Ehrhart polynomials of slices of rectangular prisms

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AlCoVE: an Algebraic Combinatorics Virtual Expedition

This talk is based on joint work with Daniel McGinnis
“Lattice points in slices of prisms” (arXiv:2202.11808)

The hypersimplex

The hypersimplex $\Delta_{k,n}$ is defined by

$$\Delta_{k,n} = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = k \right\}.$$

It is of fundamental importance in

- Matroid theory $\rightarrow$ uniform matroids.
- Graph theory $\rightarrow$ the Johnshon graph.
- Grassmannians $\rightarrow$ specifically TNN Grassmannian.
- The theory of alcoved polytopes $\rightarrow$ triangulations.
- Much more! (tropical geometry, coding theory, statistics of permutations, etc.)

Basic facts about the hypersimplex

Remark

The vertices of the hypersimplex $\Delta_{k,n}$ are all the $0/1$-vectors in $\mathbb{R}^n$ that have exactly k ones.

In 1977 Stanley gave a combinatorial proof of the following fact

Theorem

The volume of the hypersimplex $\Delta_{k,n}$ is

$$\mathrm{vol}(\Delta_{k,n}) = \frac{1}{(n-1)!} A(n-1, k-1),$$

where $A(n-1,k-1) = \{\sigma \in \mathfrak{S}_{n-1}$ having $k-1$ descents$\}$.

It follows from his proof that the hypersimplex admits a certain unimodular triangulation.

Lattice points → Ehrhart polynomials

A vast generalization of the volume is the Ehrhart polynomial. To each polytope $\mathscr{P} \subseteq \mathbb{R}^n$ associate the function

$$t \mapsto \#(t\mathscr{P} \cap \mathbb{Z}^n).$$

This happens to be a polynomial that we denote $\operatorname{ehr}(\mathscr{P}, t)$. If $d := \dim \mathscr{P}$ and

$$\operatorname{ehr}(\mathscr{P}, t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0,$$

then

- $a_d = \operatorname{vol}(\mathscr{P})$,
- $a_{d-1} = \frac{1}{2} \operatorname{vol}(\partial \mathscr{P})$,
- $a_0 = 1$.
- $a_1, \ldots, a_{d-2}$ can be negative in general. ☹

h^*-polynomials

If we consider the *Ehrhart series*,

$$\mathrm{Ehr}(\mathscr{P}, x) = \sum_{j=0}^{\infty} \mathrm{ehr}(\mathscr{P}, j) x^j = \frac{h^*(x)}{(1-x)^{d+1}}$$

Stanley showed in 1993 that the numerator is a polynomial with nonnegative integer coefficients.

$$h^*(\mathscr{P}, x) = h_0 + h_1 x + \cdots + h_d x^d.$$

Remark (Major problems)

- Find conditions that h^*-polynomials of lattice polytopes must satisfy (inequalities, for example).
- Find combinatorial interpretations of the coefficients of the h^*-polynomial, at least for particular families of polytopes.

What about the hypersimplex?

Let $W(\ell, n, m+1)$ denote the number of permutations $\sigma \in \mathfrak{S}_n$ that have exactly $m+1$ cycles and "weight" ℓ (using some definition of weight).

Theorem (F. '21)

Consider the hypersimplex $\Delta_{k,n}$. The coefficient of degree m of its Ehrhart polynomial is given by

$$[t^m]\,\mathrm{ehr}(\Delta_{k,n}, t) = \frac{1}{(n-1)!}\sum_{\ell=0}^{k-1} W(\ell, n, m+1) A(n-1, k-\ell-1),$$

which in particular is positive.

Ehrhart in another basis

Regarding the h^*-polynomial we have the following combinatorial interpretation.

Theorem (Early '17 - Kim '20)

Consider the hypersimplex $\Delta_{k,n}$. The coefficient of degree m of its h^*-polynomial is given by

$$[x^m]h^*(\Delta_{k,n}, x) = \# \left\{ \begin{array}{c} \textit{decorated ordered set partitions} \\ \textit{of type } (k,n) \textit{ and winding number } m \end{array} \right\},$$

What is a slice of a prism?

Definition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$. The **rectangular prism** $\mathscr{R}_{\mathbf{c}}$ is defined as the polytope

$$\mathscr{R}_{\mathbf{c}} = \{x \in \mathbb{R}^n : 0 \leq x_i \leq c_i \text{ for each } i \in [n]\}.$$

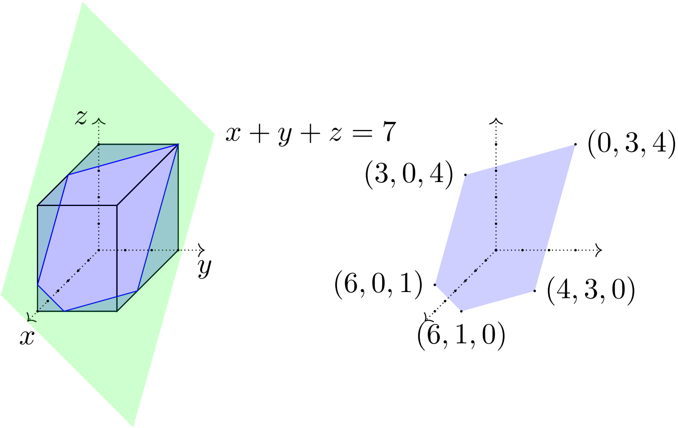
For each positive integer k, the k-th slice $\mathscr{R}_{k,\mathbf{c}}$ is defined as:

$$\mathscr{R}_{k,\mathbf{c}} = \left\{ x \in \mathscr{R}_{\mathbf{c}} : \sum_{i=1}^{n} x_i = k \right\}.$$

Example (The basic example)

Consider $\mathbf{c} = (1, \ldots, 1) \in \mathbb{Z}^n_{>0}$. The k-th slice of $\mathscr{R}_{\mathbf{c}}$ is precisely the hypersimplex $\Delta_{k,n}$.

Example



If you consider the 3-dimensional rectangular prism of sides 6, 3 and 4 and you intersect it with the hyperplane $x+y+z=7$ you get the polytope on the right.

Fat slices

The preceding type of slice is what we informally call a "thin slice". Consider two nonnegative integers $a < b$ and the polytope $\mathscr{R}'_{a,b,\mathbf{c}}$ defined by

$$\mathscr{R}'_{a,b,\mathbf{c}} := \left\{ x \in \mathscr{R}_{\mathbf{c}} : a \leq \sum_{i=1}^{n} x_i \leq b \right\}.$$

We say that this is a "fat slice" of the prism $\mathscr{R}_{\mathbf{c}}$.

Example

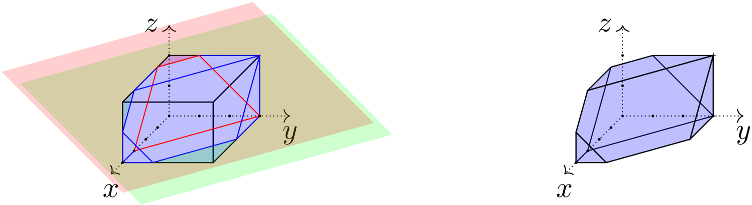


Figure: $\mathscr{R}'_{3,5,(4,3,2)}$

Fat Slices

A fat slice can be easily converted into a thin slice while preserving the Ehrhart polynomial.

Proposition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$ *and* $0 \leq a < b$. *Then, the fat slice* $\mathscr{R}'_{a,b,\mathbf{c}}$ *has the same Ehrhart polynomial as the thin slice* $\mathscr{R}_{k,\mathbf{c}'}$ *where* $k = b$ *and* $\mathbf{c}' = (\mathbf{c}, b - a) \in \mathbb{Z}^{n+1}_{>0}$.

Basic properties of these polytopes

Remark

- Slices of prisms are *alcoved polytopes*.
- The edges of a slice of a prism are all parallel to some vector of the form $e_i - e_j$. Hence, they are all *generalized permutohedra* or *base polymatroids*.
- They are all *polypositroids*.

Conjecture (F., Jochemko, Schröter '21)

All positroids are Ehrhart positive.

Algebras of Veronese type

Definition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$ and $k > 0$. The **algebra of Veronese type** $\mathscr{V}(\mathbf{c}, k)$ is defined as the the graded algebra over a field $\mathbb{F}$ generated by all the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\alpha_1 + \cdots + \alpha_n = k$ and $\alpha_i \leq c_i$ for all i.

Theorem (Hibi and De Negri '97)

There is an isomorphism between $\mathscr{V}(\mathbf{c}, k)$ *and the Ehrhart ring of* $\mathscr{R}_{k,\mathbf{c}}$.

A consequence of the above result is that the Hilbert function of $\mathscr{V}(\mathbf{c}, k)$ coincides with $\operatorname{ehr}(\mathscr{R}_{k,\mathbf{c}}, t)$ and moreover, the numerator of the Hilbert series is $h^*(\mathscr{R}_{k,\mathbf{c}}, x)$.

Weighted Permutations and Compatibility

Definition

A **weighted permutation** is a pair (σ, w) where $\sigma \in \mathfrak{S}_n$ and w assigns weight to the cycles of σ. The **total weight** of (σ, w) is the sum of the weights $w(\mathfrak{c})$ of all cycles $\mathfrak{c}$ of σ.

Example: $(1\ 3\ 6)^{7}(2\ 5)^{0}(4)^{2}$.

Definition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$. A weighted permutation (σ, w) is said to be **$\mathbf{c}$-compatible** if for each cycle $\mathfrak{c}$ of σ, we have

$$w(\mathfrak{c}) < \sum_{i \in \mathfrak{c}} c_i.$$

Example

Let $\mathbf{c} = (2, 4, 6, 8)$, $\sigma = (1\ 3)^{6}(2\ 4)^{11}$.
Then (σ, w) is $\mathbf{c}$-compatible because

$$w((1\ 3)) = 6 < \sum_{i \in (1\ 3)} c_i = 2 + 6 = 8$$

and

$$w((2\ 4)) = 11 < \sum_{i \in (2\ 4)} c_i = 4 + 8 = 12.$$

Also, the total weight is $w(\sigma) = 6 + 11 = 17$

Ehrhart Polynomial for Slices of Prisms

Definition

We define $W(\ell, n, m+1, \mathbf{c})$ to be the number of $\mathbf{c}$-compatible weighted permutations (σ, w) where $\sigma \in \mathfrak{S}_n$ has $m+1$ cycles and $w(\sigma) = \ell$.

Theorem (F. and McGinnis '22)

The coefficient of degree m of the Ehrhart polynomial for the prism slice $\mathscr{R}_{k,\mathbf{c}}$ is given by

$$[t^m]\,\mathrm{ehr}(\mathscr{R}_{k,\mathbf{c}}, t) = \frac{1}{(n-1)!} \sum_{\ell=0}^{k-1} W(\ell, n, m+1, \mathbf{c}) A(m, k-\ell-1)$$

which in particular are positive.

Flag Eulerian Numbers

Definition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$. A **$\mathbf{c}$-colored permutation on $[n]$** is a pair $(\sigma, \mathbf{s})$ where $\sigma \in \mathfrak{S}_n$ and $\mathbf{s}$ is a function $\mathbf{s} : [n] \to \mathbb{Z}_{\geq 0}$ such that $s_i := \mathbf{s}(i) \leq c_i - 1$ for each i. The set of all such $\mathbf{c}$-colored permutations is denoted by $\mathfrak{S}_n^{(\mathbf{c})}$

Definition

The set of **descents** of a $\mathbf{c}$-colored permutation $(\sigma, \mathbf{c})$ is given by

$$\mathrm{Des}(\sigma, \mathbf{s}) := \{i \in [n-1] : s_i > s_{i+1} \text{ or } s_i = s_{i+1} \text{ and } \sigma_i > \sigma_{i+1}\}.$$

The **flag descent number** of a $\mathbf{c}$-colored permutation $(\sigma, \mathbf{s}) \in \mathfrak{S}_n^{(\mathbf{c})}$ is defined by

$$\mathrm{fdes}(\sigma, \mathbf{s}) := s_n + \sum_{i \in \mathrm{Des}(\sigma, \mathbf{s})} c_{i+1}.$$

Flag Eulerian Numbers

Definition

Let $\mathbf{c} = (c_1, \ldots, c_n)$. We define the **flag Eulerian number** $A_{n,k}^{(\mathbf{c})}$ by

$$A_{n,k}^{(\mathbf{c})} := \# \left\{ (\sigma, \mathbf{s}) \in \mathfrak{S}_n^{(\mathbf{c})} : \mathrm{fdes}(\sigma, \mathbf{s}) = k - 1 \right\}.$$

Theorem

The volume of the fat slice $\mathscr{R}'_{k-1,k,\mathbf{c}}$ *is given by*

$$\mathrm{vol}(\mathscr{R}'_{k-1,k,\mathbf{c}}) = \frac{1}{n!} A_{n,k}^{(\mathbf{c})}(n, k-1)$$

Remark

The case that $\mathbf{c} = (r, \ldots, r)$, reduces to a result by Han and Josuat-Vergès (2016), and when $r = 1$ we recover Laplace's result on hypersimplices.

Ordered Set Partitions

Definition

A *decorated ordered set partition* of type (k, n) consists of a cyclically ordered partition ξ of $[n]$ and a function $w : P(\xi) \to \mathbb{Z}_{\geq 0}$ such that

$$\sum_{\mathfrak{p} \in P(\xi)} w(\mathfrak{p}) = k.$$

For a vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$, we say that a decorated ordered set partition ξ is $\mathbf{c}$-*compatible* if

$$w(\mathfrak{p}) < \sum_{i \in \mathfrak{p}} c_i$$

for all $\mathfrak{p} \in P(\xi)$.

Example

Let ξ be the following cyclically ordered partition of $[8]$

$$(\{1,3,6\},\{2,5\},\{4,7,8\}).$$

Let w be given by

$$w(\{1,3,6\}) = 1, \quad w(\{2,5\}) = 2, \quad w(\{4,7,8\}) = 4.$$

Then ξ and w make up a decorated ordered set partition of type $(7,8)$. It is $(2,1,4,5,2,3,1,1)$-compatible for instance because

$$w(\{1,3,6\}) = 1 < 2+4+3 = 9, \quad w(\{2,5\}) = 2 < 1+2 = 3.$$

$$w(\{4,7,8\}) = 4 < 5+1+1 = 7.$$

Winding Number

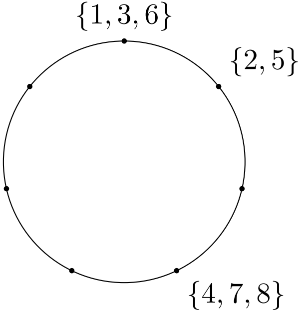
Let ξ and w again be given by

$$(\{1,3,6\},\{2,5\},\{4,7,8\})$$

and

$$w(\{1,3,6\}) = 1, \quad w(\{2,5\}) = 2, \quad w(\{4,7,8\}) = 4.$$

This decorated ordered set partition can be visualized as follows.



The h^*-coefficients

Definition

The **winding number** of a decorated ordered set partition of type (k, n) is the integer m such that

$$mk = \lambda_1 + \cdots + \lambda_n$$

where λ_i is the clockwise distance from the set containing i to the set containing $i+1$.

Theorem (F. and McGinnis '22)

The coefficient of degree m of the h^*-polynomial for $\mathscr{R}_{k,\mathbf{c}}$ is given by

$$[x^m]h^*(\mathscr{R}_{k,\mathbf{c}}, x) = \#\left\{ \begin{array}{c} \mathbf{c}\text{-compatible decorated ordered set partitions} \\ \text{of type } (k,n) \text{ and winding number } m \end{array} \right\},$$

Two conjectures regarding roots

Conjecture (F. and McGinnis '22)

All the complex roots of the polynomial

$$p_{n,m,\mathbf{c}}(x) = \sum_{\ell=0}^{\infty} W(\ell, n, m+1, \mathbf{c}) x^{\ell}.$$

lie on the unit circle $|z| = 1$*.*

Conjecture (F. and McGinnis '22)

The h^**-polynomial of a slice of a prism is always real-rooted. Moreover, if* $\mathbf{c} = (c_1, \ldots, c_n)$ *and* $\mathbf{c}' = (c_1, \ldots, c_{n-1}, c_n - 1, 1)$*, then*

$$h^*(\mathscr{R}_{k,\mathbf{c}}, x) \preceq h^*(\mathscr{R}_{k,\mathbf{c}'}, x)$$

namely, these two polynomials interlace.

THANK YOU!