Geometric vertex decomposition and liaison of toric ideals of graphs

Jenna Rajchgot

McMaster University

Joint work with Patricia Klein (Minnesota)
and with
Mike Cummings, Sergio Da Silva, Adam Van Tuyl (McMaster)

AICoVE
June 6-7, 2022

Two related viewpoints

Some ideals and varieties are popular among both commutative algebraists and algebraic combinatorialists:

- ideals of $k \times k$ minors of a generic $m \times n$ matrix $\leftrightarrow$ open patch of a Grassmannain Schubert variety
- one-sided mixed ladder determinantal ideals $\leftrightarrow$ Schubert determinantal ideals for vexillary (i.e. 2143-avoiding) permutations
- two-sided mixed ladder determinantal ideals $\leftrightarrow$ certain Kazhdan-Lusztig ideals

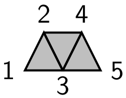
We will see that certain techniques used on the two sides of these correspondences are related.

Squarefree monomial ideals

Recall the Stanley-Reisner correspondence between a squarefree monomial ideal $I_\Delta \subseteq \mathbb{C}[x_1, \ldots, x_n]$ and simplicial complex Δ on vertices $\{1, \ldots, n\}$:

$$x_{i_1} \cdots x_{i_r} \in I_\Delta \iff \{i_1, \ldots, i_r\} \notin \Delta.$$

Example: $I_\Delta = \langle x_1x_4, x_2x_5, x_1x_5 \rangle \subseteq \mathbb{C}[x_1, \ldots, x_5]$

$\Delta =$ 

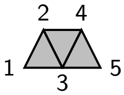
Vertex decomposability

Definition: Given a simplicial complex Δ and a vertex v of Δ, define

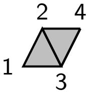
- $\mathrm{lk}_\Delta(v) := \{F \in \Delta \mid F \cup \{v\} \in \Delta, F \cap \{v\} = \varnothing\}$. link of v
- $\mathrm{del}_\Delta(v) = \{F \in \Delta \mid F \cap \{v\} = \varnothing\}$. deletion of v

Example.

$I_\Delta = \langle x_1x_4, x_1x_5, x_2x_5\rangle \subseteq \mathbb{C}[x_1, \ldots, x_5]$. $\Delta =$



$I_{\mathrm{del}_\Delta(5)} = \langle x_1x_4, x_5\rangle$. $\mathrm{del}_\Delta(5) =$



$I_{\mathrm{lk}_\Delta(5)} = \langle x_1, x_2, x_5\rangle$. $\mathrm{lk}_\Delta(5) =$

Definition: A pure simplicial complex Δ is vertex decomposable if

1. Δ is a simplex of $\Delta = \varnothing$; or
2. $\exists$ vertex $v \in \Delta$ s.t. $\mathrm{lk}_\Delta(v)$ and $\mathrm{del}_\Delta(v)$ are vertex decomposable .

Theorem: If Δ is vertex decomposable then $\mathbb{C}[x_1, \ldots, x_n]/I_\Delta$ is Cohen-Macaulay.

Geometric vertex decomposition (Knutson-Miller-Yong '09)

Set-up: Let $S = \mathbb{C}[x_1, \ldots, x_n]$, $y = x_i$, and let $<$ be a lex order with $y > x_j$, $j \neq i$. Consider an ideal

$$I = \langle yq_1 + r_1, yq_2 + r_2, \ldots, yq_\ell + r_\ell, h_1, \ldots, h_k \rangle$$

where the given gens. are a Gröbner basis and y doesn't divide any term of any q_i, r_i, h_i.

Definition/Theorem: If $C_{y,I} = \langle q_1, q_2, \ldots, q_\ell, h_1, \ldots, h_k \rangle$ and $N_{y,I} = \langle h_1, \ldots, h_k \rangle$, then

$$\text{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

and this intersection is called a geometric vertex decomposition.

Some consequences:

- the given gens of $C_{y,I}$, $N_{y,I}$ are Gröbner bases for $<$;
- in the homogeneous case, the Hilbert series of S/I can be obtained from those of S/C, S/N.

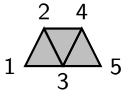
More motivation:

- lex resembles vertex decomposition
- used to study Schubert determinantal ideals for vexillary perms

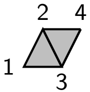
Geometric vertex decomposition: an example

Let $I = \langle x_1x_5 - x_3x_6, x_2x_5 - x_3x_4, x_1x_4 - x_2x_6 \rangle$. Let $<$ be Lex with $x_5 > x_1 > x_2 > x_3 > x_4 > x_6$.

- $\text{in}_< I = \langle x_1x_4, x_1x_5, x_2x_5 \rangle$

$\Delta =$ 

- $\text{in}_{x_5} I = \langle x_1x_5, x_2x_5, x_1x_4 - x_2x_6 \rangle = \langle x_5, x_1x_4 - x_2x_6 \rangle \cap \langle x_1, x_2 \rangle = (N_{x_5,I} + \langle x_5 \rangle) \cap C_{x_5,I}$.

$\text{in}_<(N_{x_5,I} + \langle x_5 \rangle) = I_{\text{del}_\Delta(5)} = \langle x_1x_4, x_5 \rangle$. $\text{del}_\Delta(5) =$ 

$\text{in}_<(C_{x_5,I} + \langle x_5 \rangle) = I_{\text{lk}_\Delta(5)} = \langle x_1, x_2, x_5 \rangle$. $\text{lk}_\Delta(5) =$ 

Geometrically vertex decomposable ideals

Definition (Klein-R '20): An unmixed ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is geometrically vertex decomposable if

1. $I = \langle 1 \rangle$ or I is generated by indeterminates, or
2. for some $y = x_i$, we have $\text{in}_y I = \langle C_{y,I} \rangle \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition with $N_{y,I}$ and $C_{y,I}$ geometrically vertex decomposable.

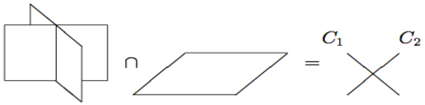
Examples: Stanley-Reisner ideals of vertex decomposable complexes, determinantal ideals, ladder determinantal ideals, Schubert determinantal ideals, defining ideals of lower bound cluster algebras, certain toric ideals of graphs

Proposition (Klein-R): If I is geometrically vertex decomposable, then I is radical and $\mathbb{C}[x_1, \ldots, x_n]/I$ is Cohen-Macaulay.

Gorenstein liaison

Let C_1 and C_2 be equidimensional subschemes of $\mathbb{P}^n$. Liaison theory asks: if $X = C_1 \cup C_2$ is "nice", do "good properties" of C_1 transfer to C_2?

Example 2.4. If X is the complete intersection in $\mathbb{P}^3$ of a surface consisting of the union of two planes with a surface consisting of one plane then X links a line C_1 to a different line C_2.



From Migliore-Nagel's "Liaison and related topics."

For us today, "nice (enough)" will mean that C_1 and C_2 share no common component and that X is Gorenstein. An example of a "good property" is the Cohen-Macaulay property.

Gorenstein Liaison

Definition: Let $V_1, V_2, X \subseteq \mathbb{P}^n$ be subschemes defined by I_{V_1}, I_{V_2}, and I_X, respectively with X arithmetically Gorenstein. If $I_X \subseteq I_{V_1} \cap I_{V_2}$ and if $[I_X : I_{V_1}] = I_{V_2}$ and $[I_X : I_{V_2}] = I_{V_1}$, then V_1 and V_2 are directly algebraically G-linked by X.

Definition:
A subscheme $V \subseteq \mathbb{P}^n$ (or its saturated and homogeneous ideal I_V) is glicci if there is a sequence of G-links from V to a complete intersection.

Theorem: Glicci $\implies$ Cohen-Macaulay

Open question: Is every arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^n$ glicci?

We will aim to use geometric vertex decomposition to study this question in some combinatorial settings!

Geometric vertex decomposition and Gorenstein liaison

Gorla, Migliore, Nagel:

- many generalized determinantal ideals are glicci
- use liaison to obtain Gröbner bases

Theorem (Nagel-Römer '07): Stanley-Reisner ideals of vertex decomposable simplicial complexes are glicci.

Theorem (Klein-R '20): A homogeneous, saturated, and unmixed geometrically vertex decomposable (gvd) ideal is glicci.

Examples of ideals that are both gvd and glicci: determinantal ideals, ladder determinantal ideals, Schubert determinantal ideals, defining ideals of lower bound cluster algebras, certain toric ideals of graphs

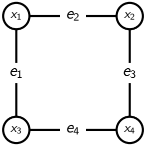
Toric ideals of graphs

Definition: Let $G = (V(G), E(G))$ be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G) = \{e_1, \ldots, e_t\}$ where each $e_i = \{x_j, x_k\}$. Consider the homomorphism $\varphi_G : \mathbb{C}[E(G)] \to \mathbb{C}[V(G)]$:

$$\varphi_G(e_i) = x_j x_k \text{ where } e_i = \{x_j, x_k\} \text{ for all } i \in \{1, \ldots, t\}.$$

The toric ideal of the graph G, denoted I_G, is ker φ_G.

Example:

$G =$ 

$$\varphi_G(e_1) = x_1x_3, \ \varphi_G(e_2) = x_1x_2, \ \varphi_G(e_3) = x_2x_4, \ \varphi_G(e_4) = x_3x_4.$$

$$\ker \varphi_G = \langle e_1e_3 - e_2e_4 \rangle$$

Toric ideals of graphs and Gorenstein liaison

Theorems: Let G be a finite simple graph and let I_G be its toric ideal.

- If there is a monomial order $<$ such that $\text{in}_< I_G$ is squarefree, then $\mathbb{C}[e_1, \dots, e_t]/I_G$ is normal. (Sturmfels)
- If $\mathbb{C}[e_1, \dots, e_t]/I_G$ is normal, then it is also Cohen-Macaulay. (Hochster)

$\implies$ if there is a monomial order $<$ such that $\text{in}_< I_G$ is squarefree then $\mathbb{C}[e_1, \dots, e_t]/I_G$ is Cohen-Macaulay.

Question: If there is a monomial order such that $\text{in}_< I_G$ is squarefree, must I_G be geometrically vertex decomposable, hence glicci?

Theorem (Constantinescu-Gorla '17): Toric ideals of bipartite graphs are glicci.

Some results on gvd of toric ideals of graphs

Let G be a finite simple graph and let $I_G \subseteq \mathbb{C}[E(G)]$ be its toric ideal.

Theorem (Cummings-Da Silva- R- Van Tuyl '22):

1. Suppose that G is bipartite. Then I_G is geometrically vertex decomposable.
2. Suppose that I_G has a universal Gröbner basis consisting of quadratic binomials. Then I_G is geometrically vertex decomposable.

Theorem (Cummings-Da Silva- R- Van Tuyl '22): Let H be obtained from G by attaching a cycle of even length to G along a single edge.

1. If $\mathbb{C}[E(G)]/I_G$ is Cohen-Macaulay, then I_H is glicci.
2. If I_G is geometrically vertex decomposable then so is I_H.

Further results on liaison of toric ideals of graphs

Let G be a finite simple graph and let $I_G \subseteq \mathbb{C}[E(G)]$ be its toric ideal.

Theorem (Cummings-Da Silva- R- Van Tuyl '22): Suppose that

- there is an edge $y \in E(G)$ contained in a 4-cycle of G; and
- $\text{in}_<(I_G)$ is a square-free monomial ideal for some lexicographic monomial order $<$ with $y > e$ for all $e \in E(G)$ with $e \neq y$.

Then I_G is glicci.

Definition: A graph G is gap-free if for any two edges $e_1 = \{u, v\}$ and $e_2 = \{w, x\}$ with $\{u, v\} \cap \{w, x\} = \varnothing$, there is an edge $e \in E(G)$ that is is adjacent to both e_1 and e_2, i.e, one of the edges $\{u, w\}, \{u, x\}, \{v, w\}, \{v, x\}$ is also in G.

Using the above theorem and a result of D'Ali on gap free graphs we get:

Corollary (Cummings-Da Silva- R- Van Tuyl '22): Suppose G is a gap free graph which contains a 4-cycle. Then I_G is glicci.

Thank you!