The characters of local permutation statistics

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UCSD
Outline

- Local permutation statistics
- Local class functions
- Application to pattern enumeration
- Path Murnaghan-Nakayama Rule
Def: A permutation statistic is a function $f : \mathfrak{S}_n \rightarrow \mathbb{R}$.

- $\text{exc}(w) = \# \{1 \leq i \leq n : w(i) > i\}$
- $\text{inv}(w) = \# \{1 \leq i < j \leq n : w(i) > w(j)\}$
- $\text{des}(w) = \# \{1 \leq i \leq n - 1 : w(i) > w(i + 1)\}$
- $\text{maj}(w) = \sum_{w(i) > w(i+1)} i$
- $\text{cyc}(w) = \text{number of cycles in } w$
Permutation Statistics

**Def:** A *permutation statistic* is a function $f : \mathfrak{S}_n \rightarrow \mathbb{R}$.

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- $\text{cyc}(w)$ = number of cycles in $w$

**Intuition:** A statistic $f : \mathfrak{S}_n \rightarrow \mathbb{R}$ is *k-local* if $f(w)$ is determined by the restriction of $w$ to $k$-element subsets of $[n]$. 
Partial Permutations

A partial permutation on $[n]$ of size $k$ is a bijection $S \sim T$ between two size $k$ subsets $S, T \subseteq [n]$.

$$\mathcal{G}_{n,k} = \{\text{all partial permutations on } [n] \text{ of size } k\}$$

**Ex:** Inside $\mathcal{G}_{5,2}$ we have the “two-line notation”

$$2 \leftrightarrow 5, \ 3 \leftrightarrow 1 \ \Leftrightarrow \ (23, 51).$$

For $(I, J) \in \mathcal{G}_{n,k}$ with $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, the indicator statistic $1_{I, J}: \mathcal{G}_n \to \mathbb{R}$ is

$$1_{I, J}(w) = \begin{cases} 1 & w(i_1) = j_1, \ldots, w(i_k) = j_k \\ 0 & \text{otherwise} \end{cases}$$
Local Statistics

Def: A permutation statistic $f : \mathcal{S}_n \to \mathbb{R}$ is $k$-local if there are constants $c_{I,J} \in \mathbb{R}$ such that

$$f(w) = \sum_{(I,J) \in \mathcal{S}_{n,k}} c_{I,J} \cdot 1_{I,J}(w) \quad \text{for all } w \in \mathcal{S}_n.$$

- $\text{exc}(w) = \# \{1 \leq i \leq n : w(i) > i\}$ is 1-local.
- $\text{inv}(w) = \# \{1 \leq i < j \leq n : w(i) > w(j)\}$ is 2-local.
- $\text{des}(w) = \# \{1 \leq i \leq n - 1 : w(i) > w(i + 1)\}$ is 2-local.
- $\text{maj}(w) = \sum_{w(i) > w(i + 1)} i$ is 2-local.
- $\text{cyc}(w) = \text{number of cycles in } w$ has no nontrivial locality.
Local Statistics: Basic Facts

**Def:** A permutation statistic \( f : \mathcal{S}_n \rightarrow \mathbb{R} \) is \( k \)-local if there are constants \( c_{I,J} \in \mathbb{R} \) such that

\[
f(w) = \sum_{(I,J) \in \mathcal{S}_{n,k}} c_{I,J} \cdot 1_{I,J}(w) \quad \text{for all } w \in \mathcal{S}_n.
\]

- The 0-local statistics are constant functions \( \mathcal{S}_n \rightarrow \mathbb{R} \).
- Any \( k \)-local statistic is also \( (k + 1) \)-local.
- Any \( f : \mathcal{S}_n \rightarrow \mathbb{R} \) is \( (n - 1) \)-local.

**Idea:** The locality of a statistic measures its ‘degree’.
Locality = Degree

Def: A permutation statistic $f : \mathcal{S}_n \rightarrow \mathbb{R}$ is $k$-local if there are constants $c_{I,J} \in \mathbb{R}$ such that

$$f(w) = \sum_{(I,J) \in \mathcal{S}_{n,k}} c_{I,J} \cdot 1_{I,J}(w) \quad \text{for all } w \in \mathcal{S}_n.$$  

- A linear combination of $k$-local statistics is $k$-local.
- If $f : \mathcal{S}_n \rightarrow \mathbb{R}$ is $k$-local and $g : \mathcal{S}_n \rightarrow \mathbb{R}$ is $\ell$-local, then

$$(f \cdot g)(w) = f(w) \cdot g(w)$$

is $(k + \ell)$-local.
Locality = Degree

**Def:** A permutation statistic $f : S_n \rightarrow \mathbb{R}$ is $k$-local if $f = \sum_{(I,J) \in S_{n,k}} c_{I,J} \cdot 1_{I,J}$.

**Artin-Wedderburn Theorem:** We have an isomorphism

$$
\Psi : \mathbb{R}[S_n] \rightarrow \bigoplus_{\lambda \vdash n} \text{End}_\mathbb{R}(V^\lambda)
$$

given by $\Psi : \alpha \mapsto (\alpha : V^\lambda \rightarrow V^\lambda)_{\lambda \vdash n}$.

**Ubiquitous:** [Ellis-Friedgut-Pipel, Even-Zohar, Huang-Guestrin-Guibas, ...]

$$
f \text{ is } k\text{-local } \iff \Psi(\sum_{w \in S_n} f(w) \cdot w) \text{ supported on } \{\lambda \vdash n : \lambda_1 \geq n - k\}.
$$
Characters are Local Statistics

**Artin-Wedderburn Theorem:** We have an isomorphism

\[ \Psi : \mathbb{R}[\mathfrak{S}_n] \to \bigoplus_{\lambda \vdash n} \text{End}_\mathbb{R}(V^\lambda) \]

given by \( \Psi : \alpha \mapsto (\alpha : V^\lambda \to V^\lambda)_{\lambda \vdash n} \).

**Ubiquitous:** A statistic \( f : \mathfrak{S}_n \to \mathbb{R} \) is \( k \)-local if and only if \( \Psi(\sum_{w \in \mathfrak{S}_n} f(w) \cdot w) \) is supported on \( \{ \lambda \vdash n : \lambda_1 \geq n - k \} \).

**Cor:** Given \( \lambda \vdash n \) with \( \lambda_1 \geq n - k \), the irreducible character

\[ \chi^\lambda : \mathfrak{S}_n \to \mathbb{R} \]

is \( k \)-local.
Vanilla Permutation Patterns

Let \( v = [v(1), \ldots, v(k)] \in \mathfrak{S}_k \) and \( w = [w(1), \ldots, w(n)] \in \mathfrak{S}_n \).

**Def:** Given \( S = \{i_1 < \cdots < i_k \} \subseteq [n] \), the permutation \( w \) matches the pattern \( v \) at \( S \) if

\[
[w(i_1), \ldots, w(i_k)] \text{ is order-isomorphic to } [v(1), \ldots, v(k)].
\]

Define \( N_v : \mathfrak{S}_n \to \mathbb{R} \) by

\[
N_v(w) := \# \{ S \subseteq [n] : w \text{ matches } v \text{ at } S \}.
\]

**Ex:** If \( v = [2, 1] \) then \( \text{inv}(w) = N_v(w) \).
(Bi)Vincular Permutation Patterns

Let $v = [v(1), \ldots, v(k)] \in S_k$ and $w = [w(1), \ldots, w(n)] \in S_n$.

**Def:** Given $S = \{i_1 < \cdots < i_k\} \subseteq [n]$, and subsets $A, B \subseteq [k - 1]$ the permutation $w$ matches the bivincular pattern $(v, A, B)$ at $S$ if

- $[w(i_1), \ldots, w(i_k)]$ is order-isomorphic to $[v(1), \ldots, v(k)]$ and
- for all $a \in A$, we have $i_{a+1} = i_a + 1$ and
- for all $b \in B$, the $b^{th}$ and $(b + 1)^{st}$ smallest elements of $\{w(i_1), \ldots, w(i_k)\}$ are consecutive.

Define $N_{v,A,B} : S_n \to \mathbb{R}$ by

$$N_{v,A,B}(w) := \#\{S \subseteq [n] : w \text{ matches } (v, A, B) \text{ at } S\}.$$

**Ex:** If $v = [2, 1]$, $A = \{1\}$, $B = \emptyset$ then $\text{des}(w) = N_{v,A,B}(w)$. 
Weighted (Bi)Vincular Permutation Patterns

Let \( v = [v(1), \ldots, v(k)] \in S_k \) and \( w = [w(1), \ldots, w(n)] \in S_n \).

**Def:** Given subsets \( A, B \subseteq [k - 1] \) and polynomials \( f, g \in \mathbb{R}[x_1, \ldots, x_k] \) we define

\[
N_{v,A,B}^{f,g} : \mathbb{S}_n \to \mathbb{R}
\]

by the rule

\[
N_{v,A,B}^{f,g}(w) = \sum_S f(i_1, \ldots, i_k) \cdot g(w(i_1), \ldots, w(i_k))
\]

where \( S = \{i_1 < \cdots < i_k\} \) ranges over all matches of \((v, A, B)\) in \( w \).

**Ex:** If \( v = [2, 1], A = \{1\}, B = \emptyset, f = x_1, g = 1 \) then

\( \text{maj}(w) = N_{v,A,B}^{f,g}(w) \).
Pattern Counting is Local

**Def:** If $\Upsilon = \{(v, A, B, f, g)\}$ is a family of weighted bivincular patterns, let $N_\Upsilon : \mathcal{S}_n \to \mathbb{R}$ be

$$N_\Upsilon(w) := \sum_{(v,A,B,f,g) \in \Upsilon} N^f,g_{v,A,B}(w).$$

**Fact:** If the largest pattern $v$ in $\Upsilon$ has size $k$, then $N_\Upsilon$ is $k$-local.

**Goal:** [After Gaetz-Ryba] Study patterns restricted to some cycle type $K_\lambda \subseteq \mathcal{S}_n$. 

▶ “That is pure hell.” – anonymous senior combinatorialist
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Class Functions and Reynolds

A function $f : \mathcal{G}_n \rightarrow \mathbb{R}$ is a class function if $f(vwv^{-1}) = f(w)$ for all $v, w \in \mathcal{G}_n$.

**Def:** The Reynolds operator $R$ acts on maps $f : \mathcal{G}_n \rightarrow \mathbb{R}$ by

$$Rf(w) = \frac{1}{n!} \sum_{v \in \mathcal{G}_n} f(vwv^{-1}).$$

The map $Rf : \mathcal{G}_n \rightarrow \mathbb{R}$ is a class function.

**Idea:** $Rf$ is the best class function approximation to $f$. 
Let $\Upsilon$ be a set of weighted bivincular patterns $(v, A, B, f, g)$.

- Let $k$ be the largest size $|v|$ of a pattern $v$ with $(v, A, B, f, g) \in \Upsilon$.
- Let $q$ be the largest size of $|A| + |B|$ for $(v, A, B, f, g) \in \Upsilon$.
- Let $p$ be the largest value of $|v| - |A| - |B| + \deg f + \deg g$ for $(v, A, B, f, g) \in \Upsilon$.

Let $m_i(w) =$ number of $i$-cycles in $w$.

**Thm:** [HR] For any $d \geq 0$, the statistic $RN^d_\Upsilon$ on $S_n$ is a rational function in $\mathbb{R}(n, m_1, m_2, \ldots, m_{kd})$. If $\deg(n) = 1$ and $\deg(m_i) = i$ the rational degree of $RN^d_\Upsilon$ is $\leq dp$. Also, the statistic

$$n(n - 1) \cdots (n - dq + 1) \cdot RN^d_\Upsilon \in \mathbb{R}[n, m_1, m_2, \ldots, m_{kd}]$$

is a polynomial in $n, m_1, m_2, \ldots, m_{kd}$.
**Thm:** [HR] For any \( d \geq 0 \), the statistic \( RN_\Gamma^d \) is a rational function in \( \mathbb{R}(n, m_1, m_2, \ldots, m_{kd}) \). If \( \deg(n) = 1 \) and \( \deg(m_i) = i \) the rational degree of \( RN_\Gamma^d \) is \( \leq dp \). Also, the statistic

\[
n(n-1) \cdots (n-dq+1) \cdot RN_\Gamma^d \in \mathbb{R}[n, m_1, m_2, \ldots, m_{kd}]
\]

is a polynomial in \( n, m_1, m_2, \ldots, m_{kd} \).

**Rmk:** Proven in the vanilla case by Gaetz-Ryba (single pattern) and Gaetz-Pierson (multiple patterns). Proof uses Jones Duality between \( S_n \) and \( P_r(n) \) acting on \( (\mathbb{C}^n)^\otimes r \). Unclear how to extend their methods to vincular or weighted cases.
Proof Idea

**Classical Science:** Understand atoms $\Rightarrow$ Understand molecules.

\[
f = \sum_{(I,J) \in \mathcal{G}_{n,k}} c_{I,J} \cdot 1_{I,J}
\]

**Atoms of $k$-Local Statistics:** $1_{I,J}$ for $(I, J) \in \mathcal{G}_{n,k}$.

\[
Rf = \sum_{(I,J)\in\mathcal{G}_{n,k}} c_{I,J} \cdot R 1_{I,J}
\]

**Atoms of $k$-Local Class Fcns:** $R 1_{I,J}$ for $(I, J) \in \mathcal{G}_{n,k}$. 
Let $\text{ch}_n : \text{Class}(\mathfrak{S}_n) \to \Lambda_n$ be the Frobenius characteristic

$$
\text{ch}_n : f \mapsto \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) \cdot p_{\lambda(w)}
$$

where $\lambda(w) \vdash n$ is the cycle type of $w$.

**Defn:** If $(I, J) \in \mathfrak{S}_{n,k}$ is a partial permutation, the atomic symmetric function is

$$A_{n,I,J} := n! \cdot \text{ch}_n(R1_{I,J}).$$

**Q:** What is the Schur expansion of $A_{n,I,J}$?
Characters and Atomic Symmetric Functions

**Defn:** If \((I, J) \in S_{n,k}\) is a partial permutation, the atomic symmetric function is \(A_{n,I,J} := n! \cdot \text{ch}_n(R 1_{I,J})\).

**Fact:** If \([I, J] = \sum_{w \in S_n} w \in \mathbb{R}[S_n], \text{ then}

\[
A_{n,I,J} = \sum_{\lambda \vdash n} \chi^\lambda[I, J] \cdot s\lambda
\]

where \(\chi^\lambda : \mathbb{R}[S_n] \rightarrow \mathbb{R}\) is the irreducible character.

**Q:** What is the Schur expansion of \(A_{n,I,J}\)?
Defn: If \((I, J) \in \mathfrak{S}_{n,k}\) is a partial permutation, the atomic symmetric function is \(A_{n,I,J} := n! \cdot \text{ch}_n(R\mathbf{1}_{I,J})\).

Fact: If \([I, J] = \sum_{w \in \mathfrak{S}_n} w \in \mathbb{R}[\mathfrak{S}_n], \text{ then} \]

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\]

where \(\chi^\lambda : \mathbb{R}[\mathfrak{S}_n] \to \mathbb{R}\) is the irreducible character.

Q: What is the Schur expansion of \(A_{n,I,J}\)?

Useless A: Apply the Murnagahan-Nakayama Rule \((n-k)!\) times.
Partial permutations \((I, J) \in S_{n,k}\) decompose into *paths* and *cycles*.

**Fact:** \(A_{n,I,J}\) only depends on the unlabeled graph.
Path and Cycle Notation

Partial permutations \((I, J) \in \mathcal{S}_{n,k}\) decompose into paths and cycles.

Fact: We have a factorization \(A_{n,I,J} = A^{\text{path}} \cdot A^{\text{cycle}}\). Also, \(A^{\text{cycle}}\) is a power sum.

\[
A_{15,I,J} = A^{\text{path}} \cdot A^{\text{cycle}} = \vec{p}_{332111} \cdot p_{211}
\]
Path Power Sums

Def: For $\mu \vdash n$, the path power sum $\vec{p}_\mu$ is the atomic symmetric function of a graph with paths of sizes $\mu_1, \mu_2, \ldots$. 
Path Power Sums

Def: For $\mu \vdash n$, the path power sum $\vec{p}_\mu$ is the atomic symmetric function of a graph with paths of sizes $\mu_1, \mu_2, \ldots$.

Facts: [HR]

- $\{\vec{p}_\mu\}$ is a basis for the space of symmetric functions.
- $\{\vec{p}_\mu\}$ is unitriangular to the power sum basis $\{p_\nu\}$. 
Classical Murnaghan-Nakayama

\[ A_{n,l,J} = \bar{p}_\mu \cdot p_\nu \]

**MN Rule:** For \( \nu \models n \), we have \( p_\nu = \sum_{\lambda \models n} \chi^\lambda_\nu \cdot s_\lambda \) where

\[ \chi^\lambda_\nu = \sum_{\nu\text{-ribbon tableaux } T \text{ shape } \lambda} (-1)^{\text{height}(T)} \]
Classical Murnaghan-Nakayama

**MN Rule:** For $\nu \models n$, we have $p_\nu = \sum_{\lambda \models n} \chi^\lambda_\nu \cdot s_\lambda$ where

$$\chi^\lambda_\nu = \sum_{\nu\text{-ribbon tableaux } T \text{ shape } \lambda} (-1)^{\text{height}(T)}$$

\[ p_{3221} = \cdots + 1 \cdot s_{431} + \cdots \]

**Q:** What about the *path* power sum $\vec{p}_\mu$?
Monotonic Ribbon Tilings

**Def:** In a *monotonic ribbon tiling* . . .

▶ the tails of the ribbons lie in distinct columns,
▶ the tail depth decreases from left to right, and
▶ each initial union of ribbons forms a partition.
Thm: [HR] For $\mu \vdash n$, we have $\vec{p}_\mu = \sum_{\lambda \vdash n} \vec{\chi}^\lambda_\mu \cdot s_\lambda$ where

$$\vec{\chi}^\lambda_\mu = \text{mult}(\mu)! \cdot \sum_{\text{monotonic tilings } \tau \text{ of } \lambda} (-1)^{\text{height}(\tau)}$$

where the ribbons are to be added in all possible orders.
Path Murnaghan-Nakayama Rule

\[ \vec{p}_{321} = 6s_6 - 4s_{51} + 2s_{33} + 2s_{411} - s_{321} \]
Atomic Expansion

**Thm:** [HR] Let \((I, J) \in \mathfrak{S}_{n,k}\) have path partition \(\mu\) and cycle partition \(\nu\). Then

\[
A_{n,I,J} = \bar{p}_\mu \cdot p_\nu = \sum_{\substack{\lambda \vdash n \\ \rho \subseteq \lambda}} \chi^\rho_\mu \cdot \chi^{\lambda/\rho}_\nu \cdot s_\lambda
\]

where \(\bar{\chi}^\rho_\mu\) counts monotonic tilings and \(\chi^{\lambda/\rho}_\nu\) counts classical tilings.

- Gives a formula for \(\chi^\lambda ([w \cdot \mathfrak{S}_r]_+)\) on coset sums in \(\mathfrak{S}_n/\mathfrak{S}_r\).
- Proves polynomiality results for pattern enumeration on conjugacy classes.
- Has applications to *probability*. 
Quasi-Random Permutations

**Defn:** [Cooper] A sequence $w^{(n)}$ of permutations in $\mathfrak{S}_n$ is *quasi-random* if for all patterns $\nu$ we have

$$\frac{N_\nu(w^{(n)})}{\binom{n}{k}} = \frac{1}{k!} + o(1) \quad \text{as } n \to \infty$$

where $|\nu| = k$.

**Thm:** [HR] Let $\mu_n$ be a sequence of conjugacy-invariant probability distributions on $\mathfrak{S}_n$ and let $\Sigma_n \sim \mu_n$. If

$$\lim_{n \to \infty} \mathbb{P} \left( \frac{m_1(\Sigma_n)}{n} = 0 \right) = 1$$

then $\Sigma_n$ is quasi-random.
Thanks for listening!!