Rigid Gorenstein toric Fano varieties arising from directed graphs

Selvi Kara, İrem Portakal, Akiyoshi Tsuchiya

Technical University of Munich

AICoVE: an Algebraic Combinatorics Virtual Expedition
June 6-7, 2022

Edge polytopes

- Let $G = (V(G), A(G))$ be a finite directed graph on the vertex set $V(G) = [n]$ with the directed edge set $A(G)$.

Edge polytopes

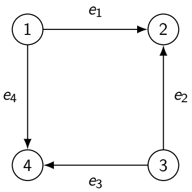
- Let $G = (V(G), A(G))$ be a finite directed graph on the vertex set $V(G) = [n]$ with the directed edge set $A(G)$.
- For a directed edge $e = (i, j) \in A(G)$, we define $\rho(e) \in \mathbb{R}^n$ by setting $\rho(e) = \mathbf{e}_i - \mathbf{e}_j$.

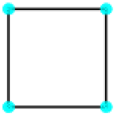
Edge polytopes

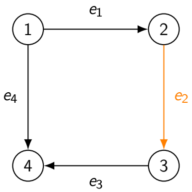
- Let $G = (V(G), A(G))$ be a finite directed graph on the vertex set $V(G) = [n]$ with the directed edge set $A(G)$.
- For a directed edge $e = (i, j) \in A(G)$, we define $\rho(e) \in \mathbb{R}^n$ by setting $\rho(e) = \mathbf{e}_i - \mathbf{e}_j$.
- The *directed edge polytope* of G, denoted by $\mathcal{A}_G$, is the lattice polytope defined as

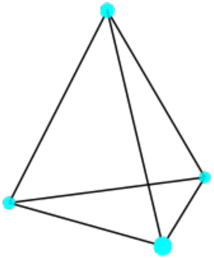
$$\mathcal{A}_G = \mathrm{conv}\{\rho(e) : e \in A(G)\} \subset \mathbb{R}^n.$$

Two edge polytopes with the same underlying graph









Fano polytopes

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope.

- We say that $\mathcal{P}$ is a *Fano* if the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{P}$ and the vertices of $\mathcal{P}$ are primitive lattice points in $\mathbb{Z}^n$.

Fano polytopes

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope.

- We say that $\mathcal{P}$ is a *Fano* if the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{P}$ and the vertices of $\mathcal{P}$ are primitive lattice points in $\mathbb{Z}^n$.
- A Fano polytope $\mathcal{P}$ is called *terminal* if every lattice point on the boundary is a vertex.

Fano polytopes

Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional lattice polytope.

- We say that $\mathcal{P}$ is a *Fano* if the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{P}$ and the vertices of $\mathcal{P}$ are primitive lattice points in $\mathbb{Z}^n$.
- A Fano polytope $\mathcal{P}$ is called *terminal* if every lattice point on the boundary is a vertex.
- A Fano polytope $\mathcal{P}$ is said to be *reflexive* if each facet of $\mathcal{P}$ has lattice distance one from the origin. Equivalently, its dual polytope

  $$\mathcal{P}^\vee = \{x \in \mathbb{R}^n : \langle x, y \rangle \geqslant -1 \text{ for all } y \in \mathcal{P}\}$$

  is a lattice polytope.

Terminal reflexive edge polytopes

Proposition ([HHMNO11])

Let G be a finite directed graph. Then the following arguments are equivalent:

1. *$\mathcal{A}_G$ is Fano;*
2. *$\mathcal{A}_G$ is terminal reflexive;*
3. *Every directed edge of G belongs to a directed cycle in G.*

T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, *Roots of Ehrhart polynomials arising from graphs*. J Algebr Comb 34, 721–749 (2011).

Terminal reflexive edge polytopes

Proposition ([HHMNO11])

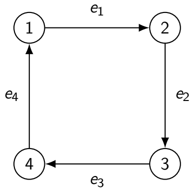
Let G be a finite directed graph. Then the following arguments are equivalent:

1. *$\mathcal{A}_G$ is Fano;*
2. *$\mathcal{A}_G$ is terminal reflexive;*
3. *Every directed edge of G belongs to a directed cycle in G.*

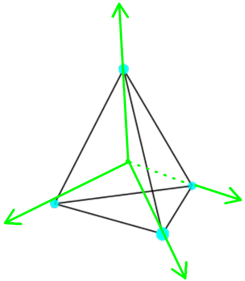
Notation: For a Fano polytope $\mathcal{A}_G$, X_G is the *normal toric Fano variety* associated to the *spanning fan* of $\mathcal{A}_G$.
In this case, X_G is a Gorenstein toric Fano variety with terminal singularities.

T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, *Roots of Ehrhart polynomials arising from graphs*. J Algebr Comb 34, 721–749 (2011).

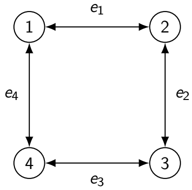
Gorenstein toric Fano variety arising from a directed graph



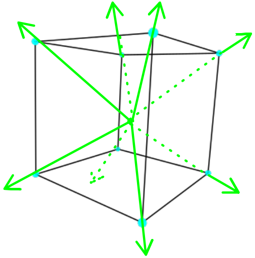
G_1



$\subseteq N_{\mathbb{R}}$



G_2



$\subseteq N_{\mathbb{R}}$

What is a deformation of an algebraic variety?

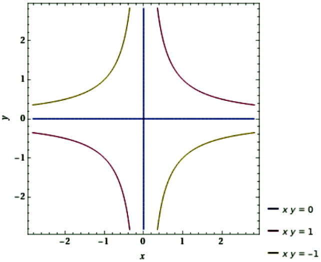
Let X be a scheme of finite type over $\mathbb{C}$ and let A be an Artinian algebra over $\mathbb{C}$. An infinitesimal deformation of X over A is defined as the following cartesian diagram:

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

What is a deformation of an algebraic variety?

Let X be a scheme of finite type over $\mathbb{C}$ and let A be an Artinian algebra over $\mathbb{C}$. An infinitesimal deformation of X over A is defined as the following cartesian diagram:

$$\begin{array}{llcccl} Example: & xy=0 & X & \hookrightarrow & \mathcal{X} & \{xy=t\} \\ & & \downarrow & & \downarrow \pi & \\ & 0 & \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) & \mathbb{C} \end{array}$$



What is a deformation of an algebraic variety?

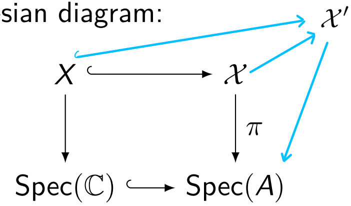
Let X be a scheme of finite type over $\mathbb{C}$ and let A be an Artinian algebra over $\mathbb{C}$. An infinitesimal deformation of X over A is defined as the following cartesian diagram:

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

where π is flat. Let $\pi' : X \to \mathcal{X}'$ be another deformation of X over $\mathrm{Spec}(A)$. We say that π and π' are isomorphic, if there exists a map $\mathcal{X} \to \mathcal{X}'$ over $\mathrm{Spec}(A)$ inducing the identity on X.

What is a deformation of an algebraic variety?

Let X be a scheme of finite type over $\mathbb{C}$ and let A be an Artinian algebra over $\mathbb{C}$. An infinitesimal deformation of X over A is defined as the following cartesian diagram:



where π is flat. Let $\pi' : X \to \mathcal{X}'$ be another deformation of X over $\mathrm{Spec}(A)$. We say that π and π' are isomorphic, if there exists a map $\mathcal{X} \to \mathcal{X}'$ over $\mathrm{Spec}(A)$ inducing the identity on X.
Let Def_X be a functor such that $Def_X(A)$ is the set of deformations of X over $\mathrm{Spec}(A)$ modulo isomorphisms.

What is rigidity?

Definition

The map π is called a first order deformation of X if $S = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$. We set $T_X^1 := \mathrm{Def}_X(\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)))$.

What is rigidity?

Definition

The map π is called a first order deformation of X if $S = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$. We set $T^1_X := \mathrm{Def}_X(\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)))$.

- The variety X is called rigid if $T^1_X = 0$.
- This implies that X has no nontrivial infinitesimal deformations.

What is rigidity?

Definition

The map π is called a first order deformation of X if $S = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$. We set $T^1_X := \mathrm{Def}_X(\mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)))$.

- The variety X is called rigid if $T^1_X = 0$.
- This implies that X has no nontrivial infinitesimal deformations.

i.e. every deformation $\pi \in \mathrm{Def}_X(A)$ over a Artin ring A is isomorphic to the trivial deformation $X \times \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A)$.

What is rigidity?

Definition

The map π is called a first order deformation of X if $S = \text{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$. We set $T^1_X := \text{Def}_X(\text{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)))$.

- The variety X is called rigid if $T^1_X = 0$.
- This implies that X has no nontrivial infinitesimal deformations.

i.e. every deformation $\pi \in \text{Def}_X(A)$ over a Artin ring A is isomorphic to the trivial deformation $X \times \text{Spec}(A) \longrightarrow \text{Spec}(A)$.

Our aim: Describe which Gorenstein toric Fano varieties arising from directed graphs are rigid purely in terms of graphs.

Combinatorial study of rigidity

- A Fano polytope is called *$\mathbb{Q}$-factorial* if it is simplicial.
- A Fano polytope is called *smooth* if the vertices of each facet form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

F. Bien, M. Brion, *Automorphisms and local rigidity of regular varieties*. Compos. Math. 104(1), 1–26 (1996)

T. de Fernex, C.D. Hacon, *Deformations of canonical pairs and Fano varieties*. J. Reine Angew. Math. 651, 97–126 (2011)

A. Higashitani, *Smooth Fano polytopes arising from finite directed graphs*. Kyoto Journal of Mathematics, 55(3):579–592, 2015.

Combinatorial study of rigidity

- A Fano polytope is called *$\mathbb{Q}$-factorial* if it is simplicial.
- A Fano polytope is called *smooth* if the vertices of each facet form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

[BB96] Every smooth toric Fano variety is rigid.

[dFH11] $\mathbb{Q}$-factorial terminal toric Fano varieties are rigid.

F. Bien, M. Brion, *Automorphisms and local rigidity of regular varieties*. Compos. Math. 104(1), 1–26 (1996)

T. de Fernex, C.D. Hacon, *Deformations of canonical pairs and Fano varieties*. J. Reine Angew. Math. 651, 97–126 (2011)

A. Higashitani, *Smooth Fano polytopes arising from finite directed graphs*. Kyoto Journal of Mathematics, 55(3):579–592, 2015.

Combinatorial study of rigidity

- A Fano polytope is called *$\mathbb{Q}$-factorial* if it is simplicial.
- A Fano polytope is called *smooth* if the vertices of each facet form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

[BB96] Every smooth toric Fano variety is rigid.
[dFH11] $\mathbb{Q}$-factorial terminal toric Fano varieties are rigid.

Theorem ([Hig15])

Let G be a finite symmetric directed graph on $[n]$. Then the following arguments are equivalent:

1. *X_G is smooth;*
2. *X_G is $\mathbb{Q}$-factorial;*
3. *G^{un} has no even cycle as subgraphs.*

F. Bien, M. Brion, *Automorphisms and local rigidity of regular varieties*. Compos. Math. 104(1), 1–26 (1996)

T. de Fernex, C.D. Hacon, *Deformations of canonical pairs and Fano varieties*. J. Reine Angew. Math. 651, 97–126 (2011)

A. Higashitani, *Smooth Fano polytopes arising from finite directed graphs*. Kyoto Journal of Mathematics, 55(3):579–592, 2015.

Combinatorial study of rigidity

The most general rigidity theorem for toric Fano varieties known to this date is the following result of Totaro:

Theorem ([Tot12])

A toric Fano variety which is smooth in codimension 2 and $\mathbb{Q}$*-factorial in codimension 3 is rigid.*

B. Totaro. *Jumping of the nef cone for Fano varieties*. J. Algebraic Geom., 21:375–396, 2012.

Combinatorial study of rigidity

The most general rigidity theorem for toric Fano varieties known to this date is the following result of Totaro:

Theorem ([Tot12])

A toric Fano variety which is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3 is rigid.

Proposition

Let $\mathcal{P}$ be a terminal reflexive polytope. Then

1. *$X_{\mathcal{P}}$ is smooth in codimension 2,*
2. *all 2-faces of $\mathcal{P}$ are triangles if and only if $X_{\mathcal{P}}$ is $\mathbb{Q}$-factorial in codimension 3.*

In particular, $X_{\mathcal{P}}$ is rigid.

B. Totaro. *Jumping of the nef cone for Fano varieties*. J. Algebraic Geom., 21:375–396, 2012.

Combinatorial study of rigidity

The most general rigidity theorem for toric Fano varieties known to this date is the following result of Totaro:

Theorem ([Tot12])

A toric Fano variety which is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3 is rigid.

Proposition

Let $\mathcal{P}$ be a terminal reflexive polytope. Then

1. *$X_{\mathcal{P}}$ is smooth in codimension 2,*
2. *all 2-faces of $\mathcal{P}$ are triangles if and only if $X_{\mathcal{P}}$ is $\mathbb{Q}$-factorial in codimension 3.*

In particular, $X_{\mathcal{P}}$ is rigid.

It remains to classify the directed edge polytopes whose 2-faces are all triangles.

B. Totaro. *Jumping of the nef cone for Fano varieties*. J. Algebraic Geom., 21:375–396, 2012.

Faces of directed edge polytopes

Lemma (Kara, , Tsuchiya)

Let G be a connected finite directed graph such that $\mathcal{A}_G$ is terminal and reflexive. Then each proper face of the directed edge polytope $\mathcal{A}_G$ is the directed edge polytope of a finite directed acyclic subgraph H of G.

S. Kara, I. Portakal, and A. Tsuchiya, Rigid Gorenstein toric Fano varieties arising from directed graphs, Collect. Math. (2022)

Faces of directed edge polytopes

Lemma (Kara, , Tsuchiya)

Let G be a connected finite directed graph such that $\mathcal{A}_G$ is terminal and reflexive. Then each proper face of the directed edge polytope $\mathcal{A}_G$ is the directed edge polytope of a finite directed acyclic subgraph H of G.

For a finite acyclic directed graph $D = ([n], A(D))$, we define a lattice polytope $\widetilde{\mathcal{A}}_D$ by

$$\widetilde{\mathcal{A}}_D := \mathrm{conv}\{\mathbf{0}, \mathbf{e}_i - \mathbf{e}_j : (i,j) \in A(D)\} \subset \mathbb{R}^n.$$

S. Kara, I. Portakal, and A. Tsuchiya, Rigid Gorenstein toric Fano varieties arising from directed graphs, Collect. Math. (2022)

Faces of directed edge polytopes

Lemma (Kara, , Tsuchiya)

Let G be a connected finite directed graph such that $\mathcal{A}_G$ is terminal and reflexive. Then each proper face of the directed edge polytope $\mathcal{A}_G$ is the directed edge polytope of a finite directed acyclic subgraph H of G.

For a finite acyclic directed graph $D = ([n], A(D))$, we define a lattice polytope $\widetilde{\mathcal{A}}_D$ by

$$\widetilde{\mathcal{A}}_D := \mathrm{conv}\{\mathbf{0}, \mathbf{e}_i - \mathbf{e}_j : (i,j) \in A(D)\} \subset \mathbb{R}^n.$$

Let $\mathcal{A}_{D_1}, \dots, \mathcal{A}_{D_r}$ be the facets of $\mathcal{A}_G$ with acyclic directed graphs $D_1, \dots, D_r$. Since the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{A}_G$, the directed edge polytope $\mathcal{A}_G$ is divided by $\widetilde{\mathcal{A}}_{D_1}, \dots, \widetilde{\mathcal{A}}_{D_r}$.

S. Kara, I. Portakal, and A. Tsuchiya, Rigid Gorenstein toric Fano varieties arising from directed graphs, Collect. Math. (2022)

Faces of directed edge polytopes

Lemma (Kara, , Tsuchiya)

Let G be a connected finite directed graph such that $\mathcal{A}_G$ is terminal and reflexive. Then each proper face of the directed edge polytope $\mathcal{A}_G$ is the directed edge polytope of a finite directed acyclic subgraph H of G.

For a finite acyclic directed graph $D = ([n], A(D))$, we define a lattice polytope $\widetilde{\mathcal{A}}_D$ by

$$\widetilde{\mathcal{A}}_D := \mathrm{conv}\{\mathbf{0}, \mathbf{e}_i - \mathbf{e}_j : (i,j) \in A(D)\} \subset \mathbb{R}^n.$$

Let $\mathcal{A}_{D_1}, \dots, \mathcal{A}_{D_r}$ be the facets of $\mathcal{A}_G$ with acyclic directed graphs $D_1, \dots, D_r$. Since the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{A}_G$, the directed edge polytope $\mathcal{A}_G$ is divided by $\widetilde{\mathcal{A}}_{D_1}, \dots, \widetilde{\mathcal{A}}_{D_r}$. In particular, each face of $\widetilde{\mathcal{A}}_{D_i}$ which does not contain the origin is a face of $\mathcal{A}_{D_i}$, hence a face of $\mathcal{A}_G$.

S. Kara, I. Portakal, and A. Tsuchiya, Rigid Gorenstein toric Fano varieties arising from directed graphs, Collect. Math. (2022)

Faces of directed edge polytopes

The directed edge polytope $\mathcal{A}_G$ is divided by $\widetilde{\mathcal{A}}_{D_1}, \ldots, \widetilde{\mathcal{A}}_{D_r}$ and each face of $\widetilde{\mathcal{A}}_{D_i}$ which does not contain the origin is a face of $\mathcal{A}_{D_i}$, hence a face of $\mathcal{A}_G$.

L. Setiabrata. Faces of root polytopes. *SIAM Journal on Discrete Mathematics*, 2021.

Faces of directed edge polytopes

The directed edge polytope $\mathcal{A}_G$ is divided by $\widetilde{\mathcal{A}}_{D_1}, \ldots, \widetilde{\mathcal{A}}_{D_r}$ and each face of $\widetilde{\mathcal{A}}_{D_i}$ which does not contain the origin is a face of $\mathcal{A}_{D_i}$, hence a face of $\mathcal{A}_G$.

Theorem ([Set21])

Let $D = (V(D), A(D))$ be a finite acyclic directed graph and $H \subset D$ a directed subgraph of D with $V(H) = V(D)$ and the directed edge set $A(H)$. Then the polytope $\mathcal{A}_H$ is a face of $\widetilde{\mathcal{A}}_D$ if and only if H is path consistent and admissible.

L. Setiabrata. Faces of root polytopes. *SIAM Journal on Discrete Mathematics*, 2021.

Rigidity purely in terms of graphs

Theorem (Kara, , Tsuchiya)

Let G be a finite directed graph such that every directed edge of G belongs to a directed cycle in G. Then the following arguments are equivalent:

1. *X_G is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3.*

Rigidity purely in terms of graphs

Theorem (Kara, , Tsuchiya)

Let G be a finite directed graph such that every directed edge of G belongs to a directed cycle in G. Then the following arguments are equivalent:

1. *X_G is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3.*
2. *G satisfies both of the following:*
   - *G has no directed subgraph C_1 whose directed edge set is*

     $$\{(i_1, i_2), (i_1, i_4), (i_3, i_2), (i_3, i_4)\};$$

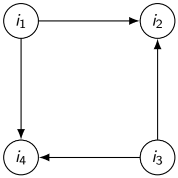
   - *For any directed subgraph C_2 of G whose directed edge set is*

     $$\{(i_1, i_2), (i_2, i_3), (i_1, i_4), (i_4, i_3)\},$$

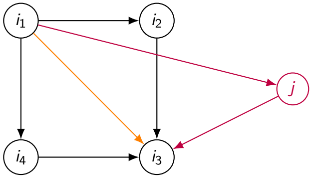
     it follows that (i_1, i_3) is a directed edge of G or there exists a vertex $j \notin \{i_2, i_4\}$ in G such that (i_1, j) and (j, i_3) are directed edges of G.

In particular, X_G is rigid.

Rigidity purely in terms of graphs



C_1



C_2

- G has no directed subgraph C_1 whose directed edge set is

$$\{(i_1, i_2), (i_1, i_4), (i_3, i_2), (i_3, i_4)\};$$

- For any directed subgraph C_2 of G whose directed edge set is

$$\{(i_1, i_2), (i_2, i_3), (i_1, i_4), (i_4, i_3)\},$$

it follows that (i_1, i_3) is a directed edge of G or there exists a vertex $j \notin \{i_2, i_4\}$ in G such that (i_1, j) and (j, i_3) are directed edges of G.

Rigidity purely in terms of graphs

Corollary (Kara, , Tsuchiya)

Let G be a finite symmetric directed graph. Then the following arguments are equivalent:

1. *X_G is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3.*
2. *The underlying undirected graph G^{un} has no 4-cycle as a subgraph.*

In particular, X_G is rigid.

Rigidity purely in terms of graphs

Corollary (Kara, , Tsuchiya)

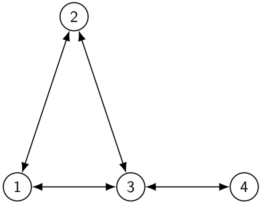
Let G be a finite symmetric directed graph. Then the following arguments are equivalent:

1. *X_G is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3.*
2. *The underlying undirected graph G^{un} has no 4-cycle as a subgraph.*

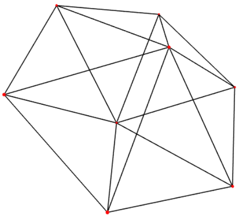
In particular, X_G is rigid.

Example

The Gorenstein toric Fano variety X_G is rigid, since G has a no C_4 as a subgraph equivalently $\mathcal{A}$ has no square 2-face.



G



Thank you for your time!

F. Bien, M. Brion.
Automorphisms and local rigidity of regular varieties.
Compos. Math. 104(1), 1–26 (1996)

T. de Fernex, C.D. Hacon
Deformations of canonical pairs and Fano varieties.
J. Reine Angew. Math. 651, 97–126 (2011)

A. Higashitani.
Smooth Fano polytopes arising from finite directed graphs.
Kyoto Journal of Mathematics, 55(3):579–592, 2015.

S. Kara, I. Portakal, A. Tsuchiya.
Rigid Gorenstein toric Fano varieties arising from directed graphs.
Collect. Math. (2022).

T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi,
Roots of Ehrhart polynomials arising from graphs.
J. Algebr. Comb. 34, 721–749 (2011).
https://doi.org/10.1007/s10801-011-0290-8

L. Setiabrata.
Faces of root polytopes.
SIAM Journal on Discrete Mathematics, 35(3):2093–2114, 2021.

B. Totaro.
Jumping of the nef cone for Fano varieties.
J. Algebraic Geom., 21:375–396, 2012.