The Immaculate Hecke Poset
AlCoVE 2022 June 7

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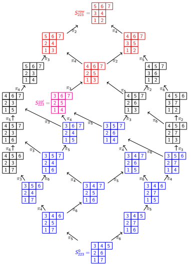


Algebraic Combinatorics Research Community project (ICERM) with

Elizabeth Niese, *Stephanie van Willigenburg*, *Julianne Vega*

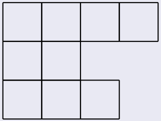
Compositions

Definition

A composition α of n is a sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $\sum_{i=1}^{k} \alpha_i = n$. Write $\alpha \vDash n$.
The diagram of α is a collection of **left-justified** boxes with α_i boxes in row i, where row 1 is the **bottom** row –"French" or Cartesian style.

Example

For $\alpha = (3, 2, 4) \vDash 9$, the diagram is



Standard Immaculate Tableaux (BBSSZ 2014)

Definition (Berg-Bergeron-Saliola-Serrano-Zabrocki 2014)

A standard immaculate tableau (SIT) of shape $\alpha \vDash n$ is a filling of the diagram of α with the n distinct entries $\{1, 2, \ldots, n\}$, such that

1. The leftmost column increases bottom to top.
2. All rows increase, left to right.

Example

$$\alpha = (3, 2, 4), \quad T = \begin{array}{|c|c|c|c|}\hline 4 & 5 & 8 & 9 \\ \hline 3 & 7 \\ \cline{1-2} 1 & 2 & 6 \\ \hline \end{array}.$$

The row-strict immaculate descent set and a partial order

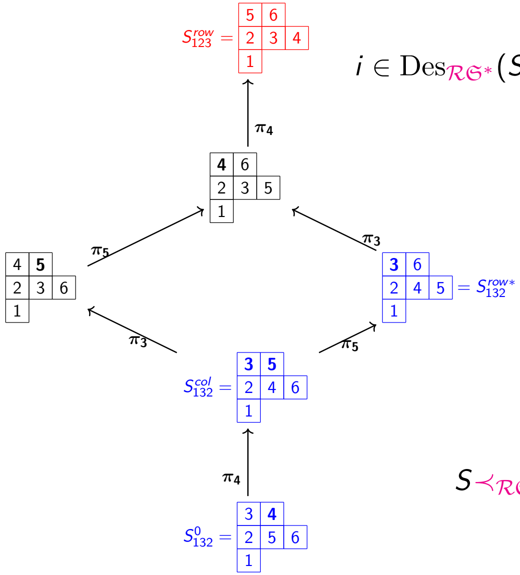
Definition (NSvWVW 2022)

The $\mathcal{RS}^*$-descent set $\mathrm{Des}_{\mathcal{RS}^*}(S)$ of a standard immaculate tableau S is

$$\mathrm{Des}_{\mathcal{RS}^*}(S) := \{i : i+1 \text{ appears weakly below } i \text{ in } S\}.$$

Theorem (NSvWVW 2022)

This defines a partial order $S \prec_{\mathcal{RS}^*_\alpha} T \iff T = s_i(S)$ on $\mathrm{SIT}(\alpha)$, where $s_i(S)$ is the operator switching i and $i+1$ in S.



$$i \in \mathrm{Des}_{\mathcal{RS}^*}(S) \iff i+1 \text{ weakly below } i.$$

$$S \prec_{\mathcal{RS}^*_\alpha} T \iff T = s_i(S).$$

Compositions of $n \leftrightarrow$ Subsets of $\{1, 2, \ldots, n-1\}$

Compositions α of n map to subsets of $[n-1] = \{1, 2, \ldots, n-1\}$.

For $\alpha = (3, 1, 4) \vDash 8$, $S = \text{set}(\alpha) = \{3, 4\} \subseteq \{1, 2, \ldots, 7\}$.

If $S = \{1, 4, 5\} \subset [7]$, then $\text{comp}(S) = (1, 3, 1, 3) \vDash 8$.

Compositions are partially ordered by refinement: β refines α if each part of α is a sum of consecutive parts of β.

Notice β refines $\alpha \iff \text{set}(\beta) \supseteq \text{set}(\alpha)$.

$\beta = (1, 2, 1, 4)$ refines $\alpha = (3, 1, 4)$: $\text{set}(\beta) = \{1, 3, 4\}$.

(Ira Gessel 1984) Quasisymmetric functions

A formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is quasisymmetric if for every $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $i_1 < i_2 < \cdots < i_k$,

$$\text{the coefficient of } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$$

is the same as

$$\text{the coefficient of } x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

Example: $\sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots$ is quasisymmetric (but not symmetric).

The set of all quasisymmetric functions forms an algebra graded by degree, $\mathrm{QSym} = \bigoplus_n \mathrm{QSym}_n$, where each QSym_n is a vector space over $\mathbb{Q}$ with bases indexed by compositions of n.

Quasisymmetric functions indexed by compositions

Definition ($\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition of n.)

The monomial quasisymmetric function indexed by α is

$$M_\alpha = \sum_{\substack{(i_1, i_2, \ldots, i_k) \\ i_1 < i_2 < \ldots < i_k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

The fundamental quasisymmetric function indexed by α is

$$F_\alpha := \sum_{\beta \text{ refines } \alpha} M_\beta.$$

Example

$M_{(2,1,1)} = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + x_2^2 x_3 x_4 + \cdots,$
$F_{(2,1,1)} = x_1 x_2 x_3 x_4 + x_1^2 x_2 x_3 + x_2^2 x_3 x_4 + \cdots = M_{(2,1,1)} + M_{(1,1,1,1)}.$
$F_{(2,2)} = M_{(2,2)} + M_{(1,1,2)} + M_{(2,1,1)} + M_{(1,1,1,1)}.$

FACT: $\{M_\alpha : \alpha \vDash n\}$ and $\{F_\alpha : \alpha \vDash n\}$ are bases for QSym_n, the **monomial** basis and the **fundamental** basis.

(Berg-Bergeron-Saliola-Serrano-Zabrocki 2014) Dual Immaculate Basis for QSym

Definition

Given $\alpha \vDash n$, an immaculate tableau of shape α is a filling D of the cells of the diagram of α such that

1. The leftmost column increases strictly, bottom to top.
2. All rows increase weakly, left to right.

Let $x^D := x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}$, where d_i is the number of i's in the tableau D; $(d_1, \ldots, d_k)$ be the content (composition) of D.

The dual immaculate function indexed by $\alpha \vDash n$ is

$$\mathfrak{S}^*_\alpha = \sum_D x^D,$$

summed over all immaculate tableaux D of shape α.

Theorem: It is a basis of QSym_n**.**

$$\mathfrak{S}^*_{12} : \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} + \cdots$$

(NSvWVW 2022) Row-strict Dual Immaculate Basis

Definition (Niese-S-vanWilligenburg-Vega-Wang 2022)

Given $\alpha \vDash n$, a row-strict immaculate tableau of shape α is a filling D of the cells of the diagram of α such that

1. The leftmost column increases weakly, bottom to top.
2. The rows increase strictly, left to right.

Let $x^D := x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}$; d_i is the number of i's in the tableau D.

The row-strict dual immaculate function indexed by $\alpha \vDash n$ is

$$\mathcal{R}\mathfrak{S}^*_\alpha = \sum_D x^D,$$

summed over all row-strict immaculate tableaux D of shape α.

Theorem: It is a basis of QSym_n.

$$\mathcal{R}\mathfrak{S}^*_{12} : \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} + \cdots$$

$$\mathfrak{S}^*_{12} : \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} + \cdots$$

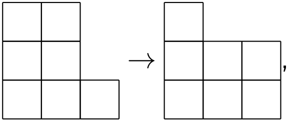
The involution ψ

Definition

There is an involutive algebra automorphism $\psi : \mathrm{QSym} \to \mathrm{QSym}$ defined by $\psi(F_\alpha) = F_{\alpha^c}$, where set$(\alpha^c)$ is the complement in $[n-1]$ of set(α).

$$\mathcal{R}\mathfrak{S}^*_\alpha = \psi(\mathfrak{S}^*_\alpha).$$

Recall: For Schur functions, $s_{\lambda^t} = \psi(s_\lambda)$, where λ^t is the *transpose* of λ: reflect the Ferrers diagram about the line $y = x$: e.g.



$(3, 2, 2) \to (3, 3, 1)$.

Descents, standard and semistandard tableaux – recap

For a standard immaculate tableau S:

[BBSSZ 2014] The $\mathfrak{S}^*$-descent set $\mathrm{Des}_{\mathfrak{S}^*}(S)$ is

$$\mathrm{Des}_{\mathfrak{S}^*}(S) := \{i : i+1 \text{ appears strictly above } i \text{ in } S\}.$$

[NSvWVW 2022] The $\mathcal{R}\mathfrak{S}^*$-descent set $\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(S)$ is

$$\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(S) := \{i : i+1 \text{ appears weakly below } i \text{ in } S\}.$$

Tableaux	s_λ Schur	$\mathfrak{S}^*_\alpha$ Dual immaculate	$\mathcal{R}\mathfrak{S}^*_\alpha$ Row-strict dual imm.
Columns bottom to top	strict $\nearrow$ **ALL Columns**	strict $\nearrow$ **Only 1st column**	weak $\nearrow$ **Only 1st column**
All Rows left to right	weak $\nearrow$	weak $\nearrow$	strict $\nearrow$
Descents for standard tableaux	i such that $i+1$ strictly above i	i such that $i+1$ strictly above i	i such that $i+1$ weakly below i
$(2,2)$	$\begin{array}{cc} 3 & 4 \\ 1 & \mathbf{2} \end{array}$, $\begin{array}{cc} 2 & 4 \\ \mathbf{1} & \mathbf{3} \end{array}$	$\begin{array}{cc} 3 & 4 \\ 1 & \mathbf{2} \end{array}$, $\begin{array}{cc} 2 & 4 \\ \mathbf{1} & \mathbf{3} \end{array}$, $\begin{array}{cc} 2 & 3 \\ \mathbf{1} & 4 \end{array}$	$\begin{array}{cc} \mathbf{3} & 4 \\ \mathbf{1} & 2 \end{array}$, $\begin{array}{cc} \mathbf{2} & 4 \\ 1 & 3 \end{array}$, $\begin{array}{cc} \mathbf{2} & \mathbf{3} \\ 1 & 4 \end{array}$
	$F_{(2,2)} + F_{(1,2,1)}$	$F_{(2,2)} + F_{(1,2,1)} + F_{(1,3)}$	$F_{(1,2,1)} + F_{(2,2)} + F_{(2,1,1)}$

The 0-Hecke algebra I

Definition

Let S_n be the symmetric group, and $\mathbb{K}$ any field. The 0-Hecke algebra $H_n(0)$ is the $\mathbb{K}$-algebra with generators $\pi_i, 1 \leq i \leq n-1$ and relations

$$\pi_i{}^2 = \pi_i; \quad \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}; \quad \pi_i\pi_j = \pi_j\pi_i, \ |i-j| \geq 2.$$

The algebra $H_n(0)$ has dimension $n!$, with basis $\{\pi_\sigma : \sigma \in S_n\}$, where $\pi_\sigma = \pi_{i_1} \cdots \pi_{i_m}$ if $\sigma = s_{i_1} \cdots s_{i_m}$ is a reduced word. The s_i are the simple reflections in the symmetric group S_n.

Theorem (Pamela Bromwich Norton 1979)

The 0-Hecke algebra admits exactly 2^{n-1} simple modules L_α, one for each composition $\alpha \vDash n$, all one-dimensional.

The 0-Hecke algebra II

For finite-dimensional $H_n(0)$-modules M, there is an analogue of the Frobenius characteristic defined on the Grothendieck ring of the symmetric groups :

$$M \mapsto \mathrm{ch}(M) = \sum_{\alpha \in \mathcal{C}(M)} F_\alpha \in \mathrm{QSym},$$

where $\mathcal{C}(M)$ is some unique subset of compositions associated to the module M.

Definition (Krob & Thibon 1997)

Let M be a finite-dimensional $H_n(0)$-module, with composition series $M = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = \mathbb{K}$ of submodules for M. Hence each successive quotient M_i/M_{i+1} is simple, and thus equal to L_{α^i} for some composition $\alpha^i \vDash n$. The quasisymmetric characteristic ch of the module M, $\mathrm{ch}(M)$, is then defined as

$$\mathrm{ch}(M) := \sum_{i=1}^{k} F_{\alpha^i}.$$

In particular $\mathrm{ch}(L_\alpha) = F_\alpha$ for each $\alpha \vDash n$.

Descent sets and 0-Hecke modules

Given: $\alpha \vDash n$, a set of standard tableaux $ST(\alpha)$ of shape α, and some notion of "descent" defined on these tableaux.

Let $s_i(T)$ be the operator switching i and $i+1$ in $T \in ST(\alpha)$. Try to define the action of the generator π_i on $T \in ST(\alpha)$ by

$$\pi_i(T) = \begin{cases} T & \text{if } i \text{ is NOT a descent of } T, \\ s_i(T) & \text{if } i \text{ is a descent of } T \text{ and } (*), \\ 0 & \text{otherwise.} \end{cases}$$

Under suitable (*) conditions, the generators will satisfy the Hecke relations, thereby defining a 0-Hecke module with basis $ST(\alpha)$.

These propitious circumstances occur for $\mathrm{Des}_{\mathfrak{S}^*}$ and $\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}$.

Partial orders on $\mathrm{SIT}(\alpha)$ from the 0-Hecke actions

The cover relation for the poset $P\mathfrak{S}^*_\alpha$ described by Berg-Bergeron-Saliola-Serrano-Zabrocki (2015), is

$$S \prec_{\mathfrak{S}^*_\alpha} T \iff \exists\, i \text{ such that } S = \pi_i^{\mathfrak{S}^*}(T).$$

with respect to the **immaculate** 0-Hecke action.

The cover relation for the poset $P\mathcal{R}\mathfrak{S}^*_\alpha$ is

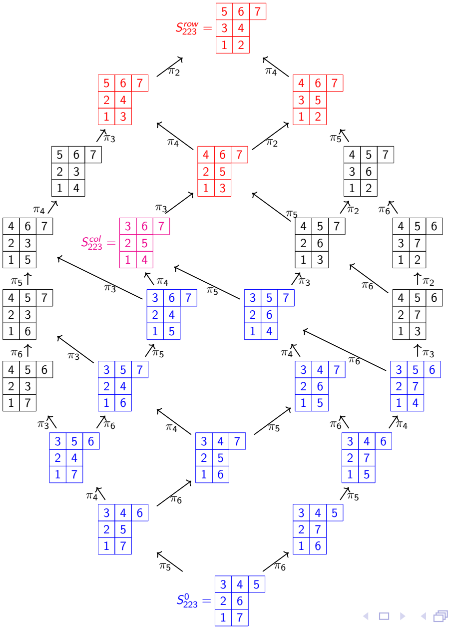
$$S \prec_{\mathcal{R}\mathfrak{S}^*_\alpha} T \iff \exists\, i \text{ such that } T = \pi_i^{\mathcal{R}\mathfrak{S}^*}(S),$$

with respect to the **row-strict immaculate** 0-Hecke action.

Lemma (NSvWVW 2022)

Let $\alpha \vDash n$ and $S, T \in \mathrm{SIT}(\alpha)$. Then $S \prec_{\mathfrak{S}^*_\alpha} T \iff S \prec_{\mathcal{R}\mathfrak{S}^*_\alpha} T$. Hence the two posets $P\mathfrak{S}^*_\alpha$ and $P\mathcal{R}\mathfrak{S}^*_\alpha$ are isomorphic; we call $P\mathcal{R}\mathfrak{S}^*_\alpha$ the immaculate Hecke poset.

The immaculate Hecke poset with the $\mathcal{RS}^*$-action



The immaculate Hecke poset is bounded and ranked

Two key straightening algorithms show that

Theorem (NSvWVW 2022)

The poset $\mathcal{PR}\mathfrak{S}^*_\alpha$ is ranked, with a unique bottom element S^0_α and a unique top element S^{row}_α. Also, for any $T \in \mathrm{SIT}(\alpha)$, there are saturated chains from S^0_α to T, and from T to S^{row}_α.
The rank of T is determined by the number of inversions of the permutation $\sigma(T)$, reading T right to left in each row, top to bottom. The rank of $\mathcal{PR}\mathfrak{S}^*_\alpha$ is $\binom{|\alpha|}{2} + \binom{\ell(\alpha)}{3} - \sum_{i=1}^{\ell(\alpha)} \binom{\alpha_i+(i-1)}{2}$.

$S^{row}_{223} =$

6	7	
3	4	5
1	2	

$\Rightarrow \sigma(S^{row}_{223}) = 7\,6\,5\,4\,3\,2\,1$, with 21 inversions.

$S^0_{223} =$

3	4	5
2	6	
1	7	

$\Rightarrow \sigma(S^0_{223}) = 5\,4\,3\,6\,2\,7\,1$, with 13 inversions.

The size of the Hecke poset: dimension formula

$\mathrm{SIT}(\alpha) := \{T \mid T \text{is a standard immaculate tableau of shape } \alpha\}.$

Proposition (BBSSZ 2014)

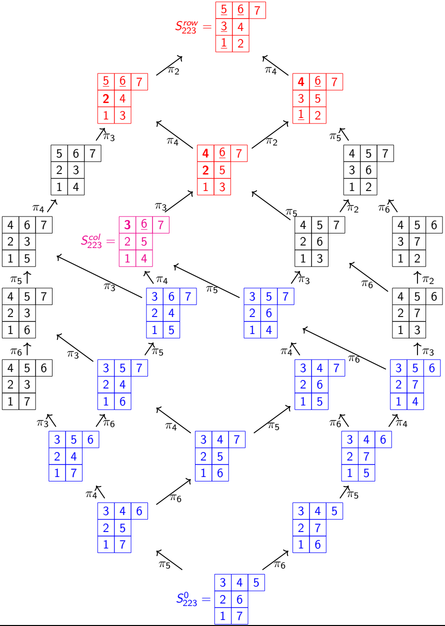
Let $\alpha \vDash n$ have ℓ parts.

$$|\mathrm{SIT}(\alpha)| = \frac{(n-1)!}{\prod_{i=1}^{\ell-1}(n - \sum_{j=1}^{i} \alpha_j) \cdot \prod_{i=1}^{\ell}(\alpha_i - 1)!}.$$

The BBSSZ formula says $|\mathrm{SIT}(223)| = \frac{7!}{(7)(5)(3)(2!)} = 24.$

Comp. α	132	223	232	323	324
$\lvert\mathrm{SIT}(\alpha)\rvert$	6	24	36	84	140
rank($\mathcal{PR}\mathfrak{S}_{\alpha}^{*}$)	4	8	9	13	16

Immaculate Hecke poset with $\mathcal{RS}^*$-action redux



- (Theorem, NSvWVW 2022) The minimal tableau S^0_{223} generates the indecomposable 0-Hecke module for the $\mathcal{RS}^*$-action on SIT(223).
- (Theorem, BBSSZ 2015) *(By reversing arrows)* The row superstandard tableau S^{row}_{223} generates the indecomposable 0-Hecke module for the $\mathfrak{S}^*$-action on SIT(223).

0-Hecke modules for $\mathfrak{S}^*_\alpha$ and $\mathcal{R}\mathfrak{S}^*_\alpha$

Theorem (Berg-Bergeron-Saliola-Serrano-Zabrocki 2015)

There is an indecomposable cyclic 0-Hecke algebra module $\mathcal{W}_\alpha = \langle S^{row}_\alpha \rangle$ whose quasisymmetric characteristic is the dual immaculate function $\mathfrak{S}^*_\alpha$: $\mathrm{ch}(\mathcal{W}_\alpha) = \mathfrak{S}^*_\alpha$. The $\mathfrak{S}^*$-action of the 0-Hecke algebra on the set $\mathrm{SIT}(\alpha)$ is defined via the $\mathfrak{S}^*$-descent set

$$\mathrm{Des}_{\mathfrak{S}^*}(S) := \{i : i+1 \text{ appears strictly above } i \text{ in } S\}.$$

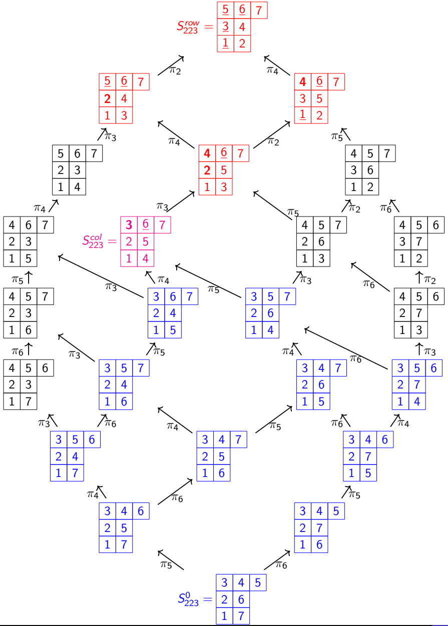
Theorem (Niese-S-vanWilligenburg-Vega-Wang 2022)

There is an indecomposable cyclic 0-Hecke algebra module $\mathcal{V}_\alpha = \langle S^0_\alpha \rangle$ whose quasisymmetric characteristic is the row-strict dual immaculate function $\mathcal{R}\mathfrak{S}^*_\alpha$: $\mathrm{ch}(\mathcal{V}_\alpha) = \mathcal{R}\mathfrak{S}^*_\alpha$. The $\mathcal{R}\mathfrak{S}^*$-action of the 0-Hecke algebra is defined via the $\mathcal{R}\mathfrak{S}^*$-descent set

$$\mathrm{Des}_{\mathcal{R}\mathfrak{S}^*}(S) := \{i : i+1 \text{ appears weakly below } i \text{ in } S\}.$$

(*) Indecomposability thanks to straightening algorithms.

0-Hecke submodules from the Hecke poset



SET(α) is the set of tableaux in SIT(α) with ALL columns increasing. It is a basis for two modules:

The column superstandard tableau S_{223}^{col} generates an indecomposable *submodule* for the $\mathcal{R}\mathfrak{S}^*$-action on SIT(223).

Reverse arrows:

$\text{Span}(\text{SIT}\backslash\text{SET})$ is invariant under $\mathfrak{S}^*$-action, and hence:

The row superstandard tableau S_{223}^{row} generates an indecomposable *quotient* module for the $\mathfrak{S}^*$-action on SIT(223).

Row-strict extended Schur functions

Theorem (NSvWVW 2022)

The set $\mathrm{SET}(\alpha)$ *is closed under the* $\mathcal{R}\mathfrak{S}^*$*-Hecke action, and*

- *coincides with interval* $[S^{col}_\alpha, S^{row}_\alpha]$ *of the poset* $\mathcal{PR}\mathfrak{S}^*(\alpha)$,
- *spans an indecomposable cyclic submodule* $\mathcal{Z}_\alpha$ *of* $\mathcal{V}_\alpha = \langle S^0_\alpha \rangle$, *generated by* S^{col}_α, *with characteristic* $\mathcal{RE}_\alpha$.
- *spans an indecomposable cyclic quotient module* $\langle S^{\mathrm{row}}_{223} \rangle / \langle SIT(\alpha) \setminus \mathrm{SET}(\alpha) \rangle$ *of* $\mathcal{W}_\alpha = \langle S^{\mathrm{row}}_{223} \rangle$, *generated by* S^{row}_α, *with characteristic* $\mathcal{E}_\alpha$.
- $\mathcal{R}\mathfrak{S}^*_\alpha - \mathcal{RE}_\alpha$, $\mathfrak{S}^*_\alpha - \mathcal{E}_\alpha$ *have associated indecomposable cyclic 0-Hecke modules.*

The sets $\{\mathcal{RE}_\alpha\}_{\alpha \vDash n}$ *(resp.* $\{\mathcal{E}_\alpha\}_{\alpha \vDash n}$*) are bases for* QSym_n *with the property that when* α *is a partition* λ, $\mathcal{RE}_\alpha$ *(resp.* $\mathcal{E}_\alpha$*) coincides with the Schur function* s_{λ^t} *(resp.* s_λ*).*

(*) Indecomposability thanks to straightening algorithms.

Generating functions

Theorem (NSvWVW 2022)

The row-strict extended Schur function $\mathcal{RE}_\alpha$ is the generating function for tableaux of shape α with all rows strictly increasing, and ALL columns weakly increasing.

Theorem (Campbell-Feldman-Light-Shuldiner-Xu 2014)

The extended Schur function $\mathcal{E}_\alpha$ is the generating function for tableaux of shape α with all rows weakly increasing, and ALL columns strictly increasing. $\mathcal{E}_\alpha$ is the dual to the shin basis of the noncommutative symmetric functions.
When $\alpha = \lambda$ is a partition, $\mathcal{E}_\alpha$ is the Schur function s_λ.

The extended Schur function $\mathcal{E}_\alpha$ was obtained in a different way by Assaf-Searles (2019), and a 0-Hecke module was constructed (also differently) by Searles (2020).

Kostka numbers and descent sets – I

Let $K^{\mathfrak{S}^*}_{\alpha,\beta}$ (resp. $K^{\mathcal{R}\mathfrak{S}^*}_{\alpha,\beta}$) denote the number of immaculate (resp. row-strict immaculate) tableaux of shape α and content β. Let $K^{\mathcal{E}}_{\alpha,\beta}$ (resp. $K^{\mathcal{R}\mathcal{E}}_{\alpha,\beta}$) denote the sizes of the respective subsets where ALL columns increase strictly (resp.weakly).

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda,\mu} m_\mu, \text{ (ALL cols } <, \text{rows} \leq)$$

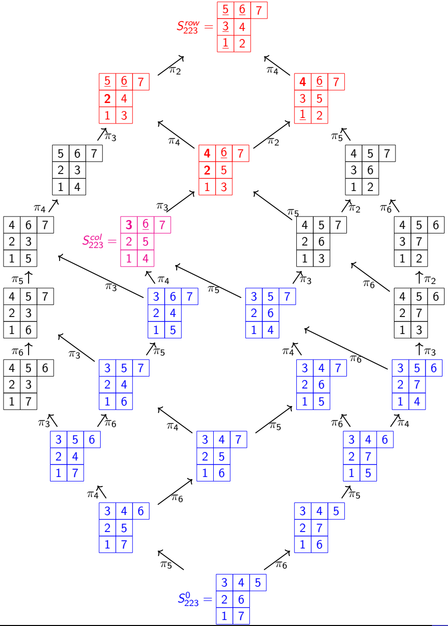
(*BBSSZ* 2014) $\mathfrak{S}^*_\alpha = \sum_{\beta \trianglelefteq \alpha} K^{\mathfrak{S}^*}_{\alpha,\beta} M_\beta,$ (1st col $<$, rows $\leq$)

(Campbell-Feldman-Light-Shuldiner-Xu 2014) (Kostka numbers)

(*Assaf & Searles* 2019) $\mathcal{E}_\alpha = \sum_{\beta \trianglelefteq \alpha} K^{\mathcal{E}}_{\alpha,\beta} M_\beta,$ (ALL cols $<$, rows $\leq$)

(*NSvWVW* 2022) $\mathcal{R}\mathfrak{S}^*_\alpha = \sum_{\beta} K^{\mathcal{R}\mathfrak{S}^*}_{\alpha,\beta} M_\beta,$ (1st col $\leq$, rows $<$)

(*NSvWVW* 2022) $\mathcal{R}\mathcal{E}_\alpha = \sum_{\beta} K^{\mathcal{R}\mathcal{E}}_{\alpha,\beta} M_\beta.$ (ALL cols $\leq$, rows $<$)



$\mathrm{SIT}^*(\alpha)$ is the set of tableaux in $\mathrm{SIT}(\alpha)$ with first column consisting of the smallest integers; $S^{\mathrm{row}*}_\alpha \in \mathrm{SIT}^*(\alpha)$ has its remaining cells filled consecutively along rows, bottom to top, left to right.

$$S^{row*}_{223} = S^{col}_{223};\ S^{row*}_{332} = \begin{array}{|c|c|c|}\hline 3 & 8 & \\ \hline 2 & 6 & 7 \\ \hline 1 & 4 & 5 \\ \hline \end{array} \neq S^{col}_{332}.$$

$\langle S^{\mathrm{row}*}_{223} \rangle$ is an invariant submodule of the $\mathfrak{S}^*$-action, with basis $\mathrm{SIT}^*(223)$. The quotient $\langle S^{\mathrm{row}}_{223} \rangle / \langle S^{\mathrm{row}*}_{223} \rangle$ is cyclic and indecomposable.

More descents

Definition

Let $\mathrm{Des}_{\mathcal{A}^*}(T) := \{i : i+1 \text{ is strictly below } i \text{ in } T\}$, $T \in \mathsf{SIT}(\alpha)$.
Let $\mathrm{Des}_{\bar{\mathcal{A}}^*}(T) := \{i : i+1 \text{ is weakly above } i \text{ in } T\}$, $T \in \mathsf{SIT}(\alpha)$.

Theorem (NSvWVW 2022)

There is a cyclic $H_n(0)$-module $\mathcal{A}^*_\alpha$, generated by the least element S^0_α of the poset $P\mathcal{R}\mathfrak{S}^*(\alpha)$, with quasisymmetric characteristic

$$\mathrm{ch}(\mathcal{A}^*_\alpha) = \sum_{T \in \mathsf{SIT}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{A}^*}(T))}.$$

There is a cyclic $H_n(0)$-module $\bar{\mathcal{A}}^*_\alpha$, generated by the top element S^{row}_α of the poset $P\mathcal{R}\mathfrak{S}^*(\alpha)$, with quasisymmetric characteristic

$$\mathrm{ch}(\bar{\mathcal{A}}^*_\alpha) = \sum_{T \in \mathsf{SIT}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\bar{\mathcal{A}}^*}(T))}.$$

Note that, as is the case with $\mathfrak{S}^*_\alpha$ and $\mathcal{R}\mathfrak{S}^*_\alpha$, the two characteristics are related by the involution ψ : $\psi(\mathrm{ch}(\bar{\mathcal{A}}^*_\alpha)) = \mathrm{ch}(\mathcal{A}^*_\alpha)$.

Partial order on SIT(α) from the $\mathcal{A}^*$- and $\bar{\mathcal{A}}^*$-actions

Proposition

All four actions are captured in the same Hecke poset.

NSvWVW 2022: Kostka numbers and descent sets – II

Let $K_{\alpha,\beta}^{\mathcal{A}^*}$ (resp. $K_{\alpha,\beta}^{\bar{\mathcal{A}}^*}$) denote the number of tableaux with first column and all rows weakly increasing (resp. strictly increasing), of shape α and content β. Let $K_{\alpha,\beta}^{\mathcal{A}_{\mathsf{SET}(\alpha)}}$ (resp. $K_{\alpha,\beta}^{\bar{\mathcal{A}}_{\mathsf{SET}(\alpha)}}$) denote the sizes of the respective subsets with ALL columns and all rows weakly increasing (resp. strictly increasing).

$$\mathrm{ch}(\mathcal{A}_\alpha^*) = \sum_{\beta} K_{\alpha,\beta}^{\mathcal{A}^*} M_\beta, \text{ (1st col } \leq, \text{rows } \leq)$$

$$\mathrm{ch}(\bar{\mathcal{A}}_\alpha^*) = \sum_{\boldsymbol{\beta} \trianglelefteq \boldsymbol{\alpha}} K_{\alpha,\beta}^{\bar{\mathcal{A}}^*} M_\beta, \text{ (1st col } <, \text{rows } <)$$

$$\mathrm{ch}(\mathcal{A}_{\mathsf{SET}(\alpha)}) = \sum_{\beta} K_{\alpha,\beta}^{\mathcal{A}_{\mathsf{SET}(\alpha)}} M_\beta, \text{ (ALL cols } \leq, \text{rows } \leq)$$

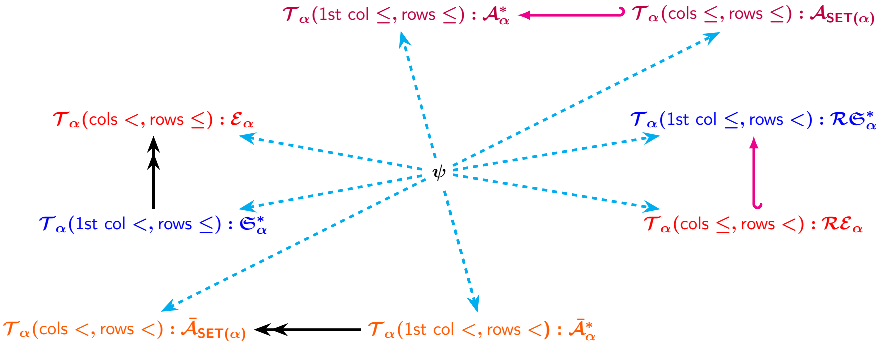
$$\mathrm{ch}(\bar{\mathcal{A}}_{\mathsf{SET}(\alpha)}) = \sum_{\boldsymbol{\beta} \trianglelefteq \boldsymbol{\alpha}} K_{\alpha,\beta}^{\bar{\mathcal{A}}_{\mathsf{SET}(\alpha)}} M_\beta. \text{ (ALL cols } <, \text{rows } <)$$

All descent sets and resulting tableaux

The unified combinatorial picture for the semistandard tableaux:

Tableaux SIT(α)	$\mathfrak{S}^*_\alpha$ Dual immaculate BBSSZ 2015	$\mathcal{RS}^*_\alpha$ **Row-strict dual imm.** NSvWVW 2022	$\mathcal{A}^*_\alpha$ ch $(\mathcal{A}^*_\alpha)$ NSvWVW 2022	$\bar{\mathcal{A}}^*_\alpha$ ch $(\bar{\mathcal{A}}^*_\alpha)$ NSvWVW 2022
1st Col bottom to top	strict $\nearrow$	weak $\nearrow$	weak $\nearrow$	strict $\nearrow$
Rows left to right	weak $\nearrow$	strict $\nearrow$	weak $\nearrow$	strict $\nearrow$
Descents for standard tableaux	i such that $i+1$ strictly above i	i such that **$i+1$ weakly below i**	i such that **$i+1$ strictly below i**	i such that **$i+1$ weakly above i**
$(2,2)$	[3 4 / 1 **2**], [2 4 / **1** **3**], [2 3 / **1** 4]	[**3** 4 / **1** 2], [**2** 4 / 1 3], [**2** **3** / 1 4]	[3 4 / 1 2], [**2** 4 / 1 3], [2 **3** / 1 4]	[**3** 4 / **1** **2**], [2 4 / **1** **3**], [**2** 3 / **1** 4]
	$F_{(2,2)}+F_{(1,2,1)}+F_{(1,3)}$	$F_{(1,2,1)}+F_{(2,2)}+F_{(2,1,1)}$	$F_{(4)}+F_{(2,2)}+F_{(3,1)}$	$F_{(1^4)}+F_{(1,2,1)}+F_{(1,1,2)}$
Tableaux SET(α)	$\mathcal{E}_\alpha$ CFLSX '14, AS '19, S '20	$\mathcal{RE}_\alpha$ NSvWVW 2022	$\mathcal{A}_{\mathrm{SET}(\alpha)}$ NSvWVW 2022	$\bar{\mathcal{A}}_{\mathrm{SET}(\alpha)}$ NSvWVW 2022
ALL Cols bottom to top	strict $\nearrow$	weak $\nearrow$	weak $\nearrow$	strict $\nearrow$
Rows left to right	weak $\nearrow$	strict $\nearrow$	weak $\nearrow$	strict $\nearrow$
$(2,2)$	$F_{(2,2)}+F_{(1,2,1)}$	$F_{(1,2,1)}+F_{(2,2)}$	$F_{(4)}+F_{(2,2)}$	$F_{(1^4)}+F_{(1,2,1)}$
Basis for QSym?	Yes	Yes	No	No
Kostka matrix	upper triangular, 1's on the diagonal	?	?	upper triangular, but 0's on the diagonal

Complete picture from the immaculate Hecke poset: 4 descent sets, 8 flavours of tableaux



The eight flavours of tableaux, their 0-Hecke modules, with characteristics related in pairs by the involution ψ, from the four descent sets. The double arrow-head indicates a quotient module, and the hooked arrow indicates a submodule.

References

0-Hecke modules for row-strict dual immaculate functions, arXiv:2202.00708

Row-strict dual immaculate functions, arXiv:2202.00706

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