The Ehrhart h*-polynomial of positroid and alcoved polytopes

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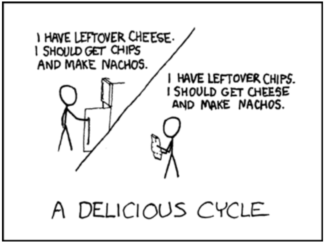

Based on arXiv:2410.01743 and arXiv:2412.02787 (with Elisabeth Bullock)

Overview

1. Ehrhart theory
2. Positroids and alcoved polytopes
3. Circuit/alcoved triangulation of connected positroid polytopes
4. h*-polynomials of positroid polytopes and alcoved polytopes
5. The relation with decorated ordered set partitions
6. Application to tree positroids and half-open positroid polytopes

Ehrhart h^*-polynomials

- Let $P \subseteq \mathbb{Z}^n$ be a d-dimensional lattice polytope. *Ehrhart polynomial* [Ehrhart '62]: $E(P,t) := \#(t \cdot P) \cap \mathbb{Z}^n$ for $t \in \mathbb{Z}_{\geq 0}$.
- *Ehrhart series*: $\mathrm{Ehr}(P,z) := \sum_{t=0}^{\infty} E(P,t)z^t = \frac{h^*(P,z)}{(1-z)^{d+1}}$.
- *Ehrhart h^*-polynomial*: the numerator of the Ehrhart series $h^*(P,z) = h_0 + h_1 z + \cdots + h_d z^d$ has degree at most d with *non-negative* coefficients [Stanley '80].

Example

Standard d-dimensional simplex Δ_d: $E(\Delta_d, t) = \binom{t+d}{d}$, $h^*(\Delta_d, z) = 1$.
Standard d-dimensional cube $□_d$: $E(□_d, t) = (1+t)^d$.

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Previously, on the h^* of hypersimplices

- Katzman ('05) computed the Ehrhart polynomial of the hypersimplices. Ferroni ('21) showed that the hypersimplices are Ehrhart positive.
- Nan Li ('11) computed the h^*-polynomial of the half-open hypersimplices.
- Nick Early ('17) conjectured that the h^*-polynomial of hypersimplices are counted by *hypersimplicial decorated ordered set partitions* (OSP), and Donghyun Kim ('20) proved it.

Question

Is there a formula for the Ehrhart series of an arbitrary alcoved polytope? Can we relate our formula to decorated ordered set partitions?

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Matroid

- A *matroid* is a pair $M = (E, \mathcal{B})$.
- E is a finite set; $\mathcal{B}$ is a collection of subsets of E, called the *bases* of M.
- The *basis exchange axiom*:

$$\text{For any } I, J \in \mathcal{B} \text{ and } i \in I \text{ there exists } j \in J \text{ such that } (I \setminus \{i\}) \cup \{j\} \in M.$$

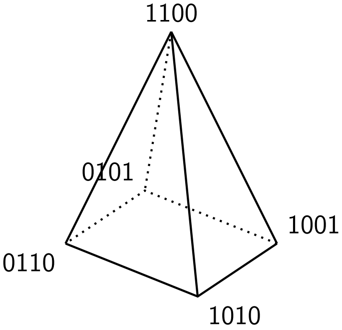
- All bases $B \in \mathcal{B}$ have the same size, called the *rank* of M.
- The *matroid polytope* is the convex hull of e_B for $B \in \mathcal{B}$ where $e_B = \sum_{i \in B} e_i$.

Matroid from matrix

Given a $k \times n$ matrix of rank k, the subset of columns giving nonzero $k \times k$ minors form the bases of a matroid.

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$\Delta_{12} = \Delta_{13} = \Delta_{14} = 1, \Delta_{23} = \Delta_{24} = 1, \Delta_{34} = 0.$



Positroid

- Postnikov first considered it in his study of the positive Grassmannian.
- Positroids are matroids given by matrices with nonnegative maximal minors.
- In bijection with several interesting classes of combinatorial objects, including Grassmann necklaces, decorated permutations, and equivalence classes of plabic graphs.

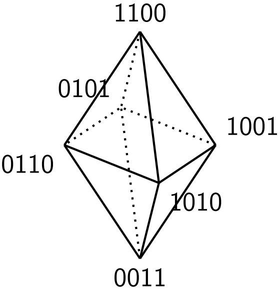


Figure: Uniform matroids are positroids. The matroid polytope of the uniform matroid $U_{k,n}$ is the hypersimplex $\Delta_{k,n}$. Here we show $\Delta_{2,4}$.

Alcoved polytopes

We follow the conventions of Lam and Postnikov.

- Consider the affine Coxeter arrangement with respect to a irreducible crystallographic root system Φ.
- The regions of the affine Coxeter arrangements are simplices called *alcoves*. A convex union of alcoves is an *alcoved polytope*.
- simple roots: $\alpha_1, \ldots, \alpha_n$; *fundamental coweights*: $\omega_1, \ldots, \omega_n$; *highest root*: $a_1\alpha_1 + \cdots + a_n\alpha_n$
- The fundamental alcove $A_\circ$ is the convex hull of $0, \omega_1/a_1, \ldots, \omega_n/a_n$.
- Positroid polytopes are exactly those matroid polytopes that are also alcoved.

Reduce to connected positroids

- A matroid which cannot be written as the direct sum of two nonempty matroids is *connected*.
- If M is a positroid such that $M = M_1 \oplus \cdots \oplus M_m$, then each M_i is a positroid [Ardila–Rincon–Williams '16].
- The matroid polytope of M is the direct product $P_M = P_{M_1} \times \cdots \times P_{M_m}$, and the Ehrhart polynomial of M is the product $E(P_M, t) = E(P_{M_1}, t) \cdots E(P_{M_m}, t)$.
- It suffices to give formulas for the h^*-polynomials of all connected positroid polytope. A connected positroid polytope on $[n]$ has dimension $n - 1$.

Grassmann necklaces

Definition

Let $k \leq n$ be a positive integer. A *Grassmann necklace* of type (k, n) is a sequence $(J_1, J_2, \ldots, J_n)$ of k-subsets $J_i \in \binom{[n]}{k}$ such that for any $i \in [n]$

- if $i \in J_i$ then $J_{i+1} = J_i - \{i\} \cup \{j\}$ for some $j \in [n]$,
- if $i \notin J_i$ then $J_{i+1} = J_i$,

where the indices i are taken modulo n.

Theorem (Postnikov '06)

There is a bijection between positroids of rank k on $[n]$ and Grassmann necklaces of type (k, n).

Circuits and cyclic left descents

- Lam and Postnikov defined a circuit to be a sequence of binary vectors $v_1 \to v_2 \to \cdots \to v_n \to v_{n+1} := v_1$ such that v_{i+1} is obtained from v_i by shifting a '1' in v_i one step to the right to the next adjacent place. The positions of the shifts in a minimal circuit give rise to a long cycle $(w) = (w_1, \ldots, w_n) \in S_n$.
- Parisi, Sherman-Bennett, Tessler, and Williams showed that the circuits of a long cycle can be recovered through its *cyclic left descents*.
- Each circuit defines a simplex (or equivalently, a type A alcove), and they together triangulate the hypercube. The set of circuits of length n with k ones triangulate the hypersimplex $\Delta_{k,n}$.

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Circuit description of alcoves

Definition (PSBTW '24)

- For a permutation $w \in S_n$, index $i \in [n]$ is a *cyclic left descent* if $i < n$ and $w^{-1}(i) > w^{-1}(i+1)$ or $i = n$ and $w^{-1}(1) < w^{-1}(n)$. Let $\mathrm{cDes}_L(w)$ denote the set of cyclic left descents of w, and $\mathrm{cdes}_L(w) = |\mathrm{cDes}_L(w)|$.
- Let $w^{(a)}$ denote the cyclic rotation of $w_1 \dots w_n$ ending at a. We define $I_r(w) = \mathrm{cDes}_L(w^{(r)})$, which only depends on the cycle (w). Then $I_{w_1}(w) \to I_{w_2}(w) \to \cdots \to I_{w_n}(w) \to I_{w_1}(w)$ is the *circuit* of $\Delta_{(w)}$, and the convex hull of their indicator vectors is called the (w)-simplex, denoted by $\Delta_{(w)}$.

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Restricted cyclic descents

- The *i-order* $<_i$ on the set $[n]$ is the total order

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-2 <_i i-1.$$

- For $i, j \in [n]$, the *cyclic interval* $[i, j]$ is

$$\begin{cases} i <_i i+1 <_i \cdots <_i j & \text{if } i \leq j \\ i <_i \cdots n <_i 1 <_i \cdots <_i j & \text{otherwise} \end{cases}$$

- The definition of cDes_L extends to the restriction of $(w_1, \ldots, w_n)$ to the alphabet given by $[i, j]$.

Example

For the long cycle $(3, 2, 4, 1, 5)$, we have $w|_{[1,3]} = (3, 2, 1)$ and $w|_{[3,1]} = (3, 4, 1, 5)$. Then $\mathrm{cDes}_L(w|_{[1,3]}) = \{1, 2\}$ and $\mathrm{cDes}_L(w|_{[3,1]}) = \{5, 1\}$.

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Circuit/alcoved triangulation of connected positroid polytopes

Theorem (J.)

Let $P_{\mathcal{J}}$ be any connected positroid polytope, where $\mathcal{J} = (J_1, \dots, J_n)$ is the associated Grassmann necklace. For any $i \in [n]$, suppose the elements of J_i are $a_1^i <_i \cdots <_i a_k^i$. Then $P_{\mathcal{J}}$ is triangulated by (w)-simplices for $w \in D_{\mathcal{J}}$, where

$$D_{\mathcal{J}} := \{(w) \in S_n \mid \mathrm{cdes}_L(w) = k+1, \mathrm{cdes}_L(w|_{[i, a_j^i]}) \leq j - 1 \text{ for all } i \in [n], j \in [k]\}$$

where $I_{w_1} \to I_{w_2} \to \cdots \to I_{w_n} \to I_{w_1}$ is the circuit of (w).

Shelling order

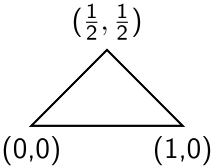
A *shelling* of a simplicial complex Γ is a linear order on its maximal faces $G_1, G_2, \ldots, G_s$ such that, for each $i \in [2, s]$, the set $G_i \cap (G_1 \cup \cdots \cup G_{i-1})$ is a union of facets of G_i.
A triangulation of a polytope P is a polytopal complex C with underlying space equal to P such that all the polytopes in C are simplices.
Stanley showed that if P has an unimodular triangulation C, then the h-vector of C is equal to the h^*-vector of P.

Weighted dual graph of the alcove triangulation

Let P be an alcoved polytope. We construct a weighted graph Γ_P such that

- vertices: closed alcoves $A \subset P$;
- edges: (A, A') if A and A' share a common facet;
- edge weights: $\text{wt}((A, A')) = \ell_i$ if the facet $F = A \cap A'$ can be transformed to a facet $F_\circ$ of the fundamental alcove $A_\circ$ under the action of the affine Weyl group such that ω_i / a_i is the vertex of $A_\circ$ that does not belong to $F_\circ$,
- where ℓ_i is the smallest positive integer such that $\ell_i \omega_i / a_i$ is an integer point.
- In type A, $\ell_i = 1$ for all i.

Example



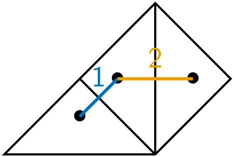


Figure: On the left, we have the fundamental alcove for B_2, and on the right, we have an alcoved polytope and its dual graph drawn in colors.

Breadth-first search order

Let $\Gamma = (V, E)$ be an undirected graph, and $v_0 \in V$ be a vertex of Γ.

- The *breadth-first search order of* Γ *with root* v_0 is the partial order $(\mathcal{P}_{v_0,\Gamma}, \prec)$ on V such that $u \prec v$ if and only if there is a shortest path from v_0 to v passing through u, for $u, v \in V$.
- For alcoved polytopes, this is the weak order (of the affine Weyl group with a specified alcove being the identity).
- Following Björner's argument, Bullock–J. showed that any linear extension of the breadth-first search order is a shelling order of the alcoved triangulation complex of P.

The h^*-polynomials of connected positroid polytopes

In type A, the edge weights of the dual graph of triangulation are all equal to one, so the sum of weights becomes the *cover*.

Theorem (J. '24)

Let $P_{\mathcal{J}}$ be any connected positroid polytope, where $\mathcal{J}$ is the associated Grassmann necklace. Let $\Gamma_{\mathcal{J}}$ be the dual graph of the circuit triangulation of $P_{\mathcal{J}}$. For any $w_0 \in D_{\mathcal{J}}$, let $(\mathcal{P}_{w_0,\Gamma_{\mathcal{J}}}, \prec)$ be the breadth-first search order on $\Gamma_{\mathcal{J}}$ with root w_0. The cover statistic of $\mathcal{P}_{w_0,\Gamma_{\mathcal{J}}}$ gives the h^*-polynomial of $P_{\mathcal{J}}$, i.e.,

$$h^*(P_{\mathcal{J}}, z) = \sum_{w \in D_{\mathcal{J}}} z^{\mathrm{cover}(w)}$$

where $\mathrm{cover}(w) = \#\{u \in D_{\mathcal{J}} \mid u \prec\!\cdot\, w\}$ *is the number of elements covered by w in $\mathcal{P}_{w_0,\Gamma_{\mathcal{J}}}$.*

The Ehrhart series of alcoved polytopes

Theorem (Bullock–J. '24)

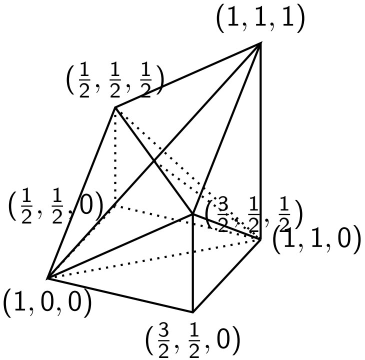
Let P be an alcoved polytope. Let Γ_P be the dual graph of the alcove triangulation of P, with edge weights given by ℓ_i's. Fix an arbitrary alcove A_0 in P, let $\mathcal{P}_{A_0}$ be the breadth-first search order of Γ_P with root A_0. The Ehrhart series of P is

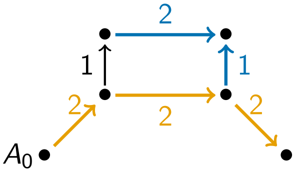
$$\mathrm{Ehr}(P, z) = \frac{\sum_{alcove\ A \subset P} z^{\mathrm{wt}(A)}}{\prod_{i=0}^{n}(1 - z^{\ell_i})}$$

where $\mathrm{wt}(A) = \sum_{A \text{ covers } A' \text{ in } \mathcal{P}_{A_0}} \mathrm{wt}((A, A'))$ is the sum of the weights of the edges between A and the alcoves it covers.

Carolina Benedetti and Kolja Knauer and Jerónimo Valencia-Porras proved the type A case in 2023 using geometric argument.

Example





The generalized hypersimplex for $\Phi = B_3$ and $k = 2$. The arrows indicate cover relations in the poset $\mathcal{P}_{A_0}$ where A_0 is chosen to be the lower left alcove. The Ehrhart series of $\Delta_2^{B_3}$ is

$$\mathrm{Ehr}(\Delta_2^{B_3}, z) = \frac{1 + z + 3z^{2} + z^{2+1}}{(1-z)^2(1-z^2)^2}.$$

Example

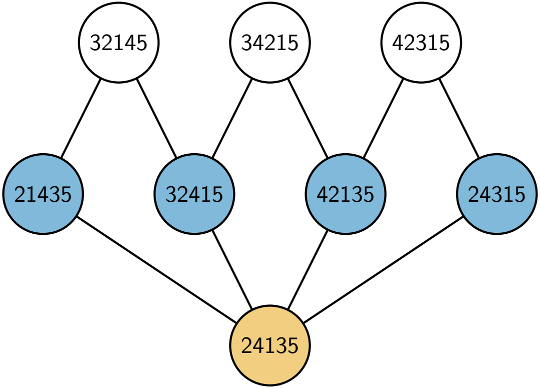


Figure: We show the graph of the circuit triangulation of the positroid polytope $P_{\mathcal{J}}$ associated to the positroid with Grassmann necklace $\mathcal{J} = (123, 235, 345, 145, 125)$, which coincides with the Hasse diagram of the poset $\mathcal{P}_{24135,\mathcal{J}}$. The h^*-polynomial of $P_{\mathcal{J}}$ is $1 + 4z + 3z^2$.

Decorated Ordered Set Partitions

Definition (Ocneanu, Early)

A *decorated ordered set partition* $((S_1)_{r_1}, \ldots, (S_d)_{r_d})$ of type (k, n) consists of a cyclically ordered set partition $(S_1, \ldots, S_d)$ of $[n]$ and a d-tuple of positive integers $(r_1, \ldots, r_d)$ that sum up to k. A decorated ordered set partition is *hypersimplicial* if $r_i \leq |S_i| - 1$ for all i. The *winding vector* of a decorated ordered set partition is an n-tuple of integers $(l_1, \ldots, l_n)$ such that $l_i = r_k + \cdots + r_{\ell-1}$ if $i \in S_k$ and $i + 1 \in S_\ell$. The *winding number* is equal to $(l_1 + \cdots + l_n)/k$. We denote the set of hypersimplicial decorated ordered set partitions of type (k, n) by $\mathrm{OSP}(\Delta_{k,n})$.

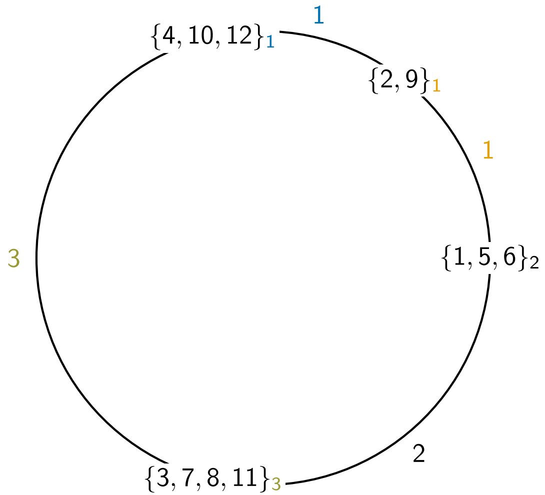


Figure: The winding vector of $((1, 5, 6)_2, (3, 7, 8, 11)_3, (4, 10, 12)_1, (2, 9)_1)$ is (6,3,3,2,0,2,0,4,6,4,3,2) and the winding number is 35/7=5. The i-th entry of the winding vector is the circular distance between i and $i+1$ in clockwise direction.

Hypersimplicial decorated ordered set partitions and h^* of the hypersimplices

Theorem (Conjectured by Early, Proof by Kim)

The number of hypersimplicial decorated ordered set partitions of type (k, n) and winding number d is equal to the d-th entry in the h^* vector of $\Delta_{k,n}$.

winding number	$\mathrm{OSP}(\Delta_{2,4})$
0	$((1234)_2)$
1	$((12)_1(34)_1)$
1	$((14)_1(23)_1)$
2	$((13)_1(24)_1)$

Table: The h^*-polynomial of the octahedron $\Delta_{2,4}$ is $1 + 2z + z^2$.

The relation between $OSP(\Delta_{2,n})$ and $h^*(\Delta_{2,n})$

- For hypersimplices of type A, our main result simplifies to

$$h^*(\Delta_{k,n}, z) = \sum_{\text{alcove } A \subset P} z^{\text{cover}(A)}$$

where $\text{cover}(A)$ is the number of elements A covers in the breadth-first order of $\Gamma_{k,n}$ with arbitrary root alcove A_0.

- Early and Kim's formula:

$$h^*(\Delta_{k,n}, z) = \sum_{\mathbf{S} \in \text{OSP}(\Delta_{k,n})} z^{\text{wind}(\mathbf{S})}$$

where $\text{wind}(\mathbf{S})$ is the winding number of $\mathbf{S}$.

Is there a bijection between Early and Kim's formula and our formula?

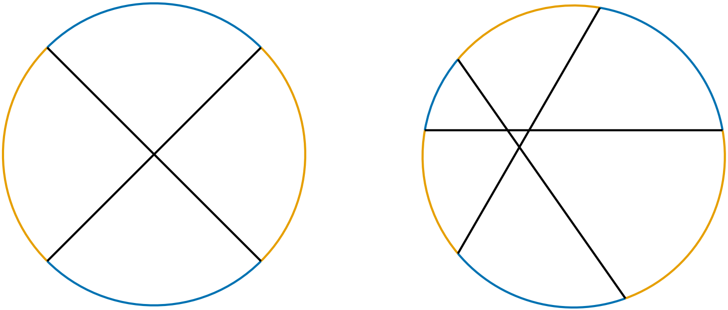
An edge labeling by chords

- Let $A, A' \subseteq \Delta_{2,n}$ be two adjacent alcoves. The hyperplane containing the facet $A \cap A'$ is defined by $y_j - y_i = 1$ for some $i \not\equiv j \pm 1 (\bmod\, n)$.
- We associate to the facet $A \cap A'$ the chord $i \leftrightarrow j$.

Lemma

For any choice of A_0, if both A', A'' are covered by A in $(\mathcal{P}_{A_0}, \prec)$, then the chords of the facets $A' \cap A, A'' \cap A$ cross in the interior of the circle.

From d chords to winding number d



For d chords that pairwise intersect in the interior of the circle, we associate an element (S_1, S_1^c) of $\mathrm{OSP}(\Delta_{2,n})$ of winding number d such that S is the union of every other arc of the circle divided by the chords.

Relation with decorated ordered set partitions when k=2

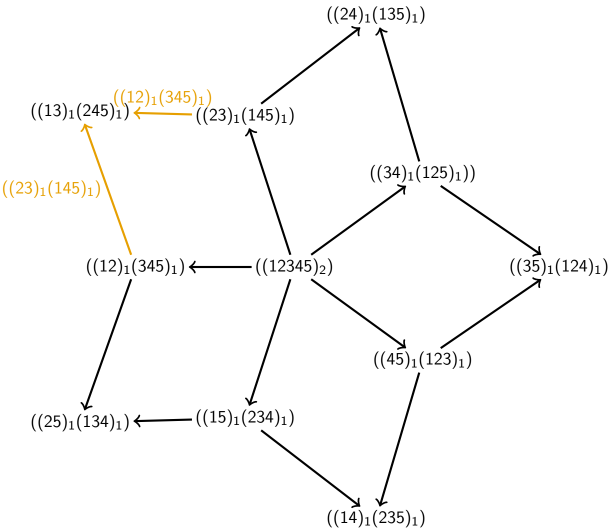
Define the map ψ_{A_0} from the alcoves in $\Delta_{2,n}$ to $\mathrm{OSP}(\Delta_{2,n})$ as follows:

- $\psi_{A_0}(A_0) = (1, 2, \ldots, n)_2$;
- If A covers $A_1, \ldots, A_d$ in $(\mathcal{P}_{A_0}, \prec)$, then $\psi_{A_0}(A) = (S_1, S_1^c)$ where S is the union of every other arc of the circle divided by the d chords corresponding to $A \cap A_1, \ldots, A \cap A_d$.

Theorem (Bullock–J. '24)

The map ψ_{A_0} is a bijection from the set of alcoves that cover d alcoves in $\mathcal{P}_{A_0}$ to $\mathbf{S} \in \mathrm{OSP}(\Delta_{2,n})$ with winding number d.

Example



Can one generalize the relation to higher k?

Tree positroids

- Positroids are also labeled by *planar bicolored graphs (plabic graphs*.
- When the plabic graph of a positroid is acyclic, we call it a *tree positroid*.
- The dual of a plabic graph is a *plabic tiling* [Oh–Postnikov–Speyer '15]. Tree positroids are those positroids whose plabic tilings are *bicolored subdivisions*, denoted by τ.

Bicolored subdivision, partial cyclic order

A *bicolored subdivision* τ is a partition of an n-gon into black and white polygons such that two adjacent polygons have different colors. We say τ has type (k, n) if any triangulation of the black polygons consists of k black triangles.

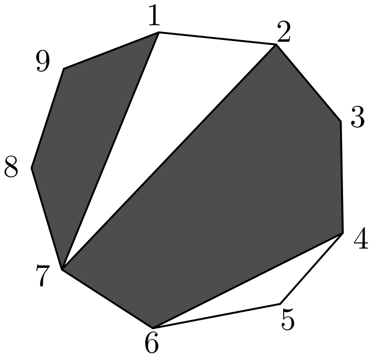


Figure: A bicolored subdivision of type $(5, 9)$.

Partial cyclic order, circular extension

- A *(partial) cyclic order* on a set X is a subset of $\binom{X}{3}$ with

$$
\begin{aligned}
(a,b,c) \in C &\implies (c,a,b) \in C && \text{(cyclicity)}\\
(a,b,c) \in C &\implies (c,b,a) \notin C && \text{(asymmetry)}\\
(a,b,c) \in C \text{ and } (a,c,d) \in C &\implies (a,b,d) \in C && \text{(transitivity)}
\end{aligned}
$$

- A cyclic order C is *total* if for all $a, b, c \in X$, either $(a, b, c) \in C$ or $(a, c, b) \in C$. A total cyclic order on $[n]$ is informally a way of placing $1, 2, \ldots, n$ on a circle.
- We say that C' extends C if $C \subseteq C'$. A total cyclic order that extends C is a *circular extension* of C.
- There exist partial cyclic orders without any circular extension [Meggido '76]. This decision problem is NP-complete.

Partial cyclic order associated with a bicolored subdivision

Let τ be a bicolored subdivision.

- For each white (resp. black) polygon P in τ, we let $v_1, \ldots, v_r$ denote its list of vertices read in clockwise (respectively, counterclockwise) order. We then associate the chain $C_{(v_1,\ldots,v_r)}$ to P.
- Define the *τ-order* C_τ to be the partial cyclic order which is the union of the chains associated to the black and white polygons.

Example

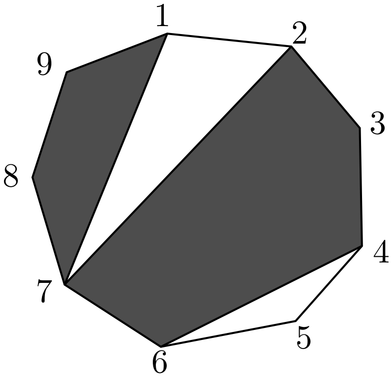


Figure: The partial cyclic order associated to this bicolored subdivision consists of chains $(1, 2, 7), (4, 5, 6), (6, 4, 3, 2, 7), (7, 1, 9, 8)$.

Triangulation by circular extensions

Proposition (PSBTW '24)

Let σ be a bicolored subdivision of type (k, n). Then

$$\Gamma_\sigma = \bigcup_{(w) \in \mathrm{Ext}(C_\sigma)} \Delta_{(w)}.$$

That is, Γ_σ is the union of (w)-simplices $\Delta_{(w)}$.

Corollary

Let τ be a bicolored subdivision and let $\mathcal{J}$ be the Grassmann necklace of the positroid defined by τ. Then we have $D_{\mathcal{J}} = \mathrm{Ext}(C_\tau)$.

Half-open

The facets of positroid polytopes are all of the form $x_{[i,j]} = k$ for some $i, j \in [n]$ and $k \in \mathbb{Z}$. We will call a facet of a positroid polytope *upper* if it is of the form $x_{[i,j]} = k$ such that the positroid polytope satisfies $x_{[i,j]} \leq k$. Ehrhart theory naturally extends to polytopes with some facets removed.

Theorem (J. '24)

Let $P_{\mathcal{J}}$ be a connected positroid polytope, where $\mathcal{J}$ is the associated Grassmann necklace. Consider the half-open positroid polytope $\tilde{P}_{\mathcal{J}} \subset [0,1)^{n-1}$ which is the projection of $P_{\mathcal{J}}$ onto the first $(n-1)$ coordinates with all upper facets removed. Then the h^*-polynomial of $\tilde{P}_{\mathcal{J}}$ is equal to $h^*(\tilde{P}_{\mathcal{J}}, z) = \sum_{w \in D_{\mathcal{J}}} z^{\mathrm{des}(w)+1}$.

Parke–Taylor polytopes are consecutive coordinate polytopes

- If we remove the 'sum of coordinates' equality from the definition of positroid polytopes, we obtain a *Parke–Taylor polytope* defined by [PSBTW '24].
- If we further require that all upper facets are of the form $x_{[i,j]} \leq 1$, then we obtain a *consecutive coordinate polytope* defined by [Ayyer–Josuat-Vergés–Ramassamy '20], whose h^*-polynomial is obtained from the h^* of the half-open polytope by dividing z.
- Both of them admit a triangulation by circular extensions of a certain partial cyclic order.

How to compute the h^*-polynomial of a closed polytope knowing the h^* of the half-open counterpart?

Let P be a polytope and let $F_1, \ldots, F_\ell$ be a collection of facets of P. Consider the restriction of the face poset of P to have coatoms $F_1, \ldots, F_\ell$. This poset $\mathcal{P}_{F_1,\ldots,F_\ell}$ describes all the faces of P in the intersections of $F_1, \ldots, F_\ell$. Let $\mu_{F_1,\ldots,F_\ell}$ be the Möbius function of this poset.

Proposition

Let P be a polytope. Let $F_1, \ldots, F_\ell$ be a collection of facets of P, and let $\tilde{P} = P \setminus (F_1 \cup \cdots \cup F_\ell)$. The h^*-polynomial of the polytope P is equal to

$$h^*(P, z) = h^*(\tilde{P}, z) - \sum_{F \in \mathcal{P}_{F_1,\ldots,F_\ell}, F \neq P} h^*(F, z)\mu_{F_1,\ldots,F_\ell}(F, P)(1 - z)^{\dim(P) - \dim(F)}.$$

Example

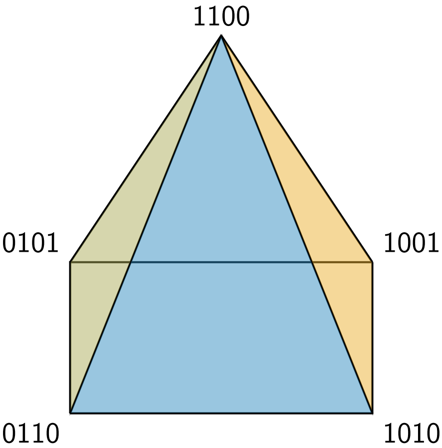


Figure: The positroid polytope associated to the Grassmann necklace $\mathcal{J} = (12, 23, 13, 14)$ is a pyramid. The orange facet corresponds to $F_1 : x_1 = 1$; the olive facet corresponds to $F_2 : x_2 = 1$; the blue facet corresponds to $F_3 : x_1 + x_2 + x_3 = 2$. These are all the upper facets of this positroid polytope.

Example

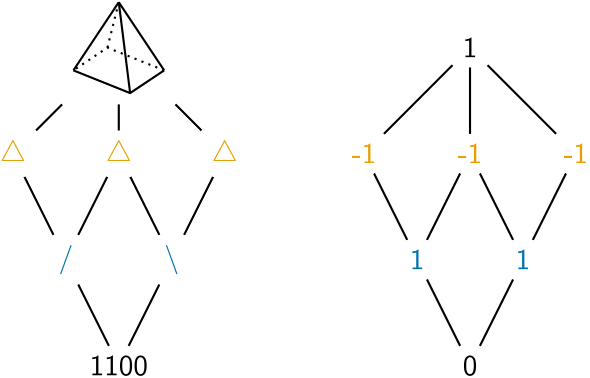


Figure: The poset $\mathcal{P}_{F_1,F_2,F_3}$ and the value of its Möbius function $\mu_{F_1,F_2,F_3}(-, P_{\mathcal{J}})$. Therefore, by inclusion-exclusion, the h^*-polynomial of $P_{\mathcal{J}}$ is $h^*(P_{\mathcal{J}}, z) = 2z^2 + 3(1-z) - 2(1-z)^2 = 1 + z$.

Thank you!

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