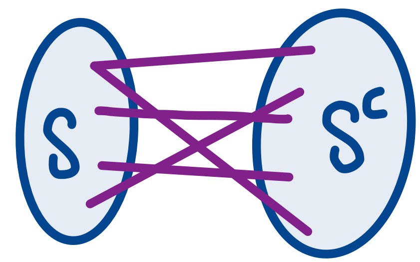
Log-concavity, matroids, and expanders

joint w/ Nima Anari, Kuikui Liu, and Shayan Oveis Gharan

Expansion of a graph $G=(V,E)$

$$h(G) = \min_{S \subseteq V} \frac{|E(S,S^c)|}{\min(|S|,|S^c|)}$$



Cheeger's inequality: For d-regular G,

$$\tfrac{1}{2}(d-\lambda_2) \le h(G) \le \sqrt{2d(d-\lambda_2)}$$

where $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{|V|}$ are eig. val. of adj. matrix A_G.

Mihail-Vazirani Conj: The edge graph of every 0-1 polytope has expansion ≥ 1.

Thm (ALOV '19) ↗ holds for matroid polytopes

Matroid: $M = ([n], \mathcal{I})$

"independent sets" $\mathcal{I}$ = nonempty collection of subsets of $[n]$ with

1) $T \in \mathcal{I},\ S \subseteq T \Rightarrow S \in \mathcal{I}$ (simplicial complex)
2) $S, T \in \mathcal{I},\ |T| > |S| \Rightarrow \exists\, i \in T \setminus S$ with $S \cup \{i\} \in \mathcal{I}$

"bases" $\mathcal{B}$ = maximal elt. of $\mathcal{I}$ "rank" $= |B|$ for any $B \in \mathcal{B}$

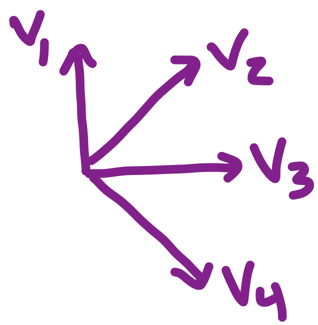
Polytope: $\operatorname{conv}\{ \mathbb{1}_B : B \in \mathcal{B} \}$ **Polynomial:** $f_{\mathcal{B}} = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$

Ex (linear) $v_1, \ldots, v_n \in \mathbb{F}^d$

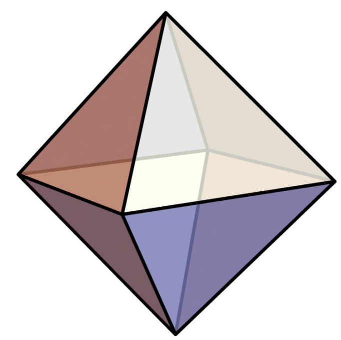
$\mathcal{I} = \{\text{linearly indep. subsets}\}$

$\mathcal{B} = \{\text{bases for span}\}$

$\text{rank} = \dim_{\mathbb{F}} \operatorname{span}\{v_1, \ldots, v_n\}$



$\mathcal{B} = \binom{[4]}{2}$



$f_{\mathcal{B}} = e_2(x_1, \ldots, x_4) = \sum_{1 \le i < j \le 4} x_i x_j$

Non-ex: $\{\{1,2\},\{3,4\}\} \neq \mathcal{B}$ for any matroid

Log concave polynomials

$$\partial^\alpha = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$$

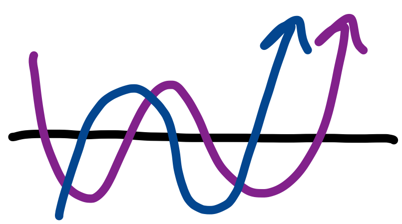
Def: $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is...

log-concave if $f \equiv 0$ or $\log(f) : \mathbb{R}_+^n \to \mathbb{R}$ is concave

strongly log-concave if $\partial^\alpha f$ is log-concave $\forall \alpha \in \mathbb{N}^n$

+homogeneous = Lorentzian (Brändén-Huh '19)

Ex: $f = \prod_{i=1}^{d} (x + r_i)$ $\quad r_1, \ldots, r_d \in \mathbb{R}_{\geq 0}$ $\quad \log(f)'' = \sum_{i=1}^{d} \frac{-1}{(x+r_i)^2} \leq 0$ $\forall x \in \mathbb{R}_+$



roots of $\frac{\partial f}{\partial x}$ in $\mathbb{R}_{\leq 0} \Rightarrow$ log-concave

More ex's: real stable poly. with coeff in $\mathbb{R}_{\geq 0}$,

mixed volume poly. $\mathrm{Vol}_d(x_1 K_1 + \ldots + x_n K_n)$ (Brunn-Minkowski)

(normalized) Schur poly. (Huh-Mészáros '19)

Matroids $\longleftrightarrow$ Log-concavity

Inspiration: Adiprasito, Huh, Katz '15

Thm (ALOV '18, Brändén-Huh '18, Backman-Eur-Simpson '19)

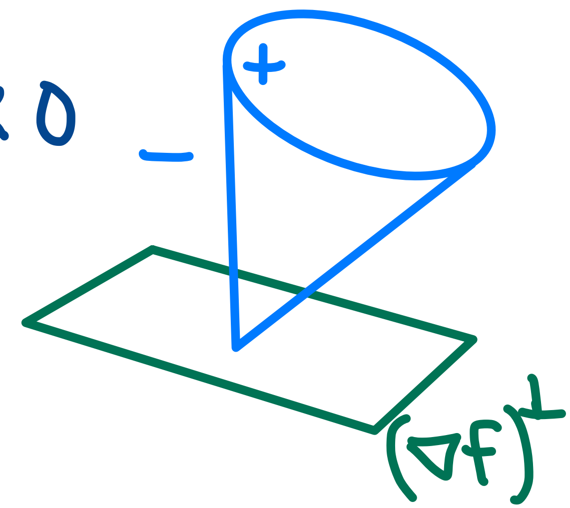
For any matroid $\sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$ is strongly log-concave.

Note: f log-concave at $a \in \mathbb{R}_+^n$

$$\Rightarrow \nabla^2 \log(f)\big|_{x=a} = \frac{f\nabla^2 f - \nabla f \nabla f^T}{f^2}\bigg|_{x=a} \preceq 0$$

$\Rightarrow$ quad. form $\nabla^2 f|_{x=a} \leq 0$ on $(\nabla f|_{x=a})^\perp$

$\Rightarrow \nabla^2 f|_{x=a}$ has ≤ 1 positive eigenvalue

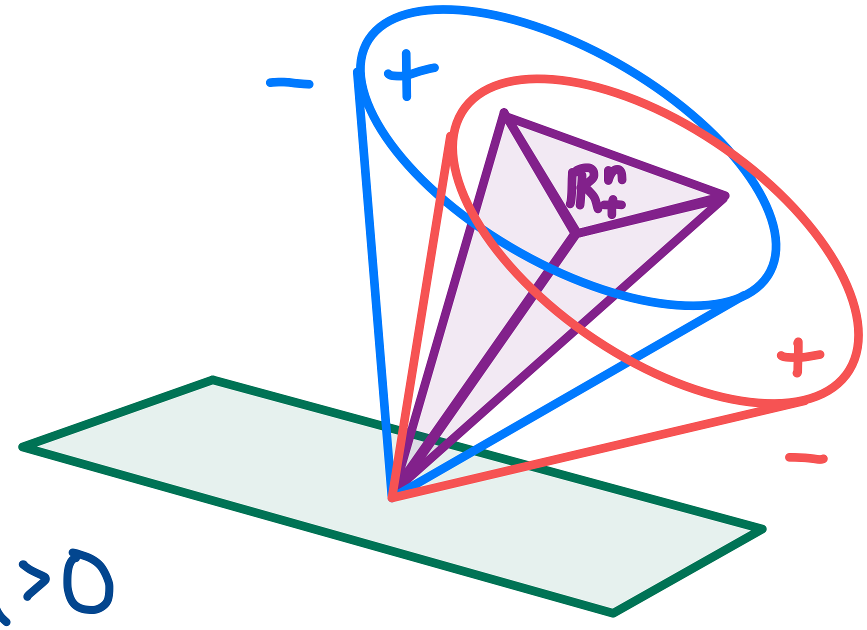


Ex: $f = \sum_{1 \leq i < j \leq 4} x_i x_j \quad \nabla^2 f = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \mathbb{1}\mathbb{1}^T - I$

Non-ex: $f = x_1x_2 + x_3x_4$

$$\nabla^2 f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Local to Global



Observation:

q_1, q_2 quadratic forms, >0 on $\mathbb{R}^n_+$, ≤ 0 on H

$\Rightarrow \lambda q_1 + \mu q_2$ has exactly one pos. eig. value $\forall \lambda, \mu > 0$

Polynomial $f \to$ graph $G(f)$ with adj. matrix $\nabla^2 f|_{x=\mathbb{1}}$

Thm: $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ homog. of deg d.

f is strongly log-concave $\Longleftrightarrow$

1) $G(\partial^\alpha f)$ is connected for all $|\alpha| \leq d-2$, and

2) $\partial^\alpha f$ has at most one pos. eig. val. for all $|\alpha| = d-2$

Markov chains

$f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ log-concave

$P = \left(\nabla^2 f|_{x=1}\right) D^{-1}$, D diag with $\mathbb{1}P = \mathbb{1}$

$\Rightarrow P$ stochastic $\lambda_2(P) \leq 0$ $\Rightarrow \lambda_2\left(\frac{1}{2}I + \frac{1}{2}P\right) \leq \frac{1}{2}$

Ex: $f = \sum_{1 \leq i < j \leq 4} x_i x_j$ $\quad P = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ $\quad \frac{1}{2}I + \frac{1}{2}P = \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{pmatrix}$

Getting bigger...

Δ = simplicial complex, pure dim $d-1$

$\Delta(k)$ = faces of dim $k-1$

weight c_S on face $S \in \Delta(d)$

Two Markov chains on $\Delta(k)$:

$\Delta(k)$: $S \qquad S\setminus\{i\} \cup \{j\} \quad \dots$

$\Delta(k-1)$: $S \xrightarrow{\text{prob } 1/k} S\setminus\{i\} \xrightarrow{\text{prob } \propto c_{S\setminus\{i\}\cup\{j\}}} S\setminus\{i\}\cup\{j\} \searrow \dots$

Idea: relate eigenvalues of $\nearrow\searrow$ and $\searrow\nearrow$ walks

High Dimensional Expanders

(developed by Lubotzky, Dinur, Mass, Kauffman, Oppenheim ...)

Thm (Kauffman, Oppenheim '18)

1) every link of Δ with dim ≥ 1 has a connected 1-skeleton
2) trans. matrix of $\nearrow\searrow$ walk on every 1-dim'l link has $\lambda_2 \leq \frac{1}{2}$

then trans. matrix of $\searrow\nearrow$ walk on $\Delta(d)$ has $\lambda_2 \leq 1 - \frac{1}{d}$.

Thm (ALOV '19) (Δ, c) satisfies (1) & (2)

$$\Longleftrightarrow f = \sum_{S \in \Delta(d)} c_S x^S \text{ is strongly log-concave}$$

Cor (ALOV '19) For any matroid of rank r this gives a random walk on $\mathcal{I}(r) = \mathcal{B}$ mixing in $O(r^2 \log(n))$ steps.

See also Cryan, Guo, Mousa '19 and ALOV+Vuong '20

$O(r(\log(r) + \log\log(n)))$ $\qquad$ $O(r\log(r))$

Conclusions

Strong log-concavity

- encompasses important examples (volumes, stable poly., matroids, etc.)
- implies discrete log-concavity of coefficients
- give distributions that can be efficiently approximately sampled