pi b_\sigma = b_{\pi|\sigma}.$$

Example

$$132 \circ m_{12/3} = m_{13/2} \qquad p_{13/2} p_1 = p_{13/2|1} = p_{13/2/4}$$

What's in a name?

Let the variables commute:

$$\rho : \text{NCSym} \to \text{Sym}$$

Theorem (Rosas-Sagan 2004)

Let π be a set partition.

$$\rho(m_\pi) = \lambda(\pi)^! m_{\lambda(\pi)} \qquad \rho(p_\pi) = p_{\lambda(\pi)}$$

$$\rho(e_\pi) = \lambda(\pi)! e_{\lambda(\pi)} \qquad \rho(h_\pi) = \lambda(\pi)! h_{\lambda(\pi)}$$

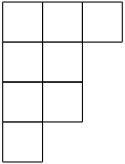
Note: The images are classical monomial, power sum, elementary and complete homogeneous symmetric functions.

Question: Where are the Schur functions?

Partitions and diagrams

An integer partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of n is a list of positive integers whose sum is n: $3221 \vdash 8$.

The diagram $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ is the array of boxes with λ_i boxes in row i from the top.



3221

Dotted Young tableaux

A dotted Young tableau (DYT) $\dot{T}$ of shape λ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase, and $1, 2, \ldots, n$ dots appear exactly once.

$$\begin{array}{|c|c|}\hline \dddot{1} & \dot{2} \\ \hline \ddot{4} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Given a DYT $\dot{T}$ we have

$$x^{\dot{T}} = x_i \text{ in position } j \text{ iff } i \text{ has } j \text{ dots above it.}$$

$$x_2 x_4 x_1$$

Rosas-Sagan Schur functions

The Rosas-Sagan Schur function in NCSym is

$$S^{RS}_{\lambda} = \sum_{\dot{T} \text{ DYT of shape } \lambda} x^{\dot{T}}.$$

Example

$S^{RS}_{21} = 2x_1x_1x_2 + 2x_1x_2x_1 + 2x_2x_1x_1 + \cdots$

$$\begin{array}{|c|c|}\hline \dot{1} & \ddot{1} \\ \hline \dddot{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline \ddot{1} & \dot{1} \\ \hline \dddot{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline \dot{1} & \dddot{1} \\ \hline \ddot{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline \dddot{1} & \dot{1} \\ \hline \ddot{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline \ddot{1} & \dddot{1} \\ \hline \dot{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline \dddot{1} & \ddot{1} \\ \hline \dot{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Rosas-Sagan Schur functions

Theorem (Rosas-Sagan 2004)

Let $\lambda \vdash n$.

- *The S_λ^{RS} are linearly independent.*
- *We have $\rho(S_\lambda^{RS}) = n! s_\lambda$.*

Note: However they are not a basis for NCSym because we only have one for each integer partition, not set partition.

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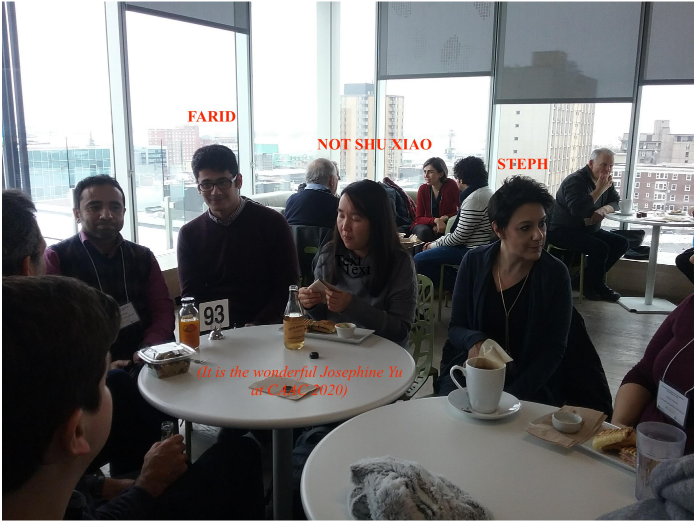
Yes there is!

The three musketeers

Aliniaeifard-Li-vW 2021



"All for one ..." – Dumas, T3M

The noncommutative Leibniz formula for $A = (a_{ij})_{1 \le i,j \le n}$ with noncommuting entries a_{ij} is

$$\mathbf{det}(A) = \sum_{\varepsilon \in \mathfrak{S}_n} \mathrm{sgn}(\varepsilon) a_{1\varepsilon(1)} a_{2\varepsilon(2)} \cdots a_{n\varepsilon(n)}$$

- product of the entries is taken top row to the bottom row
- $\mathrm{sgn}(\varepsilon)$ is the sign of permutation ε.

" … and one for all," – Dumas, T3M

Definition

Let $\lambda = \lambda_1\lambda_2\cdots\lambda_\ell$. Then the source Schur function

$$s_{[\lambda]} = \mathbf{det}\left(\frac{1}{(\lambda_i - i + j)!}h_{[\lambda_i - i + j]}\right)_{1\le i,j\le \ell}$$

where $h_{[0]} = h_\emptyset = 1$ and $h_{\text{-ve}} = 0$.

Example

$$s_{[21]} = \mathbf{det}\begin{pmatrix} \frac{1}{2!}h_{12} & \frac{1}{3!}h_{123} \\ \frac{1}{0!}h_\emptyset & \frac{1}{1!}h_1 \end{pmatrix} = \frac{1}{2!}h_{12}\frac{1}{1!}h_1 - \frac{1}{3!}h_{123}\frac{1}{0!}h_\emptyset$$

$$= \frac{1}{2}h_{12|1} - \frac{1}{6}h_{123} = \frac{1}{2}h_{12/3} - \frac{1}{6}h_{123}.$$

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We now create a set partition $\pi \vdash [n]$ from an integer partition $\lambda \vdash n$ where

$$\lambda(\pi) = \lambda$$

using tableaux T_π of shape λ such that

- every element $1, 2, \ldots, n$ appears exactly once
- rows increase left to right
- first column of rows of same length increase top to bottom.

$$\text{rows} \longleftrightarrow \text{blocks}$$

Example

$$T_\pi = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad \longleftrightarrow \qquad \pi = 134/25/78/6$$

We now create a permutation $\delta_\pi \in \mathfrak{S}_n$ using T_π.

read by row $\longleftrightarrow$ one line notation

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Let $\lambda = \lambda_1\lambda_2\cdots\lambda_\ell$. Then the Schur function in NCSym is

$$s_\pi = \delta_\pi \circ s_{[\lambda(\pi)]} = \delta_\pi \circ \mathbf{det}\left(\frac{1}{(\lambda_i - i + j)!}h_{[\lambda_i - i + j]}\right)_{1\le i,j\le \ell}.$$

Example

If $\pi = 13/2$ then $T_\pi = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$ and $\delta_\pi = 132$.

$$s_{13/2} = 132 \circ s_{[21]} = 132 \circ \left(\frac{1}{2}h_{12/3} - \frac{1}{6}h_{123}\right) = \frac{1}{2}h_{13/2} - \frac{1}{6}h_{123}$$

"… united we stand divided we fall." — Dumas, T3M

Theorem (Aliniaeifard-Li-vW 2021)

$$\mathsf{NCSym}^n = \mathrm{span}\{s_\pi \ : \ \pi \vdash [n]\} \qquad \rho(s_\pi) = s_{\lambda(\pi)}$$

$$S_\lambda^{RS} = \sum_{\delta \in \mathfrak{S}_n} \delta \circ s_{[\lambda]}$$

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Note: $\delta \circ s_\pi \neq s_{\delta(\pi)}$ in general.

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THEOREM (ALINIAEIFARD-LI-VW 2021)

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$$S_\lambda^{RS} = \sum_{\delta \in \mathfrak{S}_n} \delta \circ s_{[\lambda]}$$

Note: $\delta \circ s_\pi \neq s_{\delta(\pi)}$ in general.

THEOREM (ALINIAEIFARD-LI-VW 2021)

For $n \geq 5$ we have $n!$ different bases:

$$\mathrm{NCSym}^n = \mathrm{span}\{\delta \circ s_\pi \ : \ \pi \vdash [n]\} \qquad \rho(\delta \circ s_\pi) = s_{\lambda(\pi)}$$

"The more we have ventured the more we gain" – Dumas, T3M

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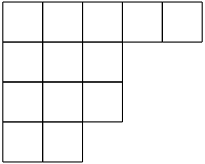
For $\lambda \vdash n, \mu \vdash m$ the skew diagram λ/μ is the array of $n - m$ boxes contained in λ but not in μ.



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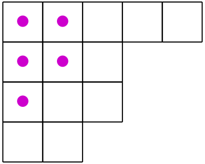

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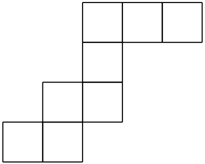

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Rosas-Sagan skew Schur functions

The Rosas-Sagan skew Schur function in NCSym is

$$S^{RS}_{\lambda/\mu} = \sum_{\dot{T} \text{ DYT of shape } \lambda/\mu} x^{\dot{T}}.$$

Theorem (Aliniaeifard-Li-vW 2021)

Let $\lambda \vdash n$ and $\mu \vdash m$.

$$\rho(S^{RS}_{\lambda/\mu}) = (n-m)! s_{\lambda/\mu}$$

$$S^{RS}_{\lambda/\mu} = \sum_{\nu \vdash (n-m)} c^{\lambda}_{\mu\nu} S^{RS}_{\nu}$$

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and the $c^{\lambda}_{\mu\nu}$ are the classical Littlewood-Richardson coefficients.

Noncommutative symmetric functions

Take the noncommutative symmetric functions (Gelfand-Krob-Lascoux-Leclerc-Thibon) with map $\mathcal{J}$ (Bergeron-Reutenauer-Rosas-Zabrocki).

$$\mathcal{J} : \sum_{j_1 \leq j_2 \leq \cdots \leq j_n} x_{j_1} x_{j_2} \cdots x_{j_n} \mapsto \frac{1}{n!} h_{[n]} \in \mathsf{NCSym}$$

Theorem (Aliniaeifard-Li-vW 2021)

For the immaculate function $\mathfrak{S}_\lambda$ of Berg-Bergeron-Saliola-Serrano-Zabrocki

$$\mathcal{J}(\mathfrak{S}_\lambda) = s_{[\lambda]}.$$

For the noncommutative ribbon Schur function $\mathbf{r}_\alpha$ of Gelfand et al.

$$\mathcal{J}(\mathbf{r}_\alpha) = r_{[\alpha]}$$

the ribbon source Schur function.

“Never fear quarrels, but seek hazardous adventures”
– Alexandre Dumas, The Three Musketeers

Here ends our Algebraic Combinatorics Virtual Expedition ...

Thank you very much!

Thank you to the organizers for a wonderful conference!