The characters of local permutation statistics

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UCSD

Outline

- Local permutation statistics
- Local class functions
- Application to pattern enumeration
- Path Murnaghan-Nakayama Rule

Permutation Statistics

Def: A *permutation statistic* is a function $f : \mathfrak{S}_n \to \mathbb{R}$.

- $\mathrm{exc}(w) = \#\{1 \le i \le n \,:\, w(i) > i\}$
- $\mathrm{inv}(w) = \#\{1 \le i < j \le n \,:\, w(i) > w(j)\}$
- $\mathrm{des}(w) = \#\{1 \le i \le n-1 \,:\, w(i) > w(i+1)\}$
- $\mathrm{maj}(w) = \sum_{w(i)>w(i+1)} i$
- $\mathrm{cyc}(w) =$ number of cycles in w

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Intuition: A statistic $f : \mathfrak{S}_n \to \mathbb{R}$ is *k-local* if $f(w)$ is determined by the restriction of w to k-element subsets of $[n]$.

Partial Permutations

A *partial permutation* on $[n]$ of size k is a bijection $S \xrightarrow{\sim} T$ between two size k subsets $S, T \subseteq [n]$.

$$\mathfrak{S}_{n,k} = \{\text{all partial permutations on } [n] \text{ of size } k\}$$

Ex: Inside $\mathfrak{S}_{5,2}$ we have the "two-line notation"

$$2 \mapsto 5,\ 3 \mapsto 1 \quad \Leftrightarrow \quad (23, 51).$$

For $(I, J) \in \mathfrak{S}_{n,k}$ with $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, the *indicator statistic* $\mathbf{1}_{I,J} : \mathfrak{S}_n \to \mathbb{R}$ is

$$\mathbf{1}_{I,J}(w) = \begin{cases} 1 & w(i_1) = j_1, \ \ldots \ , w(i_k) = j_k \\ 0 & \text{otherwise} \end{cases}$$

Local Statistics

Def: A permutation statistic $f : \mathfrak{S}_n \to \mathbb{R}$ is *k-local* if there are constants $c_{I,J} \in \mathbb{R}$ such that

$$f(w) = \sum_{(I,J)\in\mathfrak{S}_{n,k}} c_{I,J} \cdot \mathbf{1}_{I,J}(w) \quad \text{for all } w \in \mathfrak{S}_n.$$

- $\mathrm{exc}(w) = \#\{1 \le i \le n \,:\, w(i) > i\}$ is 1-local.
- $\mathrm{inv}(w) = \#\{1 \le i < j \le n \,:\, w(i) > w(j)\}$ is 2-local.
- $\mathrm{des}(w) = \#\{1 \le i \le n-1 \,:\, w(i) > w(i+1)\}$ is 2-local.
- $\mathrm{maj}(w) = \sum_{w(i)>w(i+1)} i$ is 2-local.
- $\mathrm{cyc}(w) =$ number of cycles in w has no nontrivial locality.

Local Statistics: Basic Facts

Def: A permutation statistic $f : \mathfrak{S}_n \to \mathbb{R}$ is *k-local* if there are constants $c_{I,J} \in \mathbb{R}$ such that

$$f(w) = \sum_{(I,J) \in \mathfrak{S}_{n,k}} c_{I,J} \cdot \mathbf{1}_{I,J}(w) \quad \text{for all } w \in \mathfrak{S}_n.$$

- The 0-local statistics are constant functions $\mathfrak{S}_n \to \mathbb{R}$.
- Any k-local statistic is also $(k+1)$-local.
- Any $f : \mathfrak{S}_n \to \mathbb{R}$ is $(n-1)$-local.

Idea: The locality of a statistic measures its 'degree'.

Locality = Degree

Def: A permutation statistic $f : \mathfrak{S}_n \to \mathbb{R}$ is *k-local* if there are constants $c_{I,J} \in \mathbb{R}$ such that

$$f(w) = \sum_{(I,J)\in\mathfrak{S}_{n,k}} c_{I,J} \cdot \mathbf{1}_{I,J}(w) \quad \text{for all } w \in \mathfrak{S}_n.$$

- A linear combination of k-local statistics is k-local.
- If $f : \mathfrak{S}_n \to \mathbb{R}$ is k-local and $g : \mathfrak{S}_n \to \mathbb{R}$ is ℓ-local, then

  $$(f \cdot g)(w) = f(w) \cdot g(w)$$

  is $(k + \ell)$-local.

Locality = Degree

Def: A permutation statistic $f : \mathfrak{S}_n \to \mathbb{R}$ is *k-local* if $f = \sum_{(I,J) \in \mathfrak{S}_{n,k}} c_{I,J} \cdot \mathbf{1}_{I,J}$.

Artin-Wedderburn Theorem: We have an isomorphism

$$\Psi : \mathbb{R}[\mathfrak{S}_n] \to \bigoplus_{\lambda \vdash n} \mathrm{End}_{\mathbb{R}}(V^\lambda)$$

given by $\Psi : \alpha \mapsto (\alpha : V^\lambda \to V^\lambda)_{\lambda \vdash n}$.

Ubiquitous: [Ellis-Friedgut-Pipel, Even-Zohar, Huang-Guestrin-Guibas, . . .]

$$f \text{ is } k\text{-local} \quad \Leftrightarrow \quad \Psi\left(\textstyle\sum_{w \in \mathfrak{S}_n} f(w) \cdot w\right) \text{ supported on } \{\lambda \vdash n : \lambda_1 \geq n - k\}.$$

Characters are Local Statistics

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Ubiquitous: A statistic $f : \mathfrak{S}_n \to \mathbb{R}$ is k-local if and only if $\Psi(\sum_{w \in \mathfrak{S}_n} f(w) \cdot w)$ is supported on $\{\lambda \vdash n : \lambda_1 \geq n - k\}$.

Cor: Given $\lambda \vdash n$ with $\lambda_1 \geq n - k$, the irreducible character

$$\chi^\lambda : \mathfrak{S}_n \to \mathbb{R}$$

is k-local.

Vanilla Permutation Patterns

Let $v = [v(1), \dots, v(k)] \in \mathfrak{S}_k$ and $w = [w(1), \dots, w(n)] \in \mathfrak{S}_n$.

Def: Given $S = \{i_1 < \cdots < i_k\} \subseteq [n]$, the permutation w *matches the pattern v at S* if

$$[w(i_1), \dots, w(i_k)] \text{ is order-isomorphic to } [v(1), \dots, v(k)].$$

Define $N_v : \mathfrak{S}_n \to \mathbb{R}$ by

$$N_v(w) := \#\{S \subseteq [n] \; : \; w \text{ matches } v \text{ at } S\}.$$

Ex: If $v = [2, 1]$ then $\mathrm{inv}(w) = N_v(w)$.

(Bi)Vincular Permutation Patterns

Let $v = [v(1), \ldots, v(k)] \in \mathfrak{S}_k$ and $w = [w(1), \ldots, w(n)] \in \mathfrak{S}_n$.

Def: Given $S = \{i_1 < \cdots < i_k\} \subseteq [n]$, and subsets $A, B \subseteq [k-1]$ the permutation w *matches the bivincular pattern* (v, A, B) *at* S if

- $[w(i_1), \ldots, w(i_k)]$ is order-isomorphic to $[v(1), \ldots, v(k)]$ **and**
- for all $a \in A$, we have $i_{a+1} = i_a + 1$ **and**
- for all $b \in B$, the b^{th} and $(b+1)^{st}$ smallest elements of $\{w(i_1), \ldots, w(i_k)\}$ are consecutive.

Define $N_{v,A,B} : \mathfrak{S}_n \to \mathbb{R}$ by

$$N_{v,A,B}(w) := \#\{S \subseteq [n] \,:\, w \text{ matches } (v, A, B) \text{ at } S\}.$$

Ex: If $v = [2, 1]$, $A = \{1\}, B = \varnothing$ then $\mathrm{des}(w) = N_{v,A,B}(w)$.

Weighted (Bi)Vincular Permutation Patterns

Let $v = [v(1), \ldots, v(k)] \in \mathfrak{S}_k$ and $w = [w(1), \ldots, w(n)] \in \mathfrak{S}_n$.

Def: Given subsets $A, B \subseteq [k-1]$ and polynomials $f, g \in \mathbb{R}[x_1, \ldots, x_k]$ we define

$$N^{f,g}_{v,A,B} : \mathfrak{S}_n \to \mathbb{R}$$

by the rule

$$N^{f,g}_{v,A,B}(w) = \sum_S f(i_1, \ldots, i_k) \cdot g(w(i_1), \ldots, w(i_k))$$

where $S = \{i_1 < \cdots < i_k\}$ ranges over all matches of (v, A, B) in w.

Ex: If $v = [2,1]$, $A = \{1\}, B = \varnothing$, $f = x_1, g = 1$ then $\mathrm{maj}(w) = N^{f,g}_{v,A,B}(w)$.

Pattern Counting is Local

Def: If $\Upsilon = \{(v, A, B, f, g)\}$ is a family of weighted bivincular patterns, let $N_\Upsilon : \mathfrak{S}_n \to \mathbb{R}$ be

$$N_\Upsilon(w) := \sum_{(v,A,B,f,g)\in\Upsilon} N^{f,g}_{v,A,B}(w).$$

Fact: If the largest pattern v in Υ has size k, then N_Υ is k-local.

Goal: [After Gaetz-Ryba] Study patterns restricted to some cycle type $K_\lambda \subseteq \mathfrak{S}_n$.

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- "That is pure hell." – anonymous senior combinatorialist

Class Functions and Reynolds

A function $f : \mathfrak{S}_n \to \mathbb{R}$ is a *class function* if $f(vwv^{-1}) = f(w)$ for all $v, w \in \mathfrak{S}_n$.

Def: The *Reynolds operator* R acts on maps $f : \mathfrak{S}_n \to \mathbb{R}$ by

$$Rf(w) = \frac{1}{n!} \sum_{v \in \mathfrak{S}_n} f(vwv^{-1}).$$

The map $Rf : \mathfrak{S}_n \to \mathbb{R}$ is a class function.

Idea: Rf is the *best class function approximation* to f.

Pattern Counting and Reynolds

Let Υ be a set of weighted bivincular patterns (v, A, B, f, g).

- Let k be the largest size $|v|$ of a pattern v with $(v, A, B, f, g) \in \Upsilon$.
- Let q be the largest size of $|A| + |B|$ for $(v, A, B, f, g) \in \Upsilon$.
- Let p be the largest value of $|v| - |A| - |B| + \deg f + \deg g$ for $(v, A, B, f, g) \in \Upsilon$.

Let $m_i(w)$ = number of i-cycles in w.

Thm: [HR] For any $d \geq 0$, the statistic RN_Υ^d on $\mathfrak{S}_n$ is a rational function in $\mathbb{R}(n, m_1, m_2, \ldots, m_{kd})$. If $\deg(n) = 1$ and $\deg(m_i) = i$ the rational degree of RN_Υ^d is $\leq dp$. Also, the statistic

$$n(n-1)\cdots(n-dq+1)\cdot RN_\Upsilon^d \in \mathbb{R}[n, m_1, m_2, \ldots, m_{kd}]$$

is a *polynomial* in $n, m_1, m_2, \ldots, m_{kd}$.

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is a *polynomial* in $n, m_1, m_2, \ldots, m_{kd}$.

Rmk: Proven in the vanilla case by Gaetz-Ryba (single pattern) and Gaetz-Pierson (multiple patterns). Proof uses *Jones Duality* between $\mathfrak{S}_n$ and $\mathcal{P}_r(n)$ acting on $(\mathbb{C}^n)^{\otimes r}$. Unclear how to extend their methods to vincular or weighted cases.

Proof Idea

Classical Science: Understand atoms $\Rightarrow$ Understand molecules.

$$f = \sum_{(I,J)\in\mathfrak{S}_{n,k}} c_{I,J} \cdot \mathbf{1}_{I,J}$$

Atoms of k-Local Statistics: $\mathbf{1}_{I,J}$ for $(I,J) \in \mathfrak{S}_{n,k}$.

$$Rf = \sum_{(I,J)\in\mathfrak{S}_{n,k}} c_{I,J} \cdot R\,\mathbf{1}_{I,J}$$

Atoms of k-Local Class Fcns: $R\,\mathbf{1}_{I,J}$ for $(I,J) \in \mathfrak{S}_{n,k}$.

Atomic Symmetric Functions

Let $\mathrm{ch}_n : \mathrm{Class}(\mathfrak{S}_n) \to \Lambda_n$ be the *Frobenius characteristic*

$$\mathrm{ch}_n : f \mapsto \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) \cdot p_{\lambda(w)}$$

where $\lambda(w) \vdash n$ is the cycle type of w.

Defn: If $(I, J) \in \mathfrak{S}_{n,k}$ is a partial permutation, the *atomic symmetric function* is

$$A_{n,I,J} := n! \cdot \mathrm{ch}_{\mathrm{n}}(R\,\mathbf{1}_{I,J}).$$

Q: What is the Schur expansion of $A_{n,I,J}$?

Characters and Atomic Symmetric Functions

Defn: If $(I, J) \in \mathfrak{S}_{n,k}$ is a partial permutation, the *atomic symmetric function* is $A_{n,I,J} := n! \cdot \mathrm{ch_n}(R\,\mathbf{1}_{I,J})$.

Fact: If $[I, J] = \sum_{\substack{w \in \mathfrak{S}_n \\ w(I)=J}} w \in \mathbb{R}[\mathfrak{S}_n]$, then

$$A_{n,I,J} = \sum_{\lambda \vdash n} \chi^\lambda[I, J] \cdot s_\lambda$$

where $\chi^\lambda : \mathbb{R}[\mathfrak{S}_n] \to \mathbb{R}$ is the irreducible character.

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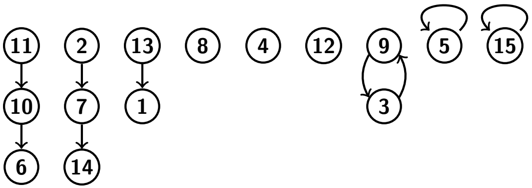
Useless A: Apply the Murnaghan-Nakayama Rule $(n-k)!$ times.

Path and Cycle Notation

Partial permutations $(I, J) \in \mathfrak{S}_{n,k}$ decompose into *paths* and *cycles*.

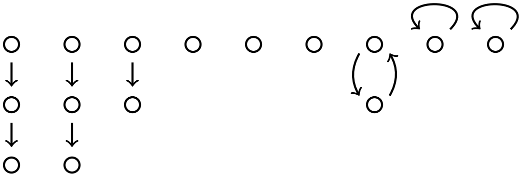


Fact: $A_{n,I,J}$ only depends on the unlabeled graph.

Path and Cycle Notation

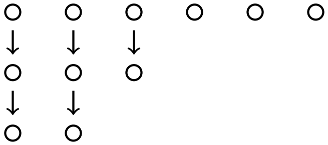
Partial permutations $(I, J) \in \mathfrak{S}_{n,k}$ decompose into *paths* and *cycles*.

Fact: We have a *factorization* $A_{n,I,J} = A^{\text{path}} \cdot A^{\text{cycle}}$. Also, A^{cycle} is a *power sum*.



$$A_{15,I,J} = A^{\text{path}} \cdot A^{\text{cycle}} = \color{red}{\vec{p}_{332111}} \cdot p_{211}$$

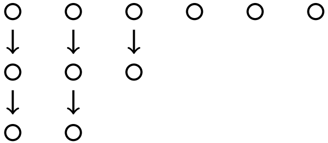
Path Power Sums



$\vec{p}_{332111}$

Def: For $\mu \vdash n$, the *path power sum* $\vec{p}_\mu$ is the atomic symmetric function of a graph with paths of sizes $\mu_1, \mu_2, \ldots$.

Path Power Sums



Def: For $\mu \vdash n$, the *path power sum* $\vec{p}_\mu$ is the atomic symmetric function of a graph with paths of sizes $\mu_1, \mu_2, \ldots$.

Facts: [HR]

- $\{\vec{p}_\mu\}$ is a basis for the space of symmetric functions.
- $\{\vec{p}_\mu\}$ is unitriangular to the power sum basis $\{p_\nu\}$.

Classical Murnaghan-Nakayama

$$A_{n,I,J} = \vec{p}_\mu \cdot p_\nu$$

MN Rule: For $\nu \models n$, we have $p_\nu = \sum_{\lambda \vdash n} \chi_\nu^\lambda \cdot s_\lambda$ where

$$\chi_\nu^\lambda = \sum_{\substack{\nu\text{-ribbon tableaux } T \\ \text{shape } \lambda}} (-1)^{\mathrm{height}(T)}$$

Classical Murnaghan-Nakayama

MN Rule: For $\nu \models n$, we have $p_\nu = \sum_{\lambda \vdash n} \chi^\lambda_\nu \cdot s_\lambda$ where

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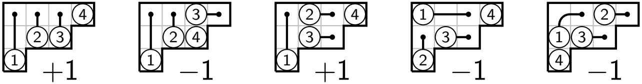
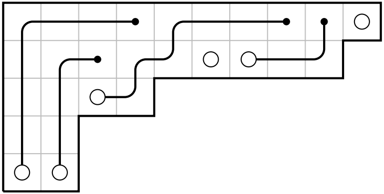


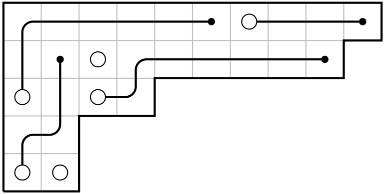
$$p_{3221} = \cdots + {\color{red}1} \cdot s_{431} + \cdots$$

Q: What about the *path* power sum $\vec{p}_\mu$?

Monotonic Ribbon Tilings



MONOTONIC

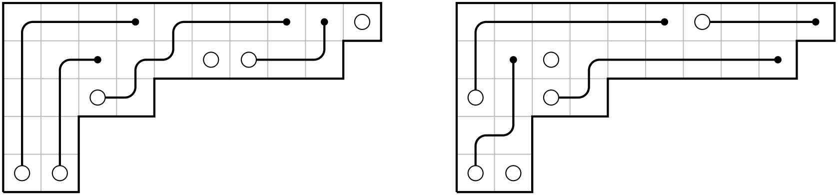


NOT MONOTONIC

Def: In a *monotonic ribbon tiling* . . .

- the tails of the ribbons lie in distinct columns,
- the tail depth decreases from left to right, and
- each initial union of ribbons forms a partition.

Path Murnaghan-Nakayama Rule

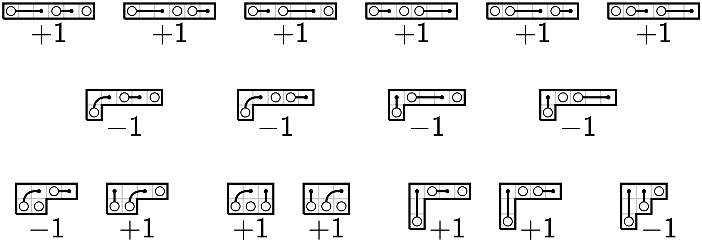


Thm: [HR] For $\mu \vdash n$, we have $\vec{p}_\mu = \sum_{\lambda \vdash n} \vec{\chi}^\lambda_\mu \cdot s_\lambda$ where

$$\vec{\chi}^\lambda_\mu = \text{mult}(\mu)! \cdot \sum_{\substack{\text{monotonic tilings } \tau \text{ of } \lambda \\ \text{ribbons of sizes } \mu}} (-1)^{\text{height}(\tau)}$$

where the ribbons are to be added **in all possible orders.**

Path Murnaghan-Nakayama Rule



$$\vec{p}_{321} = 6s_6 - 4s_{51} + 2s_{33} + 2s_{411} - s_{321}$$

Atomic Expansion

Thm: [HR] Let $(I, J) \in \mathfrak{S}_{n,k}$ have path partition μ and cycle partition ν. Then

$$A_{n,I,J} = \vec{p}_\mu \cdot p_\nu = \sum_{\substack{\lambda \vdash n \\ \rho \subseteq \lambda}} \vec{\chi}^\rho_\mu \cdot \chi^{\lambda/\rho}_\nu \cdot s_\lambda$$

where $\vec{\chi}^\rho_\mu$ counts monotonic tilings and $\chi^{\lambda/\rho}_\nu$ counts classical tilings.

- Gives a formula for $\chi^\lambda([w \cdot \mathfrak{S}_r]_+)$ on coset sums in $\mathfrak{S}_n/\mathfrak{S}_r$.
- Proves polynomiality results for pattern enumeration on conjugacy classes.
- Has applications to *probability*.

Quasi-Random Permutations

Defn: [Cooper] A sequence $w^{(n)}$ of permutations in $\mathfrak{S}_n$ is *quasi-random* if for all patterns v we have

$$\frac{N_v(w^{(n)})}{\binom{n}{k}} = \frac{1}{k!} + o(1) \qquad \text{as } n \to \infty$$

where $|v| = k$.

Thm: [HR] Let μ_n be a sequence of conjugacy-invariant probability distributions on $\mathfrak{S}_n$ and let $\Sigma_n \sim \mu_n$. If

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{m_1(\Sigma_n)}{n} = 0\right) = 1$$

then Σ_n is quasi-random.

Thanks for listening!!