Splitting the cohomology of regular semisimple Hessenberg varieties

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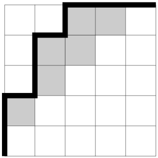
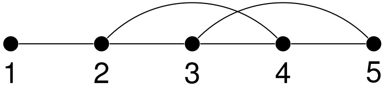
Dyck paths

- A Hessenberg function is a non-decreasing function $\mathbf{m}\colon [n] \to [n]$ such that $\mathbf{m}(i) \geq i$ for every $i \in [n]$.
- The graph associated to h is the graph with vertex set $[n]$ and set of edges $E = \{\{i,j\}; i < j \leq \mathbf{m}(i)\}$.
- These are called indifference graphs.
- Hessenberg functions can also be identified with Dyck paths.
- In the rest of the talk by *graph* we will almost always mean an indifference graph.



$\mathbf{m} = (2,4,5,5,5)$



G

Definitions

- The chromatic polynomial of a graph admits a symmetric function generalization introduced by Stanley. Given a graph G it is defined as

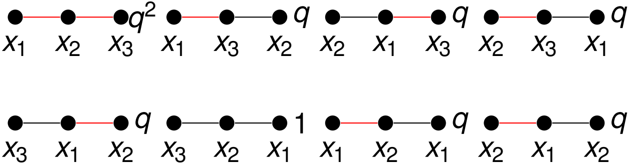
$$\mathrm{csf}(G) := \sum_{\kappa} x_{\kappa}.$$

  where the sum runs through all proper colorings of the vertices $\kappa\colon V(G) \to \mathbb{N}$ and $x_\kappa := \prod_{v \in V(G)} x_{\kappa(v)}$.
- Given a graph G the chromatic quasisymmetric function $\mathrm{csf}_q(G)$ is

$$\mathrm{csf}_q(G) := \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa}.$$

  where the sum runs through all proper colorings of the vertices $\kappa\colon V(G) \to \mathbb{N}$.

Example of csf_q



$$\mathrm{csf}_q(G) = (1 + 4q + q^2)x_1x_2x_3 + q(x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2)$$

Basis of the symmetric algebra

- Elementary basis

$$e_n = \sum_{1 \le i_1 < i_2 < \ldots < i_n} \prod_{j=1}^{n} x_{i_j}, \qquad e_\lambda = \prod e_{\lambda_i}.$$

- Power sum basis

$$p_n = \sum_{1 \le i} x_i^n, \qquad p_\lambda = \prod p_{\lambda_i}.$$

- For $\mathrm{csf}_q(G)$ in the example before we have

$$\begin{aligned} \mathrm{csf}_q(G) =& (1 + q + q^2)x_1 x_2 x_3 + q(x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3) \\ =& (1 + q + q^2)e_3 + q e_{2,1} \\ \mathrm{csf}(G) =& p_{1,1,1} - 2p_{2,1} + p_3 \end{aligned}$$

Basis of the symmetric algebra

- Complete homogeneous basis

$$h_n = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_n} \prod_{j=1}^{n} x_{i_j}, \qquad h_\lambda = \prod h_{\lambda_i}.$$

- Schur basis For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ the Schur symmetric function s_λ can be defined as

$$s_\lambda := \det(h_{\lambda_i + j - i})_{i,j=1,\ldots,l}.$$

$$\mathrm{csf}(G) = (1 + 2q + q^2)s_{1,1,1} + qs_{2,1}$$

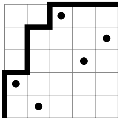
Involution ω of Λ

There is an involution on Λ given by

$$\begin{aligned} \omega \colon \Lambda &\longrightarrow \Lambda \\ e_n &\longmapsto h_n \\ h_n &\longmapsto e_n \\ s_\lambda &\longmapsto s_{\lambda^t} \\ p_n &\mapsto (-1)^{n-1} p_n \end{aligned}$$

Most of the results that follows are for $\omega(csf_q(G))$.

Stanley-Stembridge formula



$\sigma = 21534$

Definition

We define $S_{n,\mathbf{m}}$ as $\{\sigma \in S_n; \sigma(i) \leq \mathbf{m}(i) \text{ for every } i\}$.

The p-expansion of csf in terms of permutations

We have that

$$\omega(\mathrm{csf}(G_{\mathbf{m}})) = \sum_{\sigma \in S_{n,\mathbf{m}}} p_{\lambda(\sigma)},$$

where $\lambda(\sigma)$ is the cycle partition of σ.

The p-expansion of csf in terms of increasing forests

We have that

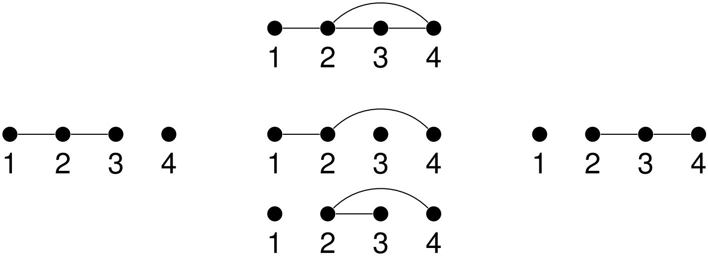
$$\omega(\mathrm{csf}(G_{\mathbf{m}})) = \sum_{F \in IF(G_{\mathbf{m}})} p_{\lambda(F)}.$$

Example

If $\mathbf{m} = (2,4,4,4)$, we have that

$$\omega(\mathrm{csf}(G_{\mathbf{m}})) = p_{1,1,1,1} + 4p_{2,1,1} + p_{2,2} + 4p_{3,1} + 2p_4$$

Below the increasing spanning forests with partition $(3,1)$.



Representations of S_n and the Frobenius character

- The irreducible representations of S_n are indexed by partitions, representing its conjugacy classes.
- For every $\lambda \vdash n$, we let $S_n \to GL(V_\lambda)$ be the irreducible representation associated with λ.
- Every other representation $S_n \to GL(W)$ admits a decomposition $W = \bigoplus_\lambda V_\lambda^{a_\lambda}$.
- We define the Frobenius character of W by $\mathrm{ch}(W) = \sum a_\lambda s_\lambda \in \Lambda_n$.

Induced representations

- For each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we let

$$S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_i} \subset S_n.$$

- We have that S_n acts on S_n/S_λ by multiplication on the left.
- This gives a representation $S_n \to GL(\mathbb{C}^{S_n/S_\lambda})$, the representation induced by the trivial representation of S_λ.
- The Frobenius character of $\mathbb{C}^{S_n/S_\lambda}$ is h_λ.

Hessenberg varieties

- For a Hessenberg function **m** and a diagonal matrix X with distinct entries, we define the Hessenberg variety

$$\mathcal{Y}_{\mathbf{m}}(X) := \{V_1 \subset \ldots \subset V_n = \mathbb{C}^n; XV_i \subset V_{\mathbf{m}(i)}\}$$

- The Hessenberg variety is a subvariety of the flag variety.
- There is a S_n-action on the cohomology $H^*(\mathcal{Y}_{\mathbf{m}}(X))$.
- $\mathrm{ch}_q(H^*(\mathcal{Y}_{\mathbf{m}}(X))) = \mathrm{csf}_q(G_{\mathbf{m}})$.

Decomposition Theorem

- Consider the map $\mathcal{Y}_{\mathbf{m}}(X) \to \mathbb{P}^{n-1} = \mathrm{Gr}(1, n)$, $V_\bullet \mapsto V_1$.
- The decomposition theorem states that we can write the cohomology $H^*(\mathcal{Y}_{\mathbf{m}}(X))$ in terms of the cohomology of subvarieties of $\mathbb{P}^{n-1}$.
- In this case, these subvarieties will be the varieties $H_i \subset \mathbb{P}^{n-1}$ that are the union of the coordinates codimension i planes.
- H_i is the union of the closures of the orbits of codimension i via the action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{n-1}$.
- Explicitly

$$H^*(\mathcal{Y}_{\mathbf{m}}(X)) = \bigoplus_{i=0}^{n-1} H^*(\widetilde{H}_i) \otimes L_i.$$

- This decomposition is compatible with the S_n-action. That is, we have a S_n-action on each summand $H^*(\widetilde{H}_i) \otimes L_i$. Moreover, each vector space L_i has an action of S_i.

- $\widetilde{H}_i$ is the union of $\binom{n}{i}$ copies of $\mathbb{P}^{n-i-1}$.
- We have that $\operatorname{ch}(H^*(\widetilde{H}_i) \otimes L_i) = (n-i)h_{n-i}\operatorname{ch}(L_i)$.
- Our goal: find a combinatorial interpretation for $\operatorname{ch}(L_i)$

Theorem

We have that $g_k(G_{\mathbf{m}}) := \omega(\operatorname{ch}(L_k))$ *(*$k = 0, \ldots, n-1$*) is equal to*

$$\sum_{\substack{\sigma=\tau_1\cdots\tau_j \in S_{n,\mathbf{m}} \\ |\tau_1| \geq n-k}} (-1)^{k-j+1} h_{|\tau_1|-n+k} p_{|\tau_2|} \cdots p_{|\tau_j|}$$

which is equivalent to

$$\sum_{\substack{F=T_1\cup\ldots\cup T_j \in IF(G_{\mathbf{m}}) \\ |T_1| \geq n-k}} (-1)^{k-j+1} h_{|T_1|-n+k} p_{|T_2|} \cdots p_{|T_j|}.$$

The symmetric function $g_k(G_{\mathbf{m}})$ *is of degree* k*. There are* q*-analogue versions.*

Corollaries

Corollary

We have that

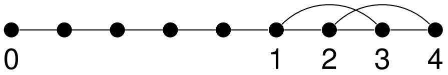
$$\operatorname{csf}(G_{\mathbf{m}}) = \sum_{i=0}^{n-1} (n-i) e_{n-i} g_i(\mathbf{m}).$$

Corollary

The symmetric function $g_k(G_{\mathbf{m}})$ is Schur-positive.

Invariance of g_k

- Let P_ℓ be the path graph with vertices $1, \ldots, \ell$. Consider the indifference graph $P_\ell \sqcup_{\ell,1} G_{\mathbf{m}}$ obtained by attaching the vertex ℓ of P_ℓ with the vertex 1 of $G_{\mathbf{m}}$. Then $g_k(P_\ell \sqcup_{\ell,1} G_{\mathbf{m}}) = g_k(G_{\mathbf{m}})$.



- Every increasing subtree of $P_\ell \sqcup_{\ell,1} G_{\mathbf{m}}$ with root 0 and size at least ℓ is of the form $P_\ell \sqcup_{\ell,1} T$ where T is an increasing tree of $G_{\mathbf{m}}$ with root 1.
- We can extend the definition of $g_k(G_{\mathbf{m}})$ for every k, by defining $g_k(G_{\mathbf{m}}) = g_k(P_\ell \sqcup_{\ell,1} G_{\mathbf{m}})$ for $\ell >> 0$.
- As consequence we have that $g_k(P_\ell) = g_k(\bullet)$.

Generating functions

- We have the following generating function for $\mathrm{csf}(P_\ell)$

$$\sum_{k\geq 0} \mathrm{csf}(P_k) z^k = \frac{\sum_{\ell\geq 0} e_k z^k}{1 - \sum_{k\geq 2}(k-1)e_k z^k}$$

- We have the following generating function for $g_k(\bullet)$.

$$\sum_{k\geq 0} g_k(\bullet) z^k = \frac{1}{1 - \sum_{k\geq 2}(k-1)e_k z^k}$$

- This actually holds for any graph $G_{\mathbf{m}}$. We have the following generating function for $g_k(G_{\mathbf{m}})$. We have that $\sum_{k\geq 0} g_k(G_{\mathbf{m}}) z^k$ is equal to

$$\sum_{i=0}^{n-1} \left(\frac{g_i(G_{\mathbf{m}}) z^i}{1 - \sum_{k\geq 2}(k-1)e_k z^k} (1 - \sum_{k=2}^{n-i-1} (k-1) e_k z^k) \right)$$

Derangements and Eulerian numbers

- We have that $\mathrm{csf}_q(P_m) = \sum_{\lambda \vdash m} A_\lambda(q) m_\lambda$, where $A_\lambda(q)$ is the Eulerian polynomial associated to λ.
- $$A_\lambda(q) = \sum q^{desc(\sigma)}$$
  σ is a permutation of $w_\lambda = (1, \dots, 1, 2, \dots, 2, \dots, \ell(\lambda), \dots, \ell(\lambda))$ such that $\sigma(i+1) \neq \sigma(i)$ and $desc(\sigma)$ is the number of indices i such that $\sigma(i) > \sigma(i+1)$.
- We have that $g_m(\bullet; q) = \sum_{\lambda \vdash m} D_\lambda(q) m_\lambda$, where $D_\lambda(q)$ is the Derangement polynomial associated to λ.
- $$D_\lambda(q) = \sum_\sigma q^{exc(\sigma)}$$
  σ is a derangement of $w_\lambda = (1, \dots, 1, 2, \dots, 2, \dots, \ell(\lambda), \dots, \ell(\lambda))$ (that is $\sigma \in S_n/S_\lambda$) and $exc(\sigma)$ are the number of indices i such that $\sigma(i) > w_\lambda(i)$.

e-positivity

Corollary

We have that $g_k(\bullet)$ is e-positive.

Conjecture

The symmetric function $g_k(\mathbf{m})$ is e-positive.

Theorem (Positivity of the leading coefficient (only for $q = 1$))

We have that the coefficient of e_k in the e-expansion of $g_k(\mathbf{m})$ is non-negative.

Corollary

Let $\mathbf{m}\colon [n] \to [n]$ be a Hessenberg function and let $\lambda \vdash n$ be a partition of length 2. The coefficient of e_λ in $\mathrm{csf}(G_{\mathbf{m}})$ is non-negative.

Thank you!