Bumpless Pipe Dream RSK, Growth diagrams, and Schubert Structure Constants

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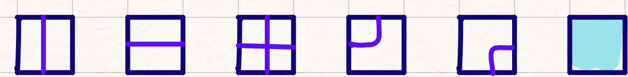
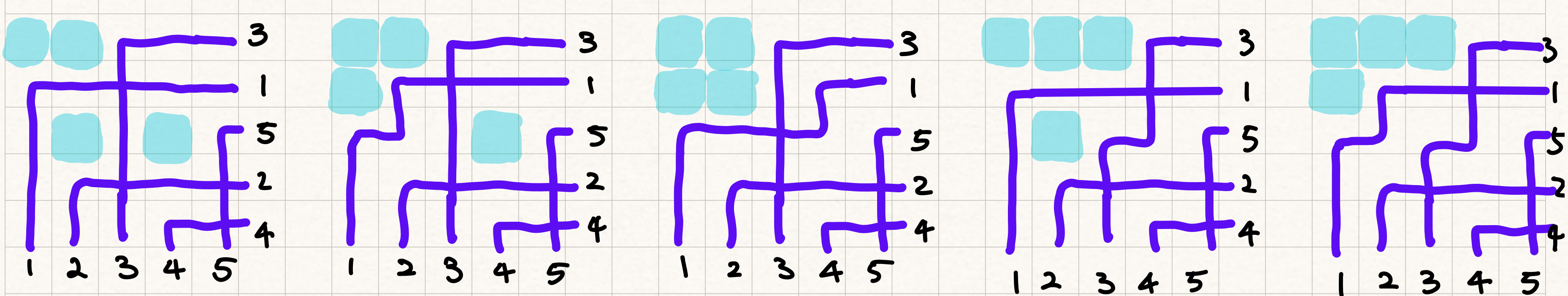
Bumpless Pipe Dreams (Lam - Lee - Shimozono '18)

• Def^n: A Schubert polynomial for $w \in S_\infty$ is defined as

$$\mathfrak{S}_\pi = \sum_{D \in BPD(\pi)} \prod_{(row, col) \in blank(D)} x_{row}$$

* Allowed tiles : 



$$\mathfrak{S}_{31524} =$$

$$x_1^2 x_3^2 + x_1^2 x_2 x_3 + x_1^2 x_2^2 + x_1^3 x_3 + x_1^3 x_2$$

Schubert Polynomials and Structure Constants

- Schubert polynomials form a basis of the polynomial ring in $x_1, x_2, \dots$

$$S_\pi S_\rho = \sum_\sigma c^\sigma_{\pi\rho} S_\sigma$$

These $c^\sigma_{\pi,\rho}$ are called the Schubert structure constants.

- When π and ρ are both permutations with a single descent at position k, $c^\sigma_{\pi\rho}$ are the well-known Littlewood-Richardson coefficients.

Classical RSK Correspondence

$$\{\text{words in } 1, 2, \dots, k\} \xleftrightarrow{\text{bij}} \left\{ (P, Q) : \begin{array}{l} P \in SSYT \\ Q \in SYT \\ P, Q \text{ same shape} \end{array} \right\}$$

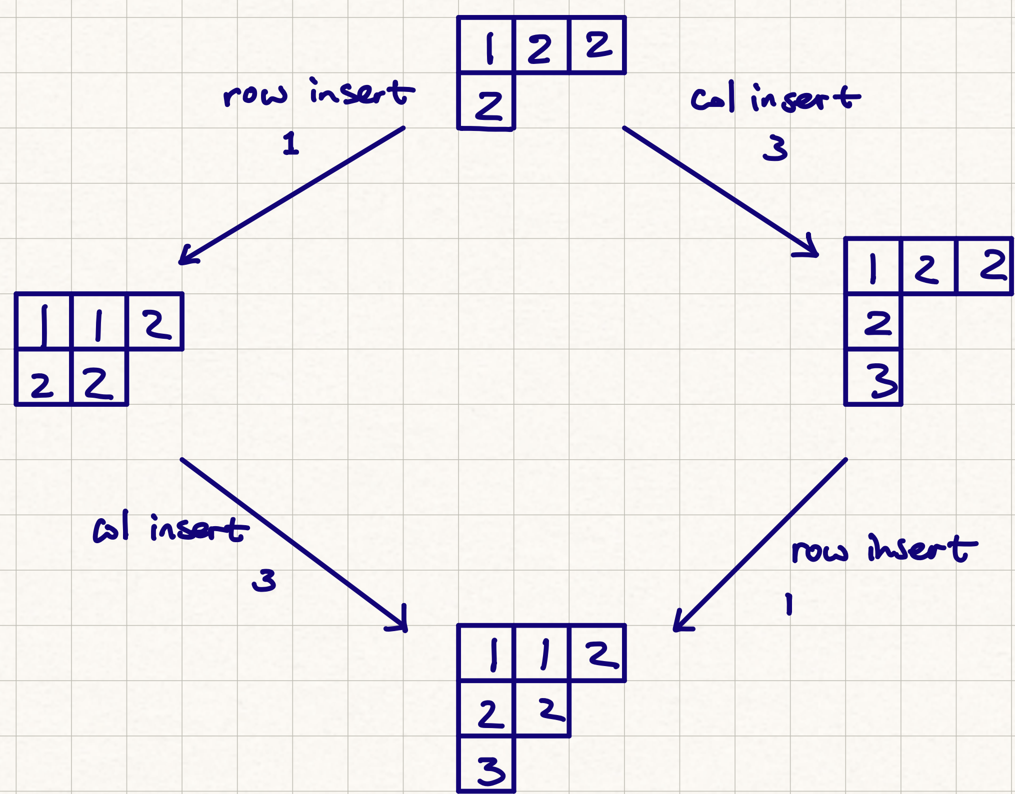
insertion tableau

recording tableau

Example 322121 $\longmapsto$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Right (row) and left (column) insertion



Row and column insertions commute in classical RSK.

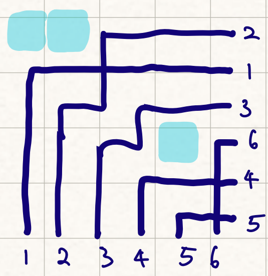
Bumpless pipe dream RSK

- Monk's rule for Schubert polynomials

$$\mathfrak{S}_{s_k}\mathfrak{S}_\pi = (x_1+\cdots+x_k)\,\mathfrak{S}_\pi = \sum_{\substack{\pi t_{ab} > \pi \\ a \le k < b}} \mathfrak{S}_{\pi t_{ab}}$$

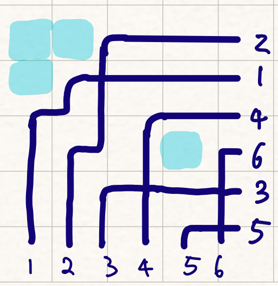
bijectively, $\underset{i \le k}{(x_i}\,,\ \underset{\mathrm{BPD}(\pi)}{\overset{\cap}{D})} \longleftrightarrow \underset{\sqcup\,\mathrm{BPD}(\pi t_{ab})}{\overset{\cap}{D'}}$

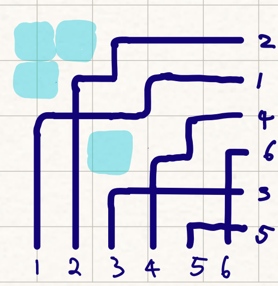
- H.–Pylyavskyy'22[+] give two insertion algorithms that generalize left and right insertions on SSYT to BPDs



left insert x_2 for $k=3$

right insert x_2 for $k=3$





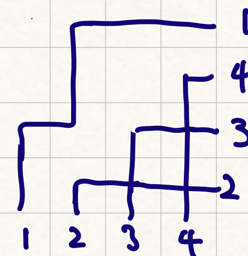
Bumpless pipe dream RSK

Def A biletter is a pair of positive integers (a,k) with $a \le k$. We write it as a_k. A biword is a word of biletters.

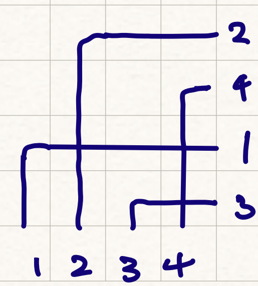
Def. Let w and v be permutations. Define $w <_k v$ iff $v = w\, t_{ab}$ where $\ell(v) = \ell(w)+1$ and $a \le k < b$.

Thm [BPD RSK] (H-Pylyavskyy)

$$\{\text{biwords of length } m\} \xleftrightarrow[\text{insertion}]{\text{left/right}} \bigsqcup_{\substack{\pi \text{ s.t.} \\ \ell(\pi)=m}} \left\{ (D, \underline{c}) : \begin{array}{l} D \in BPD(\pi) \\ \underline{c} \text{ mixed } k\text{-chain for } \pi \end{array} \right\}$$

Example: $2_3 1_2 2_2 \xleftrightarrow[\text{insert}]{\text{right}}$ (

, $1234 \underset{3}{<} 1243 \underset{2}{<} 1342 \underset{2}{<} 1432$)

$2_3 1_2 2_2 \xleftrightarrow[\text{insert}]{\text{left}}$ (

, $1234 \underset{3}{<} 1243 \underset{2}{<} 1423 \underset{2}{<} 2413$)

Lenart's growth diagram of permutations

- Lenart's growth diagram of permutations is a matrix of permutations subject to a local condition on each square:

$$
\begin{tikzcd}
\pi_{0,n} \arrow[r, dash, "k_1"'] \arrow[d, dash, "\ell_n"'] & \pi_{1,n} \arrow[r, dash, "k_2"'] \arrow[d, dash, "\ell_n"'] & \pi_{2,n} \arrow[r, dash] \arrow[d, dash, "\ell_n"'] & \cdots \arrow[r, dash, "k_m"'] & \pi_{m,n} \arrow[d, dash, "\ell_n"'] \\
\vdots & \vdots & \vdots & \iddots & \vdots \\
\pi_{0,2} \arrow[r, dash, "k_1"'] \arrow[d, dash, "\ell_2"'] & \pi_{1,2} \arrow[r, dash, "k_2"'] \arrow[d, dash, "\ell_2"'] & \pi_{2,2} \arrow[r, dash] \arrow[d, dash, "\ell_2"'] & \cdots \arrow[r, dash, "k_m"'] & {} \arrow[d, dash, "\ell_2"'] \\
\pi_{0,1} \arrow[r, dash, "k_1"'] \arrow[d, dash, "\ell_1"'] & \pi_{1,1} \arrow[r, dash, "k_2"'] \arrow[d, dash, "\ell_1"'] & \pi_{2,1} \arrow[r, dash] \arrow[d, dash, "\ell_1"'] & \cdots \arrow[r, dash, "k_m"'] & \pi_{m,1} \arrow[d, dash, "\ell_1"'] \\
\mathrm{id} = \pi_{0,0} \arrow[r, dash, "k_1"'] & \pi_{1,0} \arrow[r, dash, "k_2"'] & \pi_{2,0} \arrow[r, dash] & \cdots \arrow[r, dash, "k_m"'] & \pi_{m,0}
\end{tikzcd}
$$

Lenart's growth diagram of permutations

- Lenart's growth diagram of permutations is a matrix of permutations subject to a local condition on each square:

$$
\begin{tikzcd}
 & & & & & \pi_{m,n} \\
 & & & & & \vdots \arrow[u, dash, "\ell_n"'] \\
 & & & & & \pi_{m,1} \arrow[u, dash, "\ell_2"'] \\
\mathrm{id} = \pi_{0,0} \arrow[r, dash, "k_1"'] & \pi_{1,0} \arrow[r, dash, "k_2"'] & \pi_{2,0} \arrow[r, dash] & \cdots \arrow[r, dash, "k_m"'] & & \pi_{m,0} \arrow[u, dash, "\ell_1"']
\end{tikzcd}
$$

* Given

$$
\begin{tikzcd}
 & \sigma \\
\pi \arrow[r, dash, "\lessdot_k"'] & \rho \arrow[u, dash, "\lessdot^{\ell}"']
\end{tikzcd}
$$

$\exists!\ \delta$ that completes the Bruhat diamond.

if δ works

$$
\begin{tikzcd}
\delta \arrow[r, dash, "k"] & \sigma \\
\pi \arrow[u, dash, "\ell"] \arrow[r, dash, "k"'] & \rho \arrow[u, dash, "\ell"']
\end{tikzcd}
$$

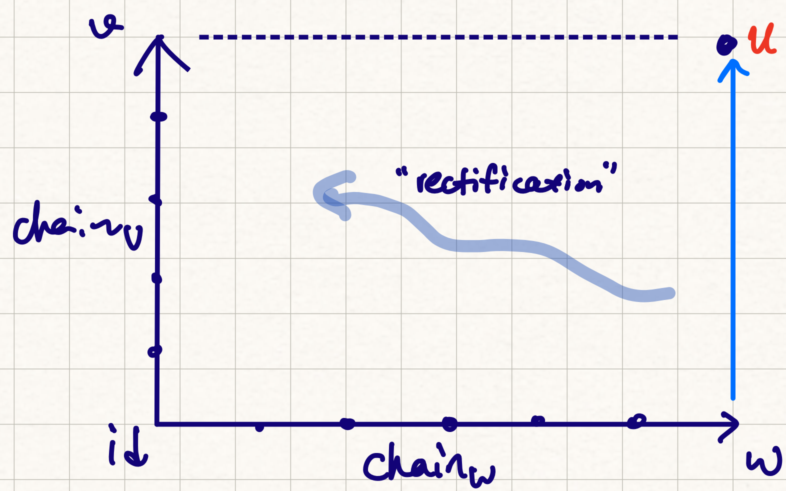
if δ does not work

$$
\begin{tikzcd}
\rho \arrow[r, dash, "k"] & \sigma \\
\pi \arrow[u, dash, "\ell"] \arrow[r, dash, "k"'] & \rho \arrow[u, dash, "\ell"']
\end{tikzcd}
$$

<u>Example</u>:

$$
\begin{tikzcd}
132 \arrow[r, dash, "1"] & 312 \\
123 \arrow[u, dash, "2"] \arrow[r, dash, "1"'] & 213 \arrow[u, dash, "2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
213 \arrow[r, dash, "1"] & 312 \\
123 \arrow[u, dash, "1"] \arrow[r, dash, "1"'] & 213 \arrow[u, dash, "1"']
\end{tikzcd}
$$

Growth diagrams and Schubert multiplication



Want to find

- chain for $id \to w$
- chain for $id \to v$

such that $\forall u \in S_\infty$,

$c^u_{v,w}$ = # chains $w \to u$ that "rectify" to the given chain of v

- Lenart '17 gave a rule for multiplying $\mathfrak{S}_v$ and $\mathfrak{S}_w$ where v has a single descent at position k, and w has all descents $\geq k$ or all descents $\leq k$.
- H.–Pylyavskyy '22+ generalize this to $\mathfrak{S}_v \cdot \mathfrak{S}_w$ where v has descents $\leq k$, and w has descents $\geq k$. (v and w have separated descent)

Connecting insertions and growth diagrams

Thm (H.-Pylyavskyy) Let $D \in BPD(\pi)$. Suppose π's first descent position is d_1 and last descent position is d_2 when $\pi \neq id$. Suppose $\ell \leq d_1 \leq d_2 \leq k$. Then left insertion of a_k and right insertion of b_ℓ commute:

$$(a_k \rightarrow D) \leftarrow b_\ell = a_k \rightarrow (D \leftarrow b_\ell)$$

Proposition (H.-Pylyavskyy) When two permutations w and v have separated descents, it is possible to construct explicit chains in mixed k-Bruhat order such that every square in the growth diagram

$$\begin{tikzcd} \delta \arrow[r, dash, "k"] \arrow[d, dash, "\ell"'] & \sigma \arrow[d, dash, "\ell"] \\ \pi \arrow[r, dash, "k"'] & \rho \end{tikzcd}$$

satisfies

$$\begin{cases} k \geq \text{last descent of } \pi \\ \ell \leq \text{first descent of } \pi. \end{cases}$$

Separated descent Schubert calculus

Thm (H.-Pylyavskyy)

Let w and v be permutation s.t

- last descent of v is $\leq k$
- first descent of w is $\geq k$

Then there exist explicit chains $ch_{\uparrow}(w)$ and $ch_{\downarrow}(v)$ s.t $\forall\, u \in S_\infty$,

$$c_{w,v}^{u} = \#\text{ growth diagrams of the form}$$