Permutations, moments, measures

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Joint work with
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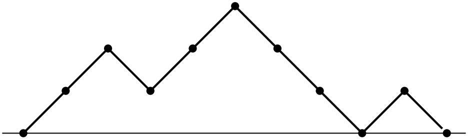
The Catalan numbers count Dyck paths, whose generating function is

$$C(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{2n}$$



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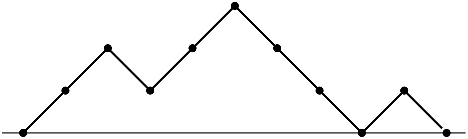
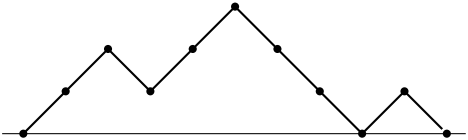
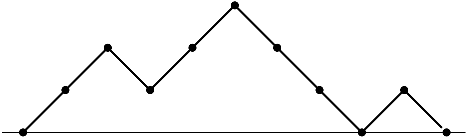
which satisfies $C = 1 + x^2 C^2$,

from which it follows that $C(x) = \cfrac{1}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{\ddots}}}$





$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$$



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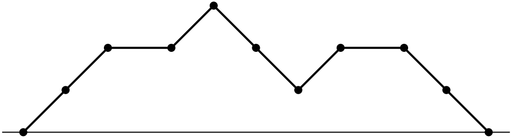
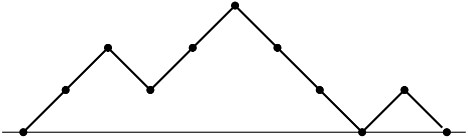


Motzkin path



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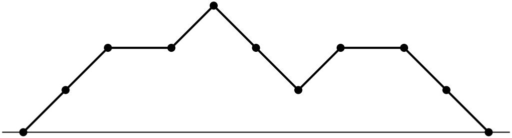
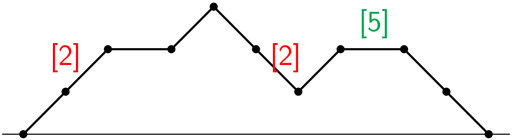


$$\cfrac{1}{1-z-\cfrac{z^2}{1-z-\cfrac{z^2}{\ddots}}}$$

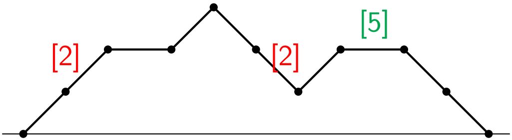
Motzkin path



Weighted Motzkin path

$$\cfrac{1}{1 - z - \cfrac{z^2}{1 - 3z - \cfrac{2^2 z^2}{\ddots\, 1 - (2n+1)z - \cfrac{n^2 z^2}{\ddots}}}} = \sum_{n \geq 0} n! \cdot z^n$$

Special case of the general correspondence by Flajolet.



Weighted Motzkin path

$$\cfrac{1}{1-\alpha_0 z-\cfrac{\beta_1 z^2}{1-\alpha_1 z-\cfrac{\beta_2 z^2}{\ddots\, 1-\alpha_n z-\cfrac{\beta_{n+1} z^2}{\ddots}}}}$$

where $\alpha_n(\cdot)$ has $\alpha_n(\mathbf{1})=2n+1$ and $\beta_n(\cdot)$ has $\beta_n(\mathbf{1})=n^2$

The Central Continued Fraction

For parameters $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$, let

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

where

$$\alpha_n = u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

and $\quad [n]_{x,y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$

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The Plan: Find a bijection taking *permutations*, carrying lots of statistics, to Motzkin paths corresponding to $\mathcal{C}(z)$, using Flajolet's general correspondence.

Consider Motzkin paths labeled as follows, where $0 \leq i < k$

- Upsteps from height $k-1$ to k have labels $pc^i d^{k-1-i}$
- Downsteps from height k to $k-1$ have labels $rh^i \ell^{k-1-i}$
- Level steps at height k have labels in

$$\{u \cdot w^i\} \ \cup \ \{s\, a^i b^{k-1-i}\} \ \cup \ \{t\, f^i g^{k-1-i}\}.$$

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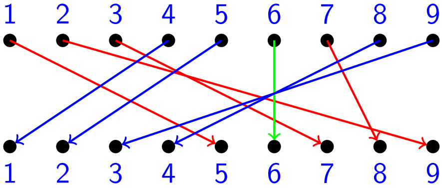
By Flajolet's correspondence, $\mathcal{C}(z)$ is the generating function for Motzkin paths thus labeled:

$$\mathcal{C}(z) = \cfrac{1}{\ddots \; 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\, z - \cfrac{p\, r\, [n+1]_{c,d}\, [n+1]_{h,\ell}\, z^2}{\ddots}}$$

Fourteen statistics on permutations $\sigma(1)\sigma(2)\ldots\sigma(n)$, based on *excedances* and *inversions*:

$\sigma(i)$: 5 9 7 1 2 6 8 4 3
i: 1 2 3 4 5 6 7 8 9



Excedances red

Anti-excedances blue

Fixed points green

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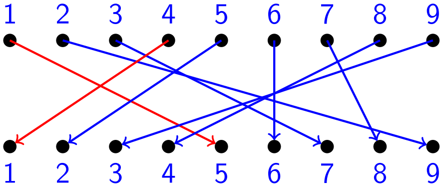

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One of the inversions red (crossing)

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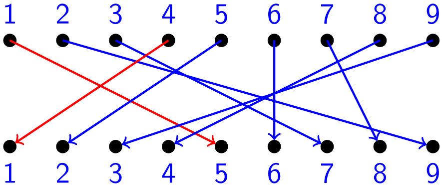

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But this gets more complicated ...

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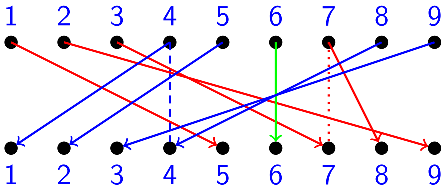

7 is a *linked* excedance: $8 = \sigma(7) > 7 > \sigma^{-1}(7) = 3$

5 9 7 1 2 6 8 4 3
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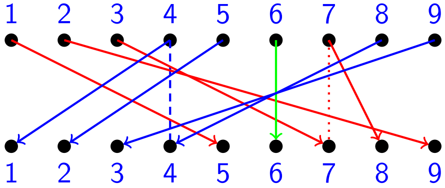

7 is a *linked* excedance: $8 = \sigma(7) > 7 > \sigma^{-1}(7) = 3$

4 is a *linked* anti-excedance: $1 = \sigma(4) < 4 < \sigma^{-1}(4) = 9$

5 9 7 1 2 6 8 4 3
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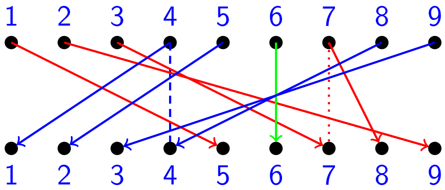

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$9 \cdots 6$ is an inversion between excedance and fixed point

1. # excedances as exc$(\sigma) := \#\{i \in [n] \mid i < \sigma(i)\}$,
2. # fixed points as fp$(\sigma) := \#\{i \in [n] \mid i = \sigma(i)\}$,
3. # anti-excedances as aexc$(\sigma) := \#\{i \in [n] \mid i > \sigma(i)\}$,
4. # linked excedances as le$(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) < i < \sigma(i)\}$,
5. # linked anti-excedances as lae$(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) > i > \sigma(i)\}$.
6. # inversions between excedances: ie$(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(j) < \sigma(i)\}$.
7. # inversions between excedances where the greater excedance is linked:ile$(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(j) < \sigma(i)$ and $\sigma^{-1}(j) < j\}$.
8. # restricted non-inversions between excedances:nie$(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(i) < \sigma(j)\}$.
9. # restricted non-inversions between excedances where the rightmost excedance is linked: nile$(\sigma) := \#\{i,j \in [n] \mid i < j < \sigma(i) < \sigma(j)$ and $\sigma^{-1}(j) < j\}$.
10. # inversions between anti-excedances: iae$(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(i) > \sigma(j)\}$.
11. # inversions between anti-excedances where the smaller anti-excedance is linked: ilae$(\sigma) := \#\{i,j \in [n] \mid j > i > \sigma(i) > \sigma(j)$ and $\sigma^{-1}(i) > i\}$.
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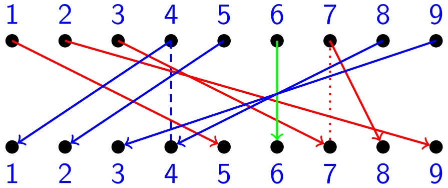
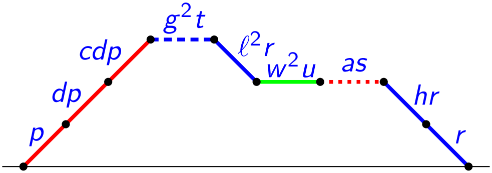
5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9

bijection

corresponding Motzkin path





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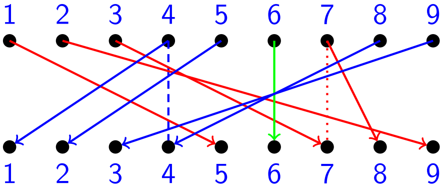

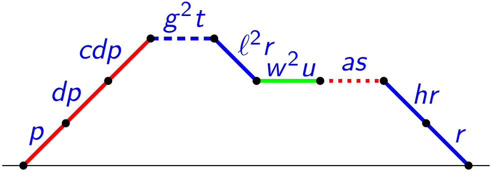

Weight of labeled Motzkin path, wt(M): Product of its labels

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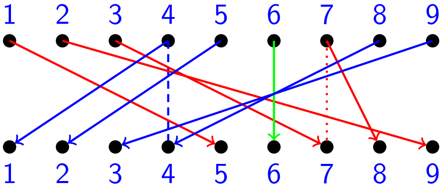

bijection

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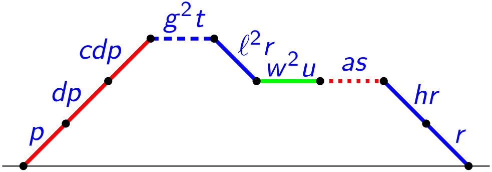

wt: $a \cdot c^2 \cdot d^2 \cdot g^2 \cdot h \cdot k^2 \cdot p^3 \cdot r^3 \cdot s \cdot t \cdot u \cdot w^2$

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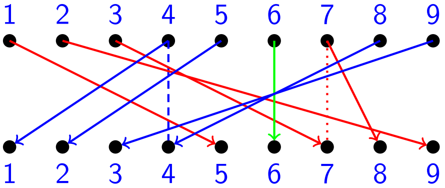

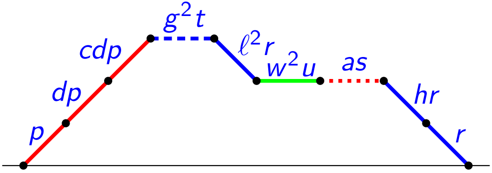

wt: $a \cdot c^2 \cdot d^2 \cdot g^2 \cdot h \cdot k^2 \cdot p^3 \cdot r^3 \cdot s \cdot t \cdot u \cdot w^2$

Above wt is one term in $[z^9]\mathcal{C}(z)$

The *weight* of a labeled Motzkin path M, $\mathrm{wt}(M)$, is the product of its labels.

Theorem: There is a bijection $\eta : \mathcal{S}_n \to \mathcal{M}_n$ such that if $M = \eta(\sigma)$ then $\mathrm{wt}(M)$ equals

$$\begin{aligned}\mathrm{stat}(\sigma) = \quad & a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} \\ & \times f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)} \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} \\ & \times p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}\end{aligned}$$

Corollary: $\mathcal{C}(z) = \sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_n} \mathrm{stat}(\sigma) z^n.$

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In short: Weight of Motzkin path goes to 14-parameter statistic on corresponding permutation

There are several related bijections in earlier literature by

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

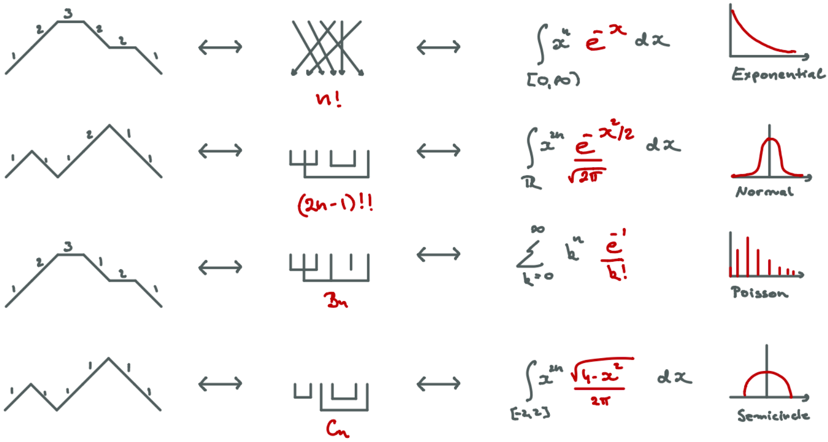
Our results generalize most of these, some modulo a bijection interchanging excedances and descents.

In a contemporaneous (yet unpublished) paper, Sokal and Zeng present a framework similar to ours, but with an additional four statistics, including some originally defined by Corteel.

Of the above, only Biane, Elizalde and Sokal-Zeng separate fixed points from anti-excedances, as we do. This leads to greater symmetry in the continued fraction, and to results not otherwise obtainable.

The number sequences arising from $\mathcal{C}$ enumerate many different combinatorial structures, such as permutations, perfect matchings and set partitions.

These basic examples happen to be moment sequences of important distributions from probability theory.



Some refinements of these objects also have meaning in probability theory.

Which structures give something probabilistically meaningful?

Moment sequences

A sequence $a_0, a_1, a_2, \ldots$ is a moment sequence of a positive measure on the real line *if and only if* all principal minors of

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

are non-negative for any n. (Hamburger, a 100 years ago)

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are non-negative for any n. (Hamburger, a 100 years ago)

In particular, $(a_n)_{n\geq 0}$ is then log-convex $\left(a_{n-1}a_{n+1} \geq a_n^2\right)$.

Moment sequences

A sequence $a_0, a_1, a_2, \ldots$ is a moment sequence of a positive measure on the real line *if and only if* all principal minors of

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

are non-negative for any n. (Hamburger, a 100 years ago)

In particular, $(a_n)_{n\geq 0}$ is then log-convex $\left(a_{n-1}a_{n+1} \geq a_n^2\right)$.

Can get strong lower bounds on growth rates of moment sequences (Haagerup–Haagerup–Ramirez-Solano, Elvey Price, Clisby–Conway–Guttmann)

$$\sum_{n\geq 0} m_n z^n \quad = \quad \mathcal{C}(z) = \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

$$\alpha_n = u\cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

Theorem: For $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$ with $pr > 0$ and c, d, h, ℓ satisfying

$$c = -d \quad \textbf{or} \quad h = -\ell \quad \textbf{or}$$
$$(c > -d \text{ and } h > -\ell) \quad \textbf{or} \quad (c < -d \text{ and } h < -\ell),$$

the sequence (m_n) is the moment sequence of some probability measure on $\mathbb{R}$. In particular if all non-negative and $pr > 0$.

Moment sequences

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With mild conditions on the parameters of $\mathcal{C}(z)$, which are easy to check, we get moment sequences.

All sequences mentioned from now on are moment sequences arising from $\mathcal{C}(z)$.

$$\mathcal{C}(z) = \cfrac{1}{\ddots \; 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\,z - \cfrac{p\,r\,[n+1]_{c,d}\,[n+1]_{h,\ell}\,z^2}{\ddots}}$$

With $s = qx$, $p = x$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{occ}_{321}(\sigma)} z^n,$$

where occ_{321} is #occurrences of the consecutive pattern 321

occurrence: 356412 not consecutive: 356412

First shown by Elizalde 2018, using a different continued fraction.

$$\mathcal{C}(z) = \cfrac{1}{\ddots \, 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\, z - \cfrac{p\, r\,[n+1]_{c,d}\,[n+1]_{h,\ell}\, z^2}{\ddots}}$$

With $s = 0$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \mathrm{Av}_{321}(n) z^n,$$

$\mathrm{Av}_{321}(n) = \#$ n-permutations avoiding consecutive pattern 321

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If $b, d, g, \ell = q$, $s = xq$, $p, u = x$, others $= 1$:

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{des}(\sigma)+1} q^{\mathrm{occ}_{2-31}(\sigma)} z^n.$$

where occ_{2-31} is #occurrences of the *vincular* pattern 2-31

2-31 occurrence: 416523 62 not adjacent: 416523

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Two more cases: Catalan and Bell numbers, both moment sequences

1–2–3   1–23

The only 3-pattern whose avoiders don't give a moment sequence is the consecutive pattern 132 (equivalently 213, 231, 312).

This is the only 3-pattern whose avoidance is not captured in $\mathcal{C}(z)$.

Theorem: The sequence of numbers of avoiders of a pattern of length 3 is a moment sequence *iff* it is a special case of $\mathcal{C}(\mathrm{z})$.

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Why are some combinatorial sequences moment sequences?

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Why are some combinatorial sequences moment sequences?

What tools from probability/analysis would it let us use?

Specializations of $\mathcal{C}(z)$ also capture a large part of the q-Askey scheme of orthogonal polynomials, here interpreted in terms of the simple concepts of excedances and inversions in permutations.

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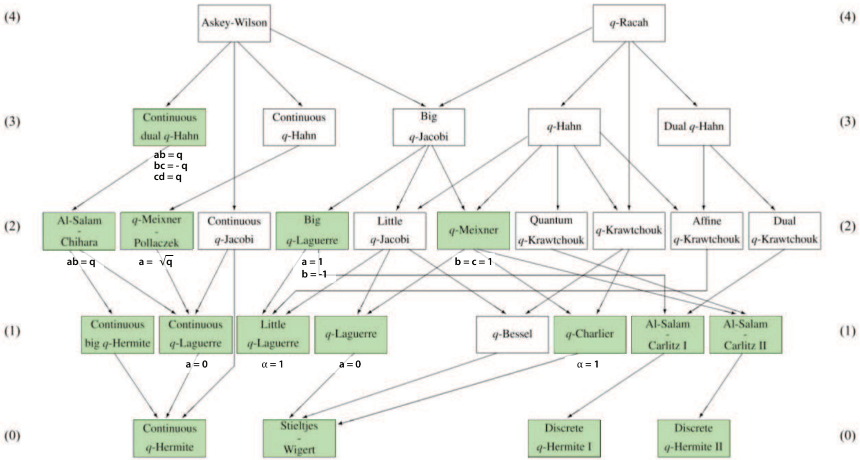
Corteel and Williams have a combinatorial interpretation with statistics on different objects (staircase tableaux) for all polynomials that specialize from the Askey-Wilson family.

SCHEME

OF

BASIC HYPERGEOMETRIC

ORTHOGONAL POLYNOMIALS



Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the *k-colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups.

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Proposition: With $s, p = kx,\ t, r = ky,\ u = (k-1)x + q$, and all other parameters set to 1, we get

$$\mathcal{C}(z) \ = \ \sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_n^k} x^{\mathrm{exc}(\sigma)} y^{\mathrm{aexc}(\sigma)} q^{\mathrm{fix}(\sigma)} z^n.$$

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Easy to refine this to distinguish linked/unlinked (anti-)excedances, because the colors embed naturally in $\mathcal{C}(z)$.

Coloring only fixed points

Because fixed points live independently in $\mathcal{C}(z)$, the following generalization is obvious:

k-arrangements: Permutations with k-colored fixed points

- 0-arrangements are derangements (no fixed points)
- 1-arrangements are permutations
- 2-arrangements were called just *arrangements* by Comtet, and coincide with Postnikov's *decorated permutations*, which are in bijection with 'certain non-negative Grassmann cells'.

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But they have many nice properties, and doubtless many more to be discovered.

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

- $A_k(0) = 1$. For $n > 0$: $A_k(n) = n \cdot A_k(n-1) + (k-1)^n$.

- $$\sum_{n\geq 0} A_k(n)\frac{x^n}{n!} = \frac{e^{(k-1)x}}{(1-x)}.$$

- $$A_k(n) = \sum_{i\geq 0}\binom{n}{i}A_{k-1}(i) \qquad \text{(successive binomial transforms)}$$

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All of this holds for $k < 0$. What does that count?

The definitions of *descents*, *excedances*, *inversions* and *major index* for colored permutations adapt trivially to k-arrangements.

Conjecture: On k-arrangements of $[n]$, excedances and descents are equidistributed, as are inversions and major index.

Encoding k-arrangements

Replacing fixed points colored i (resp. $i < k$) by $-i$ gives the *derangement (resp. permutation) form* of a k-arrangement.

Conjecture: des has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

Conjecture: The number of 3-arrangements of $[n]$ whose permutation form avoids any single pattern of length 3 is $C(n+2) - 2^n$.

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Proposition: The number of 3-arrangements of $[n]$ whose permutation form avoids any single pattern of length 3 is $C(n+2) - 2^n$.

Proved by Fu-Han-Lin. Surprisingly non-trivial.

Thanks!

N. Blitvić and E. Steingrímsson: Permutations, moments, measures

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