Promotion, rowmotion, rotation, and webs

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June 7, 2022

Catalan objects

1	2	4	7	8	9	13
3	5	6	10	11	12	14

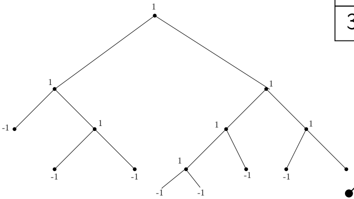
Theorem

$2 \times d$ standard Young tableaux are counted by the dth Catalan number: $C_d = \frac{1}{d+1}\binom{2d}{d} = \frac{(2d)!}{(d+1)!d!}$

The Catalan numbers for $d = 0, 1, 2, \ldots, 10$ are:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796

Catalan objects



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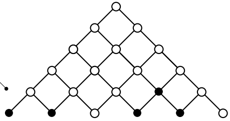

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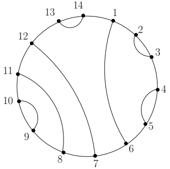

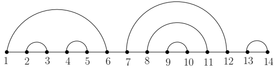

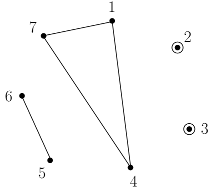

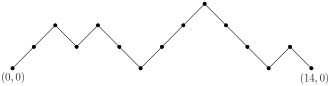

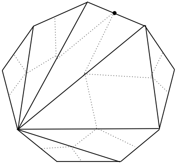

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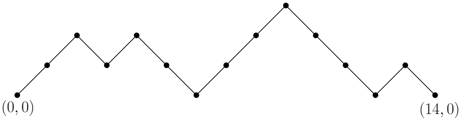
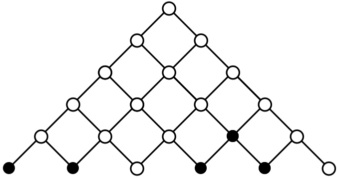
Catalan object bijections





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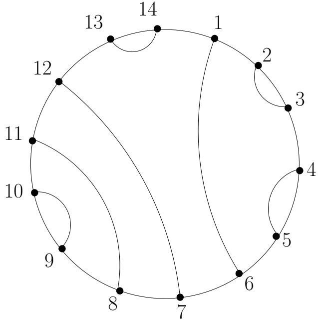

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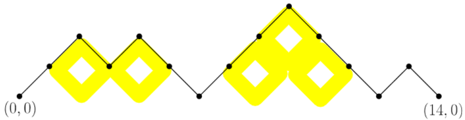

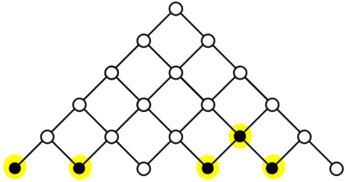

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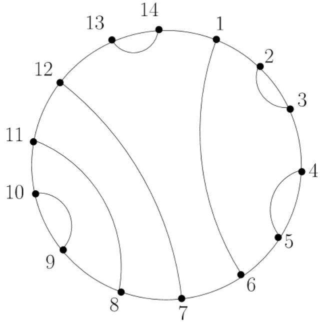

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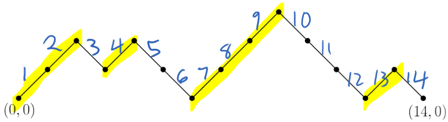

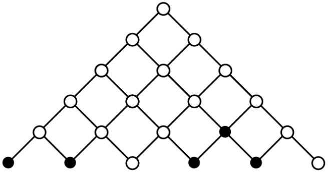

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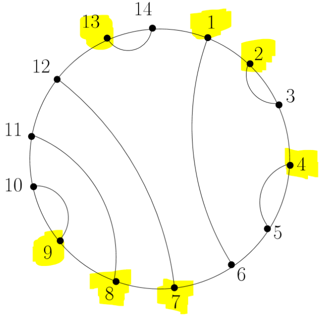

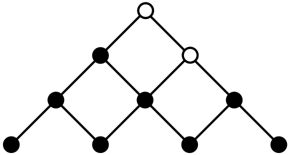

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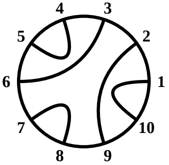

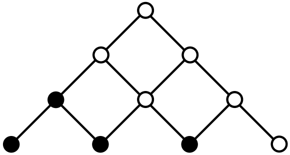

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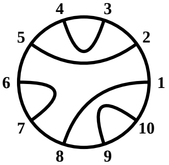

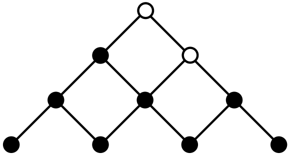

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5	6	8	9	10

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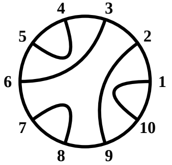
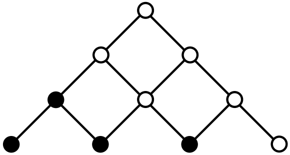


Rotation ↓



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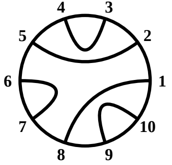

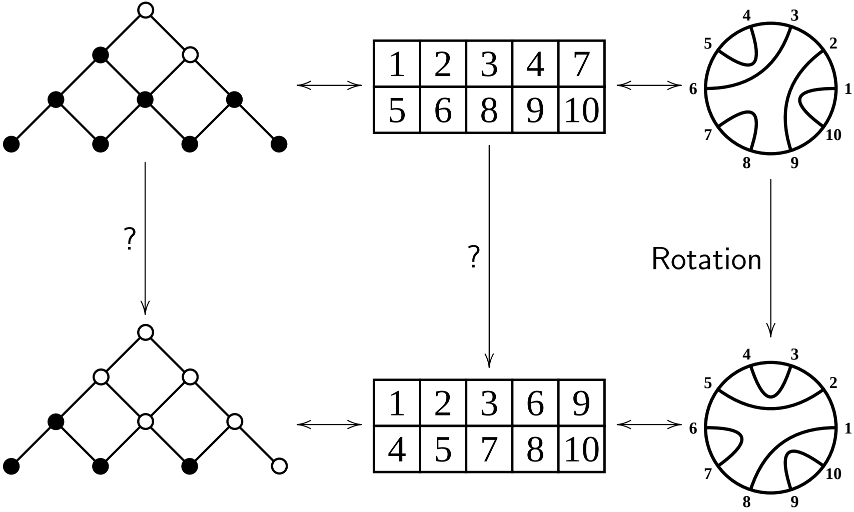
1 2 3 4 7
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?
Rotation
1 2 3 4 5 6 7 8 9 10

Promotion via jeu de taquin

1	2	3	4	7
5	6	8	9	10

Promotion via jeu de taquin

	2	3	4	7
5	6	8	9	10

Promotion via jeu de taquin

2		3	4	7
5	6	8	9	10

Promotion via jeu de taquin

2	3		4	7
5	6	8	9	10

Promotion via jeu de taquin

2	3	4		7
5	6	8	9	10

Promotion via jeu de taquin

2	3	4	7	
5	6	8	9	10

Promotion via jeu de taquin

2	3	4	7	10
5	6	8	9	

Promotion via jeu de taquin

2	3	4	7	10
5	6	8	9	11

Promotion via jeu de taquin

1	2	3	6	9
4	5	7	8	10

Promotion via Bender-Knuth involutions

1	2	3	4	7
5	6	8	9	10

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1	2	3	4	7
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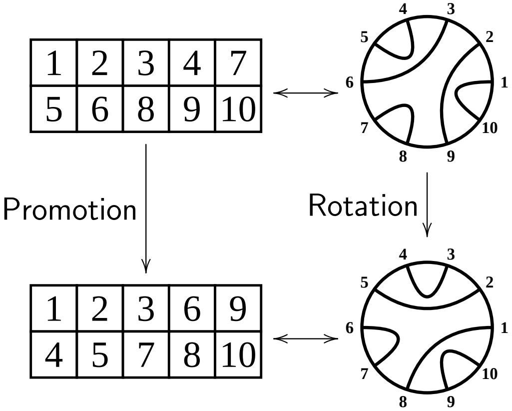
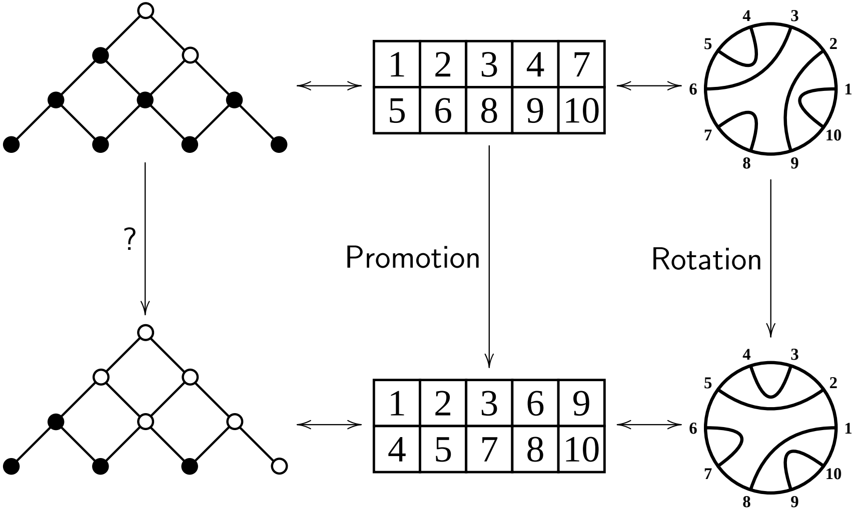
1	2	3	6	9
4	5	7	8	10

Promotion and rotation of noncrossing matchings

Theorem (D. White)

There is an equivariant bijection *between* promotion *on* $2 \times d$ *standard Young tableaux and* rotation *on non-crossing matchings of* $n = 2d$. *So promotion has order* n *(and exhibits cyclic sieving).*




1 2 3 4 7
5 6 8 9 10
4 3 5 2 6 1 7 10 8 9
?
Promotion
Rotation
1 2 3 6 9
4 5 7 8 10
4 3 5 2 6 1 7 10 8 9

Promotion and toggling

Proposition (Williams-S. 2012)

There is an equivariant bijection *between promotion on* $2 \times d$ *standard Young tableaux and toggling left to right on order ideals of the triangular poset* $\Phi^+(A_{d-1})$.

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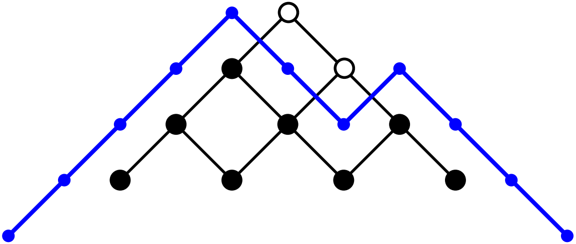
5	6	8	9	10



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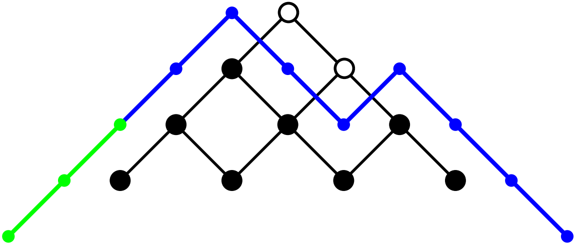

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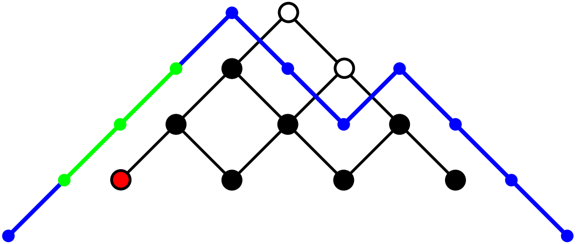

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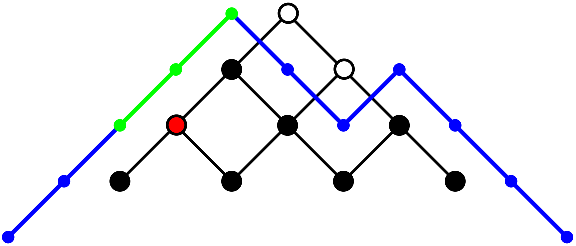

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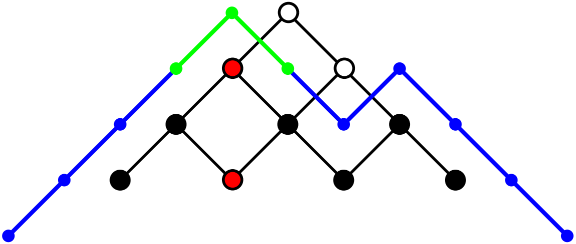

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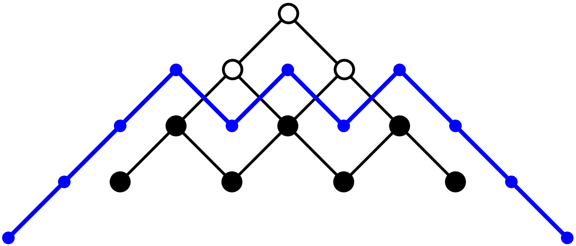

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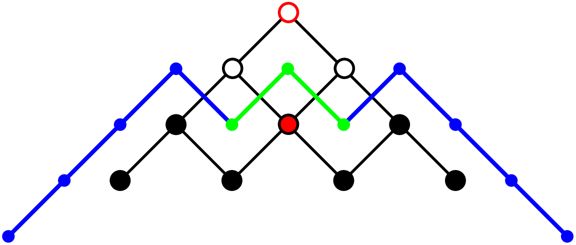

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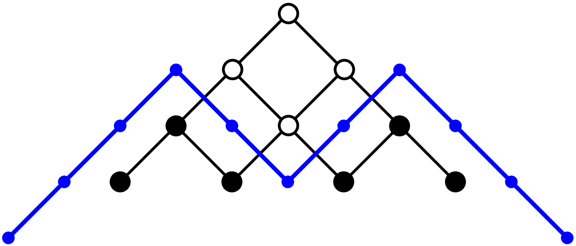

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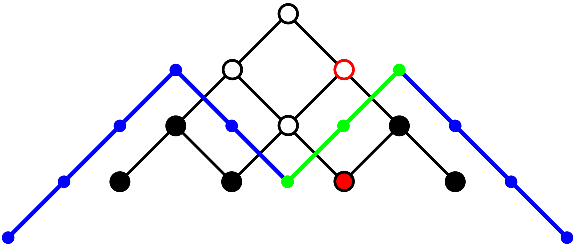

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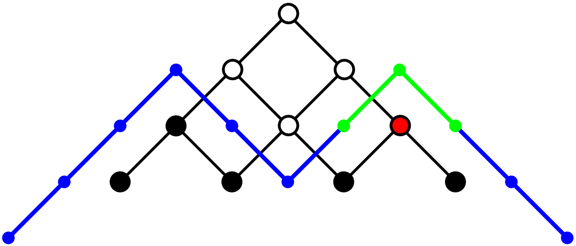

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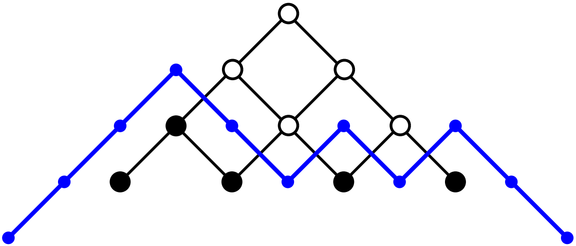

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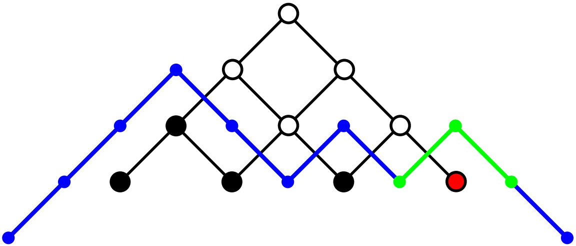

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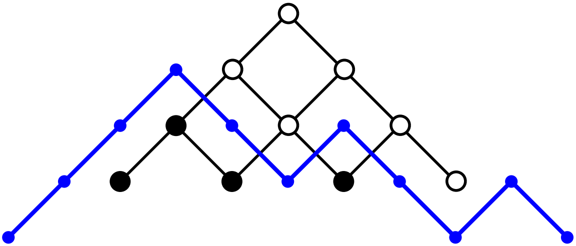

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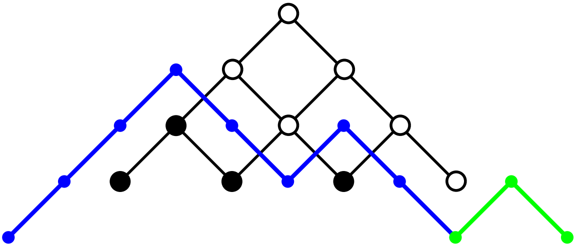

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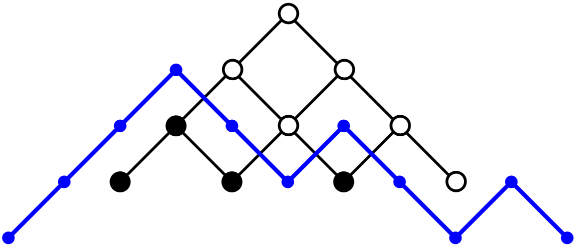

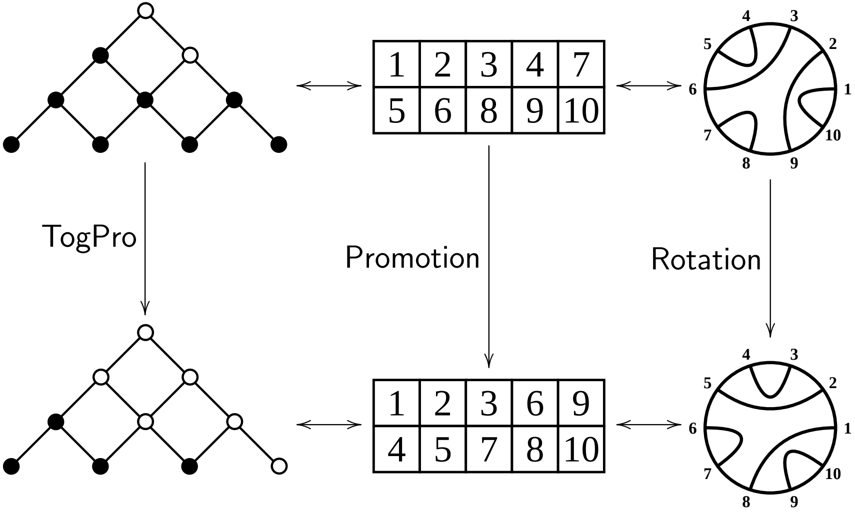
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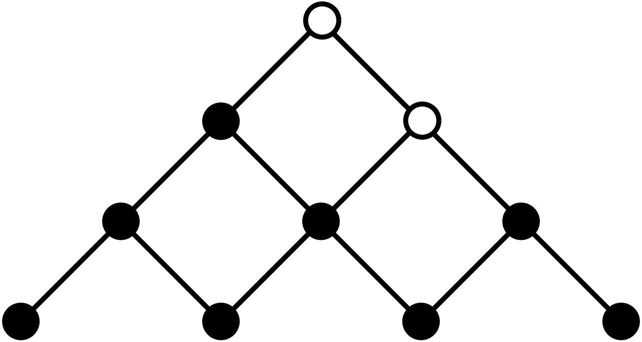
8 9

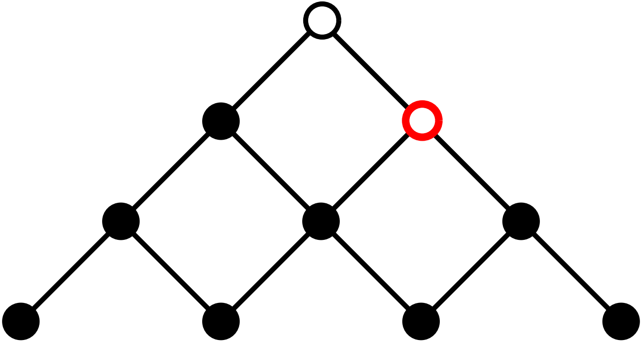
Rowmotion

An order ideal



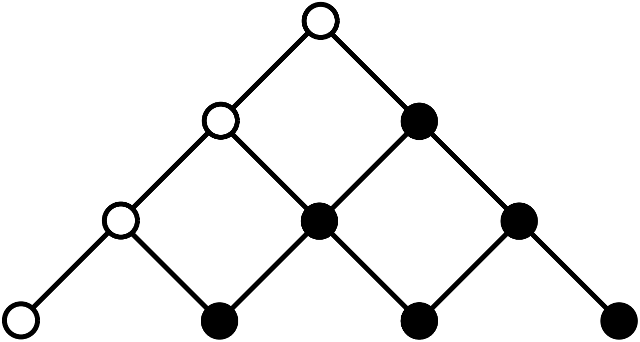
Rowmotion

Find the **minimal** elements of P not in the order ideal.



Rowmotion

Use them to generate a new order ideal.



Promotion and rowmotion are conjugate actions

Cameron and Fon-der-Flaass identified the toggle group and showed:

Theorem (Cameron-Fon-der-Flaass 1995)

Rowmotion can also be computed by toggling from top to bottom.

Theorem (Williams-S. 2012)

In any ranked poset, there is an equivariant bijection between order ideals under rowmotion and toggle-promotion (toggle left to right).

Theorem (Dilks-Pechenik-S. 2017)

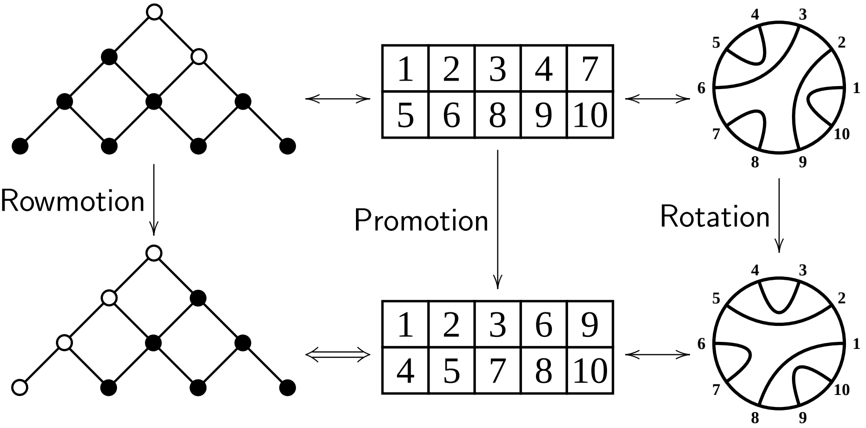
In any poset with an n-dim lattice projection, there is an equivariant bijection between the order ideals under rowmotion and 2^n toggle-promotions (toggle by hyperplanes in a given direction).

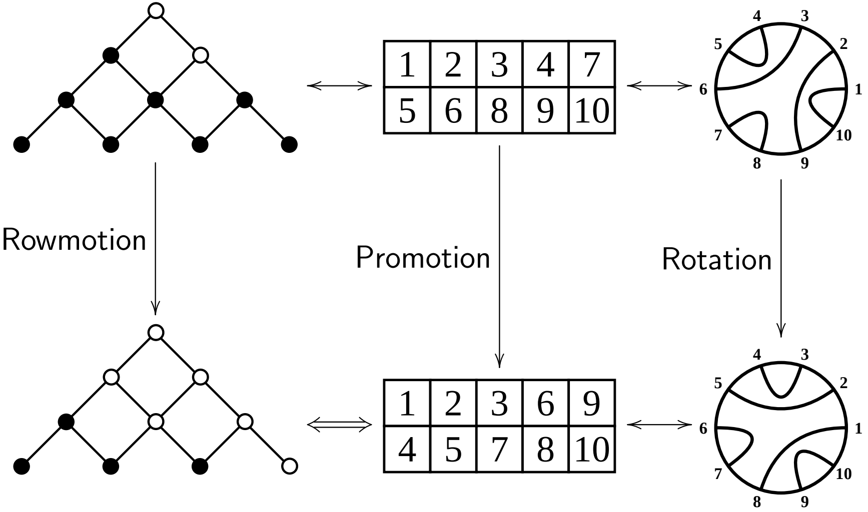
In an equivariant bijection
the orbit structure is preserved.

Promotion, rowmotion, and rotation

Corollary (Armstrong-Stump-Thomas 2013, Williams-S. 2012)

There is an equivariant bijection between promotion on $2 \times d$ standard Young tableaux and rowmotion on order ideals of $\Phi^+(A_{d-1})$. So rowmotion has order $n = 2d$ and exhibits cyclic sieving.




1 2 3 4 7
5 6 8 9 10
Rowmotion
Promotion
Rotation
1 2 3 6 9
4 5 7 8 10
4 3 5 2 6 1 7 10 8 9
4 3 5 2 6 1 7 10 8 9

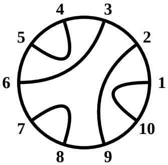
Web basis of noncrossing matchings

Theorem (Theory of SL_2 webs)

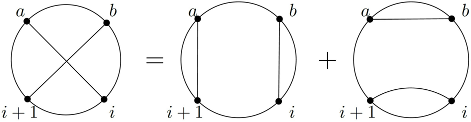
A basis for the Specht module $S^{(d,d)}$ is given by those products of matrix minors corresponding to noncrossing matchings (SL_2 webs) of $n = 2d$. The long cycle $(12 \cdots n)$ acts by rotation of diagrams.

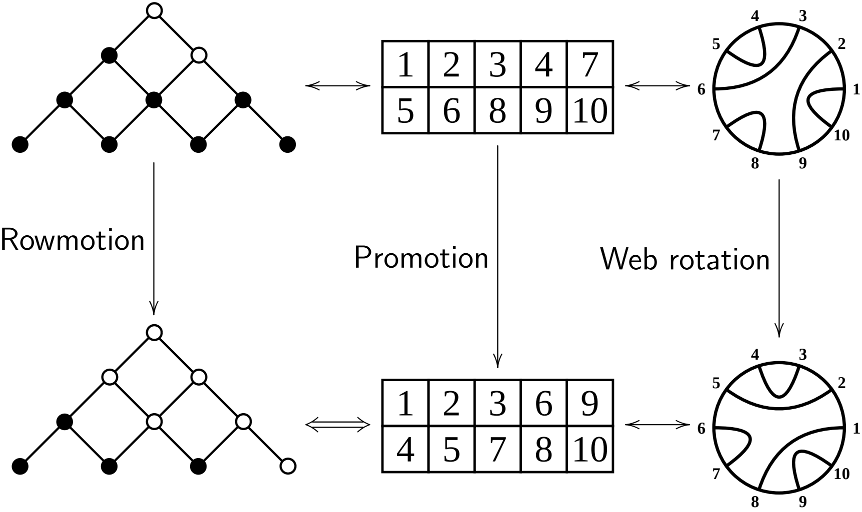
1	2	3	4	7
5	6	8	9	10

$\longleftrightarrow$



$$\begin{vmatrix} x_1 & x_{10} \\ y_1 & y_{10} \end{vmatrix} \begin{vmatrix} x_2 & x_9 \\ y_2 & y_9 \end{vmatrix} \begin{vmatrix} x_3 & x_6 \\ y_3 & y_6 \end{vmatrix} \begin{vmatrix} x_4 & x_5 \\ y_4 & y_5 \end{vmatrix} \begin{vmatrix} x_7 & x_8 \\ y_7 & y_8 \end{vmatrix}$$




1 2 3 4 7
5 6 8 9 10
Rowmotion
Promotion
Web rotation
1 2 3 6 9
4 5 7 8 10
1 2 3 4 5 6 7 8 9 10

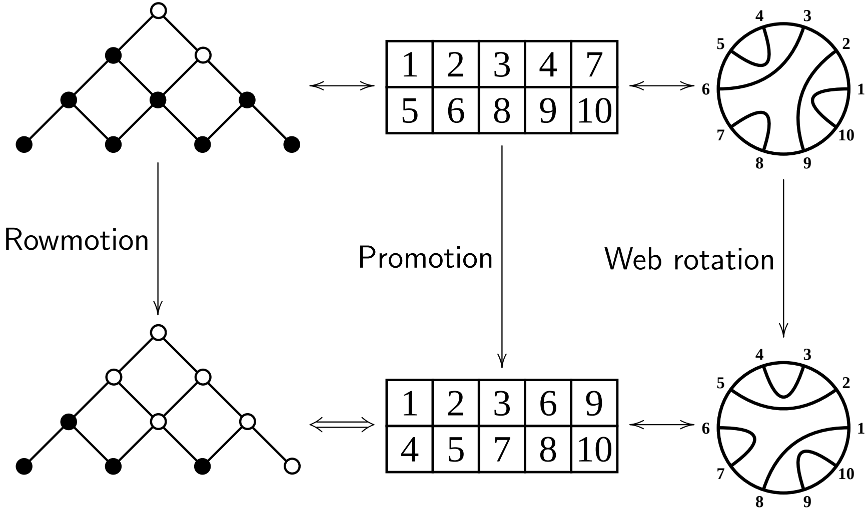
Other instances where promotion $\leftrightarrow$ rowmotion

Later in this talk:

- Increasing tableaux $\longleftrightarrow$ Order ideals of $[a] \times [b] \times [c]$
  (joint work with Dilks and Pechenik)

Not in this talk:

- Increasing labelings of $P \longleftrightarrow$ Order ideals of $\Gamma(P)$
  (joint work with Dilks and Vorland)
- P-strict labelings $\longleftrightarrow \Gamma(P)$-partitions
  (two papers joint with Bernstein and Vorland)
  - Semistandard Young tableaux $\longleftrightarrow$ Gelfand-Tsetlin patterns
  - Flagged tableaux of staircase shape $\longleftrightarrow$ Rectangular P-partitions
  - Symplectic tableaux of staircase shape $\longleftrightarrow$ Triangular flagged P-partitions
  - $\cdots$

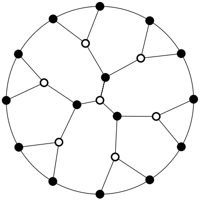

1 2 3 4 7
5 6 8 9 10
Rowmotion
Promotion
Web rotation
1 2 3 6 9
4 5 7 8 10

Other instances where promotion $\leftrightarrow$ web rotation

- Promotion on $3 \times d$ standard Young tableaux $\leftrightarrow$ rotation of SL_3 webs with $n = 3d$ boundary vertices of one color (Petersen-Pylyavskyy-Rhoades 2009)

1	3	5	9
2	6	7	11
4	8	10	12

$\longleftrightarrow$



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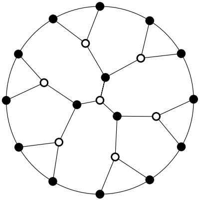


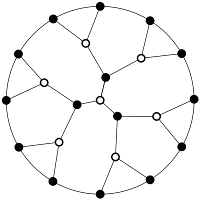
Open questions: $\rightarrow$ Rowmotion?

Other instances where promotion $\leftrightarrow$ web rotation

- Promotion on $3 \times d$ standard Young tableaux $\leftrightarrow$ rotation of SL_3 webs with $n = 3d$ boundary vertices of one color (Petersen-Pylyavskyy-Rhoades 2009)

1	3	5	9
2	6	7	11
4	8	10	12

$\longleftrightarrow$

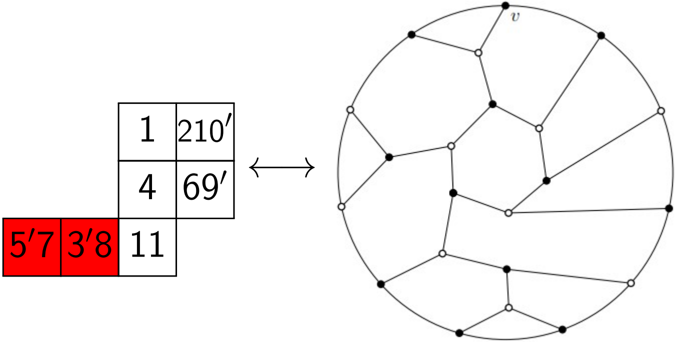


Open questions: $\rightarrow$ Rowmotion?

$\rightarrow$ Web basis for SL_m? Standard Young tableaux of shape d^m index a basis for the Specht module S^{d^m} and have promotion order $n = md$ (Haiman 1992) and cyclic sieving (Rhoades 2010), but no one knows a web basis. (There are non-diagrammatic bases that respect the S_n action.)

Other instances where promotion ↔ web rotation

- Promotion on 'rectangular' generalized oscillating tableaux of 3 rows ↔ rotation of SL_3 webs where boundary vertices may be of both colors (Rebecca Patrias 2019)

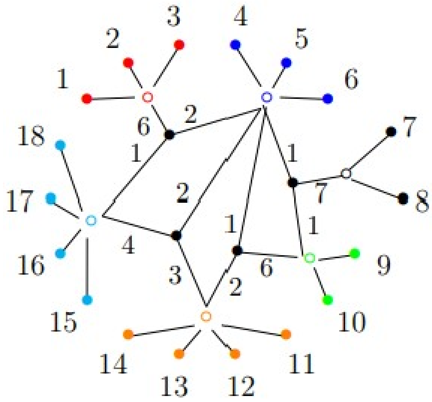


Other instances where promotion $\leftrightarrow$ web rotation

- Promotion on $d \times 2$ standard Young tableaux $\leftrightarrow$ rotation of SL_d webs with $n = 2d$ boundary vertices of one color (Chris Fraser 2022+)

1	4
2	7
3	9
5	11
6	12
8	15
10	16
13	17
14	18

$\longleftrightarrow$

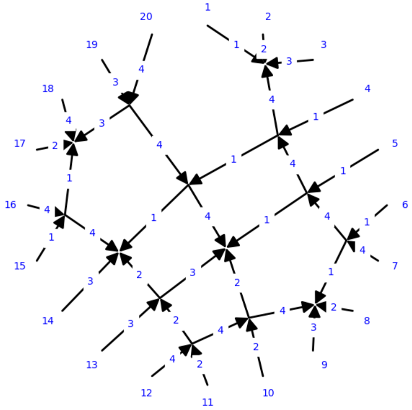


Other instances where promotion ↔ diagram rotation

- Promotion on $4 \times d$ standard Young tableaux ↔ rotation of equivalence classes of SL_4 web-like diagrams with $n = 4d$ boundary vertices (joint work in progress with Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Joshua Swanson)

1	4	5	6	15
2	8	10	11	17
3	9	13	14	19
7	12	16	18	20

$\longleftrightarrow$



Increasing tableaux

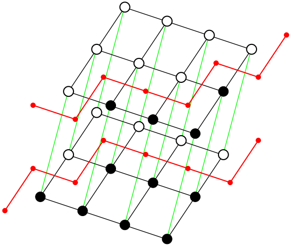
1	2	3	5	6	9	10
3	4	6	7	8	10	11

Theorem (Pechenik 2014)

$2 \times d$ packed increasing tableaux are counted by the dth small Schröder number*.*

The small Schöder numbers for $d = 0, 1, 2, \ldots, 9$ are:

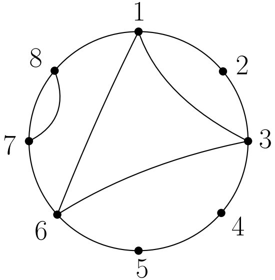
1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049

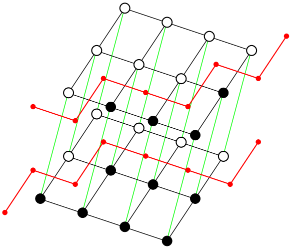


↔

1	3	7
3	6	8

↔

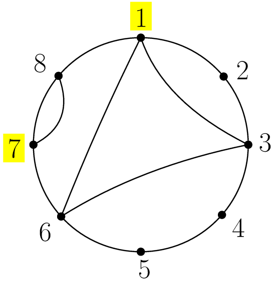


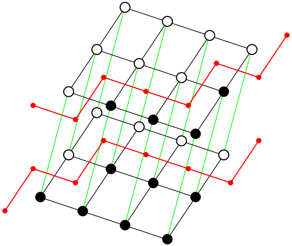


↔

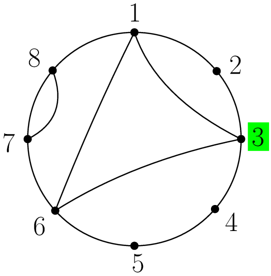
1	3	7
3	6	8

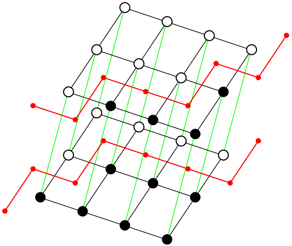
↔



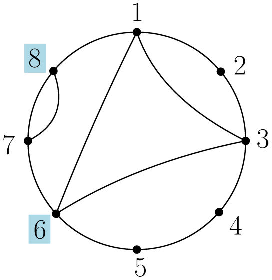


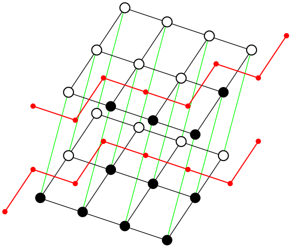
1	3	7
3	6	8





1	3	7
3	6	8

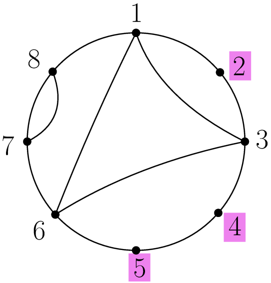


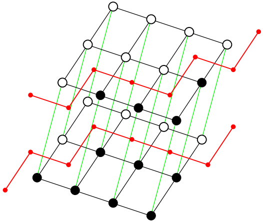


⟷

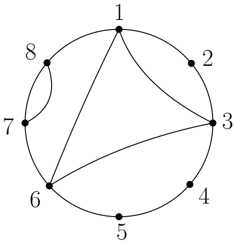
1	3	7
3	6	8

⟷

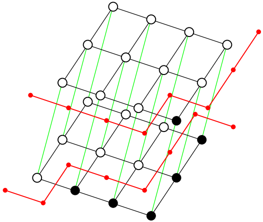




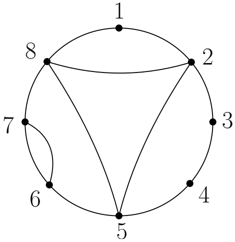
1	3	7
3	6	8

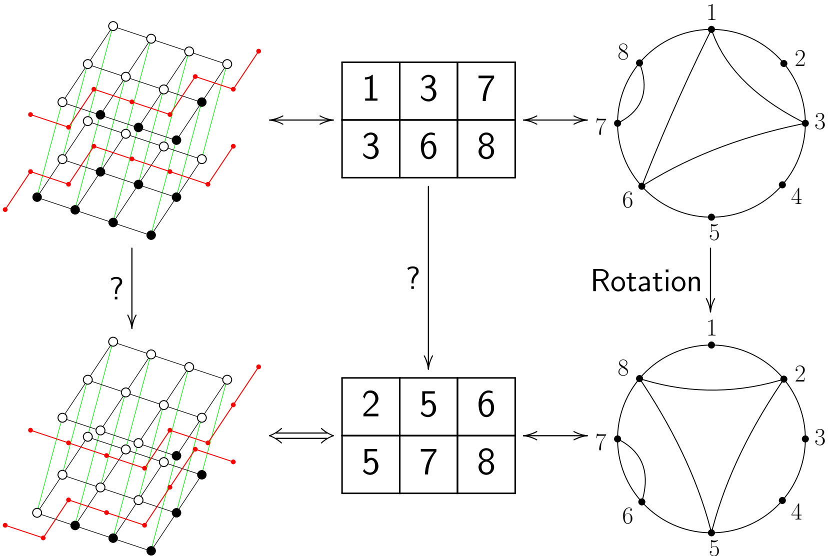


Rotation



2	5	6
5	7	8





1 3 7
3 6 8
?
?
Rotation
2 5 6
5 7 8
1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 8

K-promotion on increasing tableaux by K-jeu de taquin

1	3	7
3	6	8

K-promotion on increasing tableaux by K-jeu de taquin

	3	7
3	6	8

K-promotion on increasing tableaux by K-jeu de taquin

3		7
	6	8

K-promotion on increasing tableaux by K-jeu de taquin

3	6	7
6		8

K-promotion on increasing tableaux by K-jeu de taquin

3	6	7
6	8	

K-promotion on increasing tableaux by K-jeu de taquin

3	6	7
6	8	9

K-promotion on increasing tableaux by K-jeu de taquin

2	5	6
5	7	8

K-promotion on increasing tableaux by K-Bender-Knuth involutions

1	2	4
4	5	6

K-promotion on increasing tableaux by K-Bender-Knuth involutions

1	2	4
4	5	6

K-promotion on increasing tableaux by K-Bender-Knuth involutions

1	3	4
4	5	6

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3	4	6

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1	3	5
3	4	6

K-promotion on increasing tableaux by K-Bender-Knuth involutions

1	3	5
3	4	6

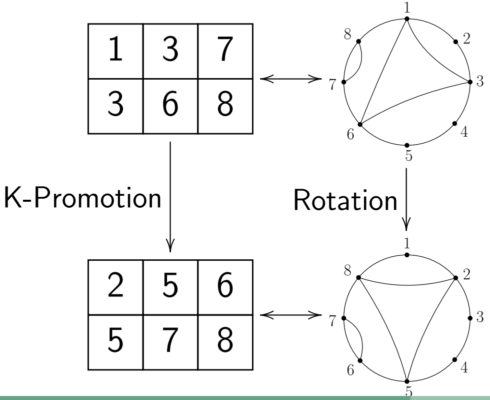
K-promotion on increasing tableaux by K-Bender-Knuth involutions

1	3	5
3	4	7

K-promotion and rotation of noncrossing partitions

Theorem (O. Pechenik 2014 (packed case))

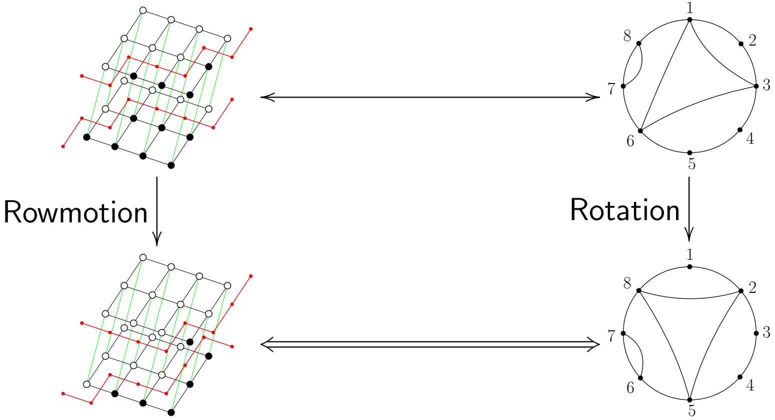
There is an equivariant bijection between K-promotion on $2 \times d$ increasing tableaux with entries at most $2d + m - 2$ and rotation on non-crossing partitions of $2d + m - 2$ into $d - 1$ non-singleton parts. So K-promotion has order $2d + m - 2$ and exhibits the CSP.



Rowmotion and rotation of noncrossing partitions

Theorem (Williams-S. '12, Cameron-Fon-der-Flaass '95, Rush-Shi '13)

There is an equivariant bijection between rowmotion on order ideals of a × b × 2 *and rotation on noncrossing partitions of* $a+b+1$ *into* $b+1$ *blocks. So rowmotion has order* $a+b+1$ *and exhibits the CSP.*

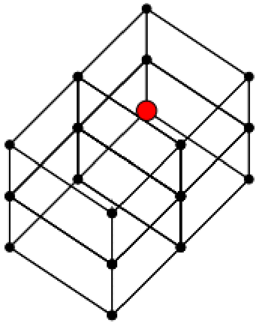


K-promotion and toggling

Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between K-promotion on $[a] \times [b]$ increasing tableaux *with entries at most* $a + b + c - 1$ *and toggling back to front on* [a] × [b] × [c].

1	2	4
4	5	6



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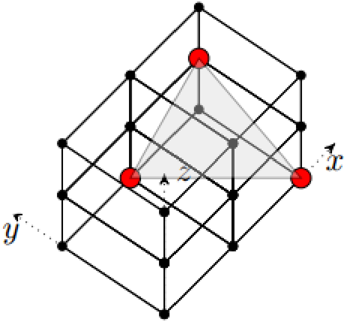

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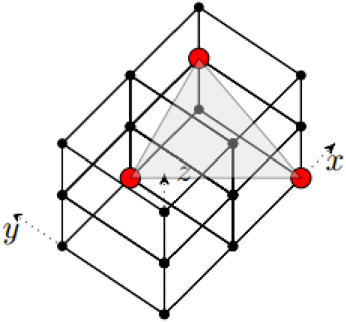

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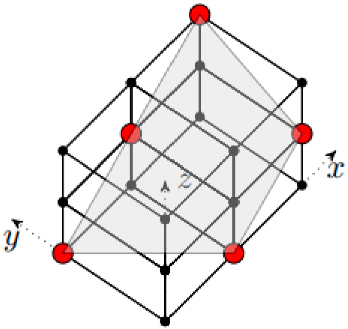
4	5	6



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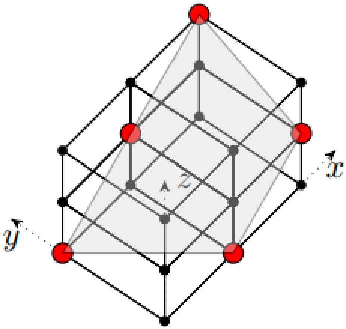

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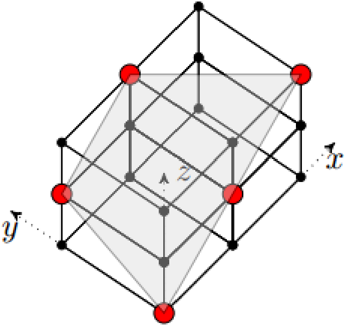

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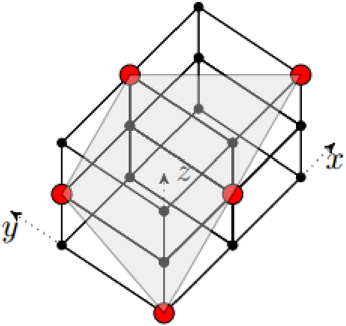
1	3	5
3	4	6

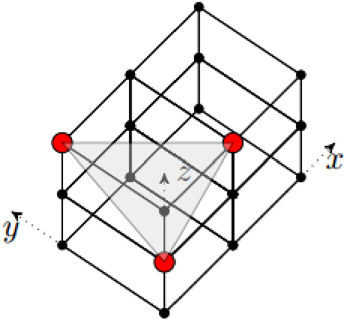


K-promotion and toggling

Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between K-promotion *on* $[a] \times [b]$ increasing tableaux *with entries at most* $a + b + c - 1$ *and* toggling back to front *on* [a] × [b] × [c].

1	3	5
3	4	6



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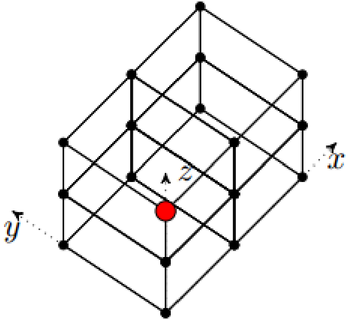
1	3	5
3	4	6



K-promotion and toggling

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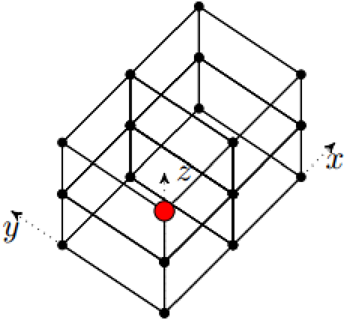
1	3	5
3	4	7

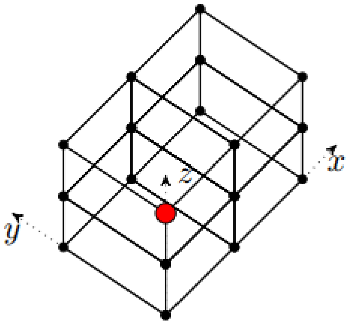


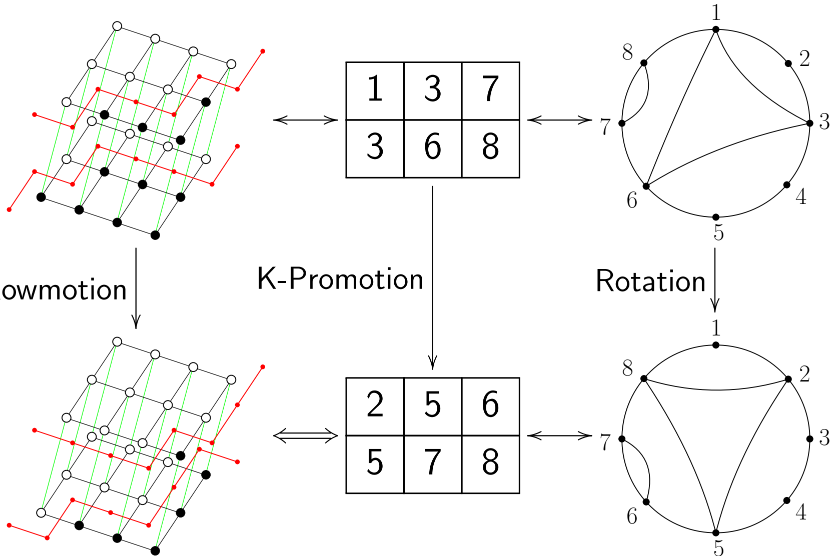
K-promotion and rowmotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between K-promotion on $[a] \times [b]$ increasing tableaux *with entries at most* $a + b + c - 1$ *and rowmotion on* $[a] \times [b] \times [c]$.

1	3	5
3	4	7





1 3 7
3 6 8
Rowmotion
K-Promotion
Rotation
2 5 6
5 7 8
1
2
3
4
5
6
7
8

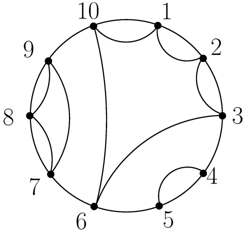
A bijection with pennant shaped standard Young tableaux

1	4	7
2	6	10
3		
5		
8		
9		

$\longleftrightarrow$

1	2	3	4	6	7	8
2	3	5	6	8	9	10

$\longleftrightarrow$



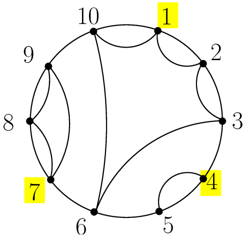
A bijection with pennant shaped standard Young tableaux

1	4	7
2	6	10
3		
5		
8		
9		

$\longleftrightarrow$

1	2	3	4	6	7	8
2	3	5	6	8	9	10

$\longleftrightarrow$



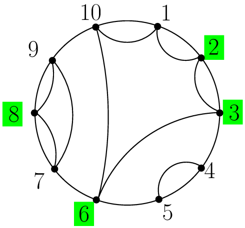
A bijection with pennant shaped standard Young tableaux

1	4	7
2	6	10
3		
5		
8		
9		

⟷

1	2	3	4	6	7	8
2	3	5	6	8	9	10

⟷



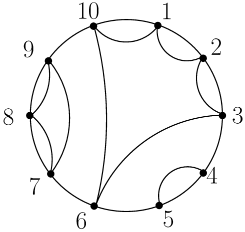
A bijection with pennant shaped standard Young tableaux

1	4	7
2	6	10
3		
5		
8		
9		

$\longleftrightarrow$

1	2	3	4	6	7	8
2	3	5	6	8	9	10

$\longleftrightarrow$



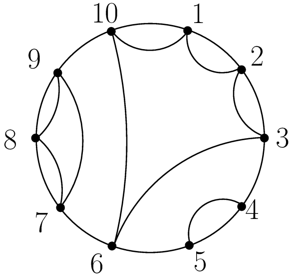
Web basis of noncrossing partitions

Theorem (Rhoades 2017, Kim-Rhoades 2022, Patrias-Pechenik-S. 2022+)

Noncrossing partitions of $n = 2d + m - 2$ into d parts with no singletons index a basis for the Specht module $S^{(d,d,1^{m-2})}$. The long cycle $(12 \cdots n)$ acts by rotation of diagrams.

1	4	7
2	6	10
3		
5		
8		
9		

$\longleftrightarrow$



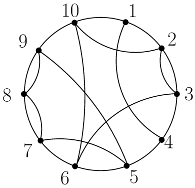
In our new proof of this theorem, we define an explicit polynomial for each noncrossing partition and show this is a basis.

Web invariant polynomials

Given a set partition π, how do we define its polynomial $[\pi]$?

Our polynomial will be a signed sum over *jellyfish tableaux*.

$$\pi = \{\{2,3,6,10\},\{5,7,8,9\},\{1,4\}\}$$



2	5	1
3	7	4
6		
	8	
10		
	9	

$$\begin{vmatrix} x_{12} & x_{13} & x_{16} & x_{110} \\ x_{22} & x_{23} & x_{26} & x_{210} \\ x_{32} & x_{33} & x_{36} & x_{310} \\ x_{52} & x_{53} & x_{56} & x_{510} \end{vmatrix} \begin{vmatrix} x_{15} & x_{17} & x_{18} & x_{19} \\ x_{25} & x_{27} & x_{28} & x_{29} \\ x_{45} & x_{47} & x_{48} & x_{49} \\ x_{65} & x_{67} & x_{68} & x_{69} \end{vmatrix} \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix}$$

$$(-1)^{\mathrm{inv}(T)}\mathrm{J}(T) = (-1)^7 M_{1,2,3,5}^{2,3,6,10} \cdot M_{1,2,4,6}^{5,7,8,9} \cdot M_{1,2}^{1,4}$$

Web invariant polynomials

Suppose $\pi = \{\{2,3,6,10\},\{5,7,8,9\},\{1,4\}\}$. Then $\mathcal{J}(\pi)$ is:

2	5	1
3	7	4
6		
10		
	8	
	9	

2	5	1
3	7	4
6		
	8	
10		
	9	

2	5	1
3	7	4
6		
	8	
	9	
10		

2	5	1
3	7	4
	8	
6		
10		
	9	

2	5	1
3	7	4
	8	
6		
	9	
10		

2	5	1
3	7	4
	8	
	9	
6		
10		

$$[\pi] = \sum_{T \in \mathcal{J}(\pi)} (-1)^{\operatorname{inv} T}\, \mathrm{J}(T) = \sum_{T \in \mathcal{J}(\pi)} \operatorname{sgn}(T)\, \mathrm{J}(T)$$

$$= M^{2,3,6,10}_{1,2,3,4} \cdot M^{5,7,8,9}_{1,2,5,6} \cdot M^{1,4}_{1,2} - M^{2,3,6,10}_{1,2,3,5} \cdot M^{5,7,8,9}_{1,2,4,6} \cdot M^{1,4}_{1,2}$$

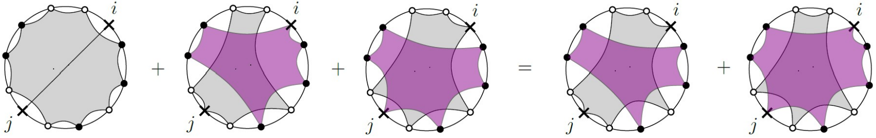
$$+ M^{2,3,6,10}_{1,2,3,6} \cdot M^{5,7,8,9}_{1,2,4,5} \cdot M^{1,4}_{1,2} + M^{2,3,6,10}_{1,2,4,5} \cdot M^{5,7,8,9}_{1,2,3,6} \cdot M^{1,4}_{1,2}$$

$$- M^{2,3,6,10}_{1,2,4,6} \cdot M^{5,7,8,9}_{1,2,3,5} \cdot M^{1,4}_{1,2} + M^{2,3,6,10}_{1,2,5,6} \cdot M^{5,7,8,9}_{1,2,3,4} \cdot M^{1,4}_{1,2}.$$

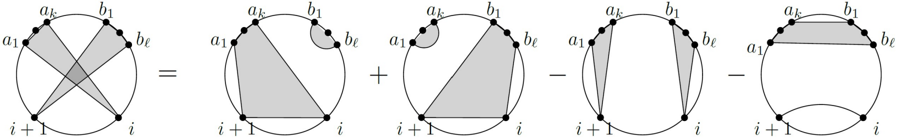
Proof that this is a basis

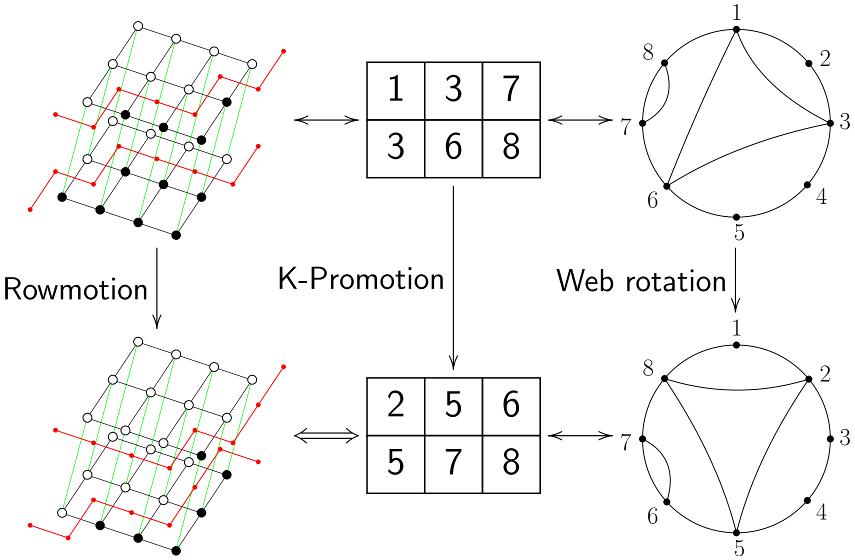
A polynomial relation for changing block sizes:

$$\left[\{A \cup B, I \cup J\}\right] + \left[\{A \cup I, B \cup J\}\right] + \left[\{A \cup J, B \cup I\}\right] = \left[\{A, B \cup I \cup J\}\right] + \left[\{A \cup I \cup J, B\}\right]$$



This specializes to an uncrossing rule:




1 3 7
3 6 8
Rowmotion
K-Promotion
Web rotation
2 5 6
5 7 8
1 2 3 4 5 6 7 8
1 2 3 4 5 6 7 8

THANKS

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