Canonical triangulation of enriched order polytopes

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This talk is based on

- S. Okada and A. Tsuchiya, Two enriched poset polytopes, arXiv:2003.12271.
- H. Ohsugi and A. Tsuchiya, Enriched chain polytopes, *Israel J. Math.* **237** (2020), 485–500.
- H. Ohsugi and A. Tsuchiya, Enriched order polytopes and enriched Hibi rings, *Eur. J. Math.* **7** (2021), 48–63.

1. Order polytopes
2. Enriched order polytopes
3. Canonical triangulations of enriched order polytopes
4. h-polynomials of canonical triangulations

Order polytopes

Ehrhart polynomial

$\mathcal{P} \subset \mathbb{R}^d$: a lattice polytope of dimension d
(i.e., the convex hull of finitely many points in $\mathbb{Z}^d$)
$m\mathcal{P} = \{m\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$: the mth dilated polytope of $\mathcal{P}$
$L_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^d|$: the Ehrhart polynomial of $\mathcal{P}$

Theorem (Ehrhart)

$L_{\mathcal{P}}(m)$ is a polynomial in m of degree d. Moreover, the leading coefficient of $L_{\mathcal{P}}(m)$ is equal to the volume of $\mathcal{P}$;

Order polytopes

$(P, <_P)$: a poset on $[d] := \{1, \ldots, d\}$.

Definition (Stanley)

The order polytope of P is

$$\mathcal{O}_P := \{\mathbf{x} \in [0,1]^d : x_i \leq x_j \text{ if } i <_P j\}.$$

Proposition (Stanley)

$\mathcal{O}_P$ is a lattice polytope of dimension d.

Vertices of $\mathcal{O}_P$

$(P, <_P)$: a poset on $[d]$.

- $F \subset [d]$ is a filter of P if for any $x \in F$ and $y \in P$, it follows that $x <_P y \Rightarrow y \in F$.

$\mathcal{F}(P)$: the set of filters of P.

- For $X \subset [d]$, set $\mathbf{e}_X := \sum_{i \in X} \mathbf{e}_i$, where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the standard basis of $\mathbb{R}^d$. In particular, $\mathbf{e}_\emptyset = \mathbf{0}$.

Theorem (Stanley)

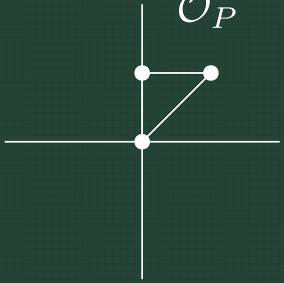
The set of vertices of $\mathcal{O}_P$ is $\{\mathbf{e}_F : F \in \mathcal{F}(P)\}$.

Example

$P =$

$$\mathcal{F}(P) = \{\emptyset, \{2\}, \{1, 2\}\}$$

$\mathcal{O}_P$



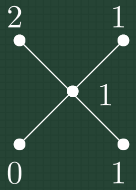
P-partition

$(P, <_P)$: a naturally labeled poset on $[d]$, i.e., $i <_P j \Rightarrow i < j$.

Definition

A map $f : P \to \mathbb{Z}_{\geq 0}$ is a P-partition if for any $i <_P j$,

$$f(i) \leq f(j).$$

$P =$ 

Remark

$\mathcal{O}_P = \mathrm{conv}\{(f(1), \ldots, f(d)) : f$: P-*partitions with* $f(i) \leq 1\}$.

The Ehrhart polynomials of $\mathcal{O}_P$

$(P, <_P)$: a naturally labeled poset on $[d]$.

Theorem (Stanley)

$$L_{\mathcal{O}_P}(m) = |\{f : P\text{-}\textit{partitions with } f(i) \leq m\}|.$$

Enriched order polytopes

Enriched P-partition

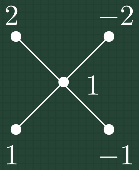
$(P, <_P)$: a naturally labeled poset on $[d]$.

Definition (Stembridge)

A map $f : P \to \mathbb{Z} \setminus \{0\}$ is an enriched P-parition if for any $i <_P j$,

- $|f(i)| \leq |f(j)|$;
- $|f(i)| = |f(j)| \Rightarrow f(j) > 0$.

$P =$



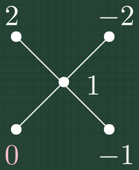
Left enriched P-partition

$(P, <_P)$: a naturally labeled poset on $[d]$.

Definition (Petersen)

A map $f : P \to \mathbb{Z}$ is a left enriched P-parition if for any $i <_P j$,

- $|f(i)| \leq |f(j)|$;
- $|f(i)| = |f(j)| \Rightarrow f(j) \geq 0$.

$P =$ 

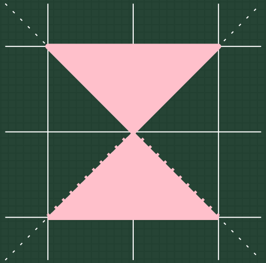
How do we define enriched order polytopes?

Let E be the set of points $\mathbf{x} \in [-1,1]^d$ such that for any $i <_P j$, the following is satisfied:

- $|x_i| \leq |x_j|$;
- $|x_i| = |x_j| \Rightarrow x_j \geq 0$.

Then E is NOT a convex polytope.

$P =$ 



Signed filters

$(P, <_P)$: a poset on $[d]$.

$$\mathcal{F}^{(e)}(P) := \left\{ (F, \varepsilon) \in \mathcal{F}(P) \times \{0, \pm 1\}^d : \varepsilon_i = \begin{cases} \pm 1 & (i \in \min(F)) \\ 1 & (i \in F \setminus \min(F)) \\ 0 & (i \notin F) \end{cases} \right\}$$

Remark

If P is naturally labeled, then

$$\mathcal{F}(P) \overset{\textit{1:1}}{\longleftrightarrow} \{f : P\text{-partition with } f(i) \leq 1\}$$

$$\mathcal{F}^{(e)}(P) \overset{\textit{1:1}}{\longleftrightarrow} \{f : \text{left enriched } P\text{-partition with } |f(i)| \leq 1\}$$

Enriched order polytopes

$(P, <_P)$: a poset on $[d]$.

- For $X \subset [d]$ and $\varepsilon \in \{0, \pm 1\}^d$, set $\mathbf{e}_X^\varepsilon := \sum_{i \in X} \varepsilon_i \mathbf{e}_i$.

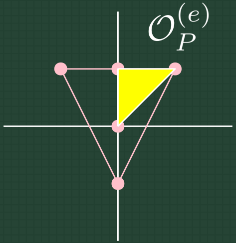
Definition (Ohsugi-T)

The enriched order polytope of P is

$$\mathcal{O}_P^{(e)} := \mathrm{conv}\{\mathbf{e}_F^\varepsilon : (F, \varepsilon) \in \mathcal{F}^{(e)}(P)\}.$$

Example

$P =$



$$\mathcal{F}^{(e)}(P) = \left\{ \begin{array}{l} (\emptyset, (0,0)), (\{2\}, (0,1)), (\{2\}, (0,-1)) \\ (\{1,2\}, (1,1)), (\{1,2\}, (-1,1)) \end{array} \right\}$$

Left enriched P-partitions and lattice points in $m\mathcal{O}_P^{(e)}$

We recall that

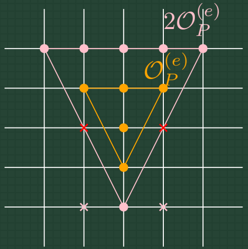
$$m\mathcal{O}_P \cap \mathbb{Z}^d = \{(f(1), \ldots, f(d)) : f : P\text{-partitions with } f(i) \leq m\}.$$

However, in general,

$$m\mathcal{O}_P^{(e)} \cap \mathbb{Z}^d \neq \{(f(1), \ldots, f(d)) : f : \text{l.e. } P\text{-partitions with } |f(i)| \leq m\}.$$

$P =$ 

$\mathcal{F}(P) = \{\emptyset, \{2\}, \{1,2\}\}$



Ehrhart polynomial of $\mathcal{O}_P^{(e)}$

$(P, <_P)$: a naturally labeled poset on $[d]$.

Theorem (Ohsugi–T, Okada–T)

$$L_{\mathcal{O}_P^{(e)}}(m) = |\{f : \textit{left enriched } P\textit{-partitions with } |f(i)| \leq m\}|.$$

Canonical triangulations of enriched order polytopes

Canonical triangulation of order polytopes

$(P, <_P)$: a poset on $[d] := \{1, \ldots, d\}$.
We regard $\mathcal{F}(P)$ as a poset by inclusion.

Theorem (Stanley)

For a chain $C = \{F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_k\}$ of $\mathcal{F}(P)$, we put

$$S_C = \mathrm{conv}\{\mathbf{e}_{F_1}, \ldots, \mathbf{e}_{F_k}\}.$$

Then the collection $\mathcal{S}_P = \{S_C : C \text{ is a chain of } \mathcal{F}(P)\}$ is a unimodular triangulation of $\mathcal{O}_P$.

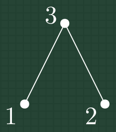
Here a lattice triangulation is unimodular if all maximal faces have the Euclidean volume $1/d!$.

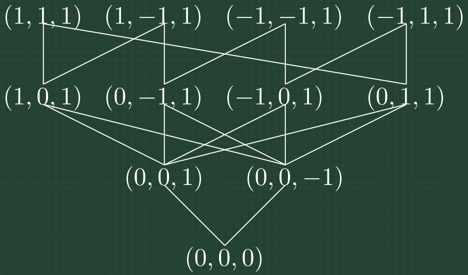
Poset structure on $\mathcal{F}^{(e)}(P)$

Definition

For $(F,\epsilon),(G,\delta)\in\mathcal{F}^{(e)}(P)$, we write $(F,\epsilon)>(G,\delta)$ if the following three conditions hold:

1. $F \supsetneq G$;
2. $\epsilon_i \geq \delta_i$ for any $i \in G$;
3. If $i \in G$ and i is minimal in F, then $\epsilon_i = \delta_i$.





Canonical triangulation of $\mathcal{O}_P^{(e)}$

Theorem (Okada–T)

For a chain $C = \{(F_1, \epsilon_1) > (F_2, \epsilon_2) \cdots > (F_k, \epsilon_k)\}$ *of* $\mathcal{F}^{(e)}(P)$, *we put*

$$S_C^{(e)} = \mathrm{conv}\{\mathbf{e}_{F_1}^{\epsilon_1}, \ldots, \mathbf{e}_{F_k}^{\epsilon_k}\}.$$

Then the collection $\mathcal{S}_P^{(e)} = \{S_C^{(e)} : C$ *is a chain of* $\mathcal{F}^{(e)}(P)\}$ *is a unimodular triangulation of* $\mathcal{O}_P^{(e)}$.

h-polynomials of canonical triangulations

Order complexes

Q : a graded poset.

Definition

The order complex $\Delta(Q)$ of Q is the simlicial complex whose faces are chains of Q.

Remark

- *$\mathcal{F}^{(e)}(P)$ is a graded poset.*
- *$\mathcal{S}_P^{(e)}$ coincides with $\Delta(\mathcal{F}^{(e)}(P))$ as simplicial complexes.*
- *$\Delta(\mathcal{F}^{(e)}(P))$ is flag and balanced.*

h-polynomials of simplicial complexes

Δ : a simplicial complex of dimension $d-1$
f_i : the number of i-dimensional faces of Δ
$f_{-1} := 1$
Define a sequence $(h_0, \ldots, h_d)$ as follows:

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

$h(\Delta, t) := \sum_{i=0}^{d} h_i t^i$: the h-polynomial of Δ

Palindromic polynomials and γ-positivity

$f(t) = \sum_{i=0}^{d} a_i t^i \in \mathbb{Z}_{>0}[t]$: a palindromic polynomial
i.e., $a_i = a_{d-i}$ for any $1 \le i \le \lfloor d/2 \rfloor$

Then there exists a unique expression

$$f(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$$

$\gamma(t) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i \in \mathbb{Z}[t]$ is called the γ-polynomial of $f(t)$.

(RR) $f(t)$ is real-rooted if all roots of $f(t)$ are real.

(GP) $f(t)$ is γ-positive if $\gamma_i \ge 0$ for all i.

(UN) $f(t)$ is unimodal if $a_0 \le \cdots \le a_k \ge \cdots \ge a_d$ with some k.

In general, (RR) $\Rightarrow$ (GP) $\Rightarrow$ (UN). If $f(t)$ is γ-positive, then

$$f(t) \text{ is real-rooted} \iff \gamma(t) \text{ is real-rooted}$$

Lemma

The h-polynomial of $\Delta(\mathcal{F}^{(e)}(P))$ is palindromic.

Left peak polynomials

$(P, <_P)$: a naturally labeled poset on $[d]$.
A permutation $\pi = \pi_1 \cdots \pi_d$ is called a linear extension of P if $i <_P j \Rightarrow \pi_i < \pi_j$.

$\mathcal{L}(P)$: the set of linear extensions of P.
For $\pi \in \mathcal{L}(P)$ with $\pi_0 = 0$, set

$$\mathrm{peak}^{(\ell)}(\pi) := |\{1 \le i \le d-1 : \pi_{i-1} < \pi_i > \pi_{i+1}\}|.$$

$W_P^{(\ell)}(t) := \sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{peak}^{(\ell)}(\pi)}$: the left peak polynomial of P.

Theorem (Ohsugi-T, Okada-T, Petersen, Stembridge)
The γ-polynomials of $\Delta(\mathcal{F}^{(e)}(P))$ equals $W_P^{(\ell)}(4t)$.

Alternative proof for the γ-positivity

P : a poset on $[d]$.
$\mathcal{G}^{(e)}(P) := \mathcal{F}^{(e)}(P) \setminus \{(\emptyset, \mathbf{0})\}$.

Remark

- $\mathcal{G}^{(e)}(P)$ *is a graded poset.*
- $\Delta(\mathcal{G}^{(e)}(P))$ *is a triangulation of* $\partial\mathcal{O}_P^{(e)}$.
- $h(\Delta(\mathcal{G}^{(e)}(P)), t) = h(\Delta(\mathcal{F}^{(e)}(P)), t)$.

Theorem (Karu)

Let Q *be a graded poset. If* Q *is Gorenstein***, namely,* $\Delta(Q)$ *is a homological sphere, then* $h(\Delta(Q), t)$ *is* γ*-positive.*