Reconstruction of polytopes and Kalai's conjecture on reconstruction of spheres

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joint work with Joseph Doolittle

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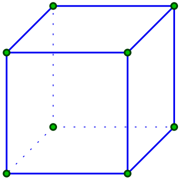
AlCoVE: an Algebraic Combinatorics Virtual Expedition
June 6, 2022

(Convex) polytope P:
convex hull of finitely many points in Euclidian space.

The graph $G(P)$:
the graph consisting of the vertices and edges of P.



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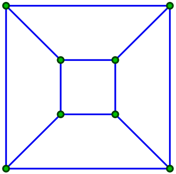


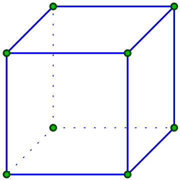
$G(P)$

Simple polytope P:
number of edges incident to each vertex equals the dimension of P.

Reconstruction of polytopes

Theorem (Blind–Mani, 1987)

If P is a simple polytope, then the graph $G(P)$ determines the entire combinatorial structure of P.



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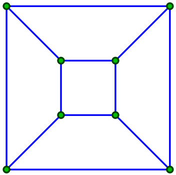

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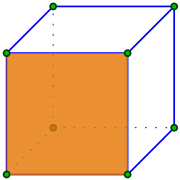

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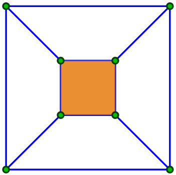

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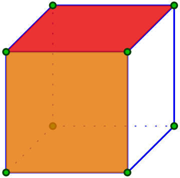

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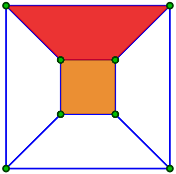

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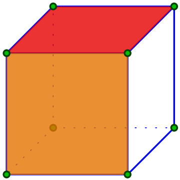

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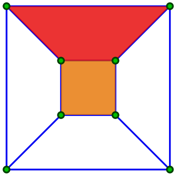

$G(P)$

Kalai, 1988: A simple constructive proof.

Reconstruction of polytopes

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If P is a simple polytope, then the graph $G(P)$ determines the entire combinatorial structure of P.

This holds for arbitrary polytopes (not only simple) in dimension 3, but not in higher dimensions.

Example

Let Δ_m be a m-dimensional simplex. The following are two non isomorphic 6-dimensional polytopes with the same graph (complete graph on 7 vertices)

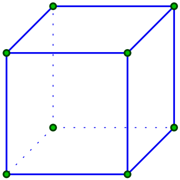
$$(\Delta_2 \times \Delta_4)^* \ncong (\Delta_3 \times \Delta_3)^*$$

Duality of polytopes

Every nonempty d-polytope P in $\mathbb{R}^d$ admits a dual polytope in $\mathbb{R}^d$:

$$P^* = \{y \in \mathbb{R}^d : x^T y \leq 1 \text{ for all } x \in P\}$$

where P is assumed to contain the origin in its interior.



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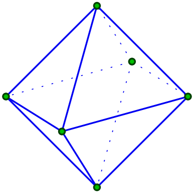

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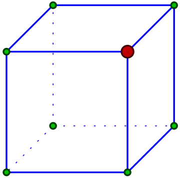

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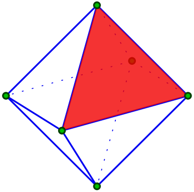

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Under this duality:

P	$\longleftrightarrow$	P^*
vertices	$\longleftrightarrow$	facets (higher dim faces)

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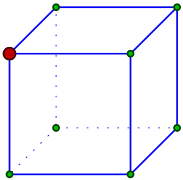

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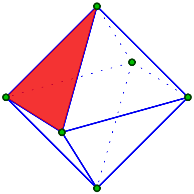

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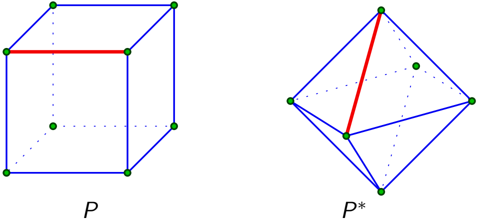

Under this duality:

$$\begin{array}{rcl} P & \longleftrightarrow & P^* \\ \text{vertices} & \longleftrightarrow & \text{facets (higher dim faces)} \\ \text{edges} & \longleftrightarrow & \text{ridges (codim 1 faces)} \end{array}$$

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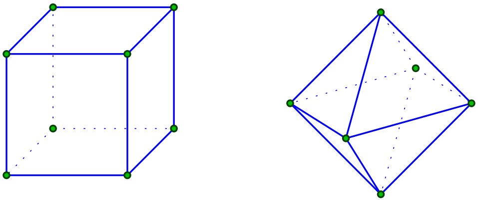

P $\qquad$ P^*

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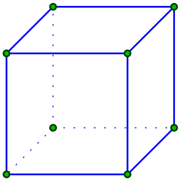
...	$\longleftrightarrow$	...

Simple vs simplicial

Simplicial polytope P:
all faces are simplices.

The facet-ridge graph $G_{FR}(P)$:
the graph whose vertices are facets of P
two facets are connected by an edge if they intersect in a ridge.

$$\begin{array}{ccc} P \text{ is simple} & \longleftrightarrow & P^* \text{ is simplicial} \\ G(P) & = & G_{FR}(P^*) \end{array}$$



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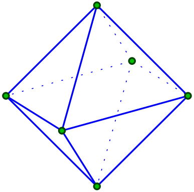

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Reconstruction of polytopes and spheres

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Simplicial spheres are completely determined by their facet-ridge graphs.

A simplicial sphere is a simplicial complex which is homeomorphic to a sphere.

Most spheres are not polytopal

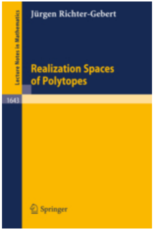
For $d \geq 3$, most d-spheres are not polytopal.

- Goodman–Pollack, 1986
- Kalai, 1988
- Pfeifle–Ziegler, 2004

Deciding polytopality of spheres is a difficult problem

Mnëv's Universality theorem: Realization spaces of polytopes can take arbitrary (semi-algebraic) shapes and thus can exhibit all kinds of pathologies.

The realizability problem for 4-polytopes is NP-hard.



Goal

Our initial goal was:
Look for a counterexample to Kalai's Conjecture among a special family of simplicial spheres which are conjectured to be polytopal.
(kill two conjectures at once)

Instead:
We proved the conjecture for this family.
(spherical subword complexes)

Rest of the talk:
Introduce subword complexes and state our main result.

Subword complexes preliminaries

Symmetric group $\mathbb{S}_{n+1}$:
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length of w: smallest r such that $w = s_{i_1} \ldots s_{i_r}$
longest element: permutation $[n+1, \; \ldots \; , 1]$
reduced expression for w: expression for w of minimal length

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In this talk: **finite Coxeter groups**
(very similar to the symmetric group)

Subword complexes

W finite Coxeter group with generating set S

$Q = (q_1, \ldots, q_m)$ a word in S

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$\pi \in W$

Definition (Knutson–Miller, 2004)

The subword complex $\Delta(Q, \pi)$ is the simplicial complex whose

faces $\longleftrightarrow$ subwords P of Q such that $Q \setminus P$ contains a reduced expression of π

Knutson–Miller. Gröbner geometry of Schubert polynomials. Ann. Math., 161(3), '05
Knutson–Miller. Subword complexes in Coxeter groups. Adv. Math., 184(1), '04

Subword complexes - Example 1

In type A_2:

$W = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

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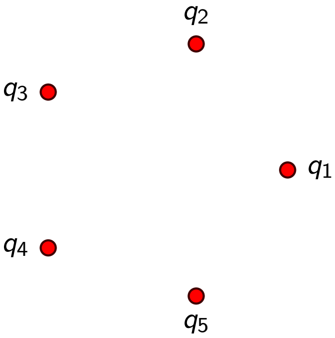

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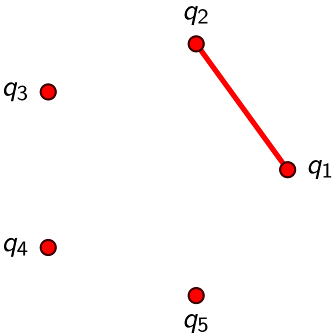

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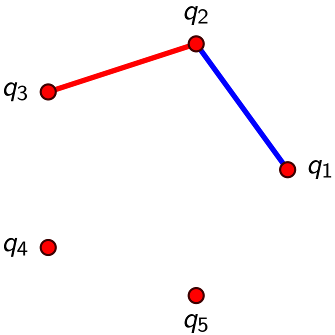

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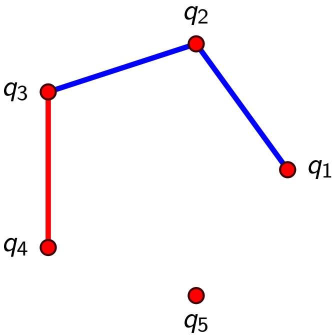

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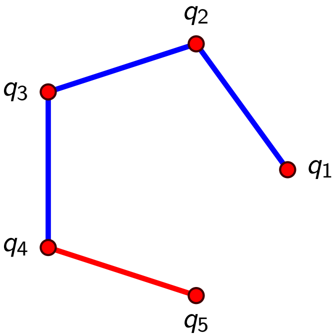

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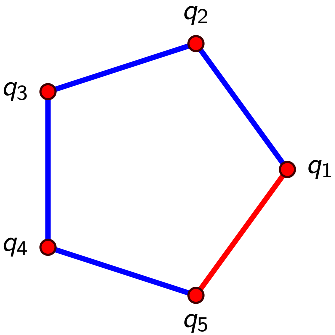

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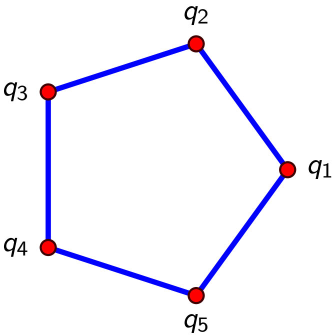

Subword complexes - Example 2

In type A_3:

$W = \mathbb{S}_4$, $S = \{s_1, s_2, s_3\} = \{(1\ 2), (2\ 3), (3\ 4)\}$

$$Q = \begin{matrix} (\ s_1, s_2, s_1, s_2, s_1, \color{red}{s_3}\ ) \\ q_1, q_2, q_3, q_4, q_5, \color{red}{q_6} \end{matrix} \quad \text{and } \pi = [3\ 2\ 1] = s_1 s_2 s_1 = s_2 s_1 s_2$$

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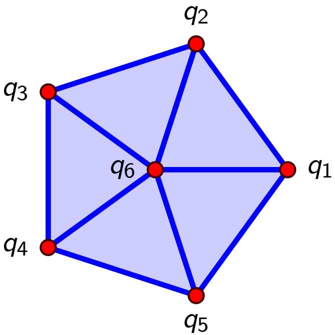

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Subword complexes are vertex decomposable spheres or balls.

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Spherical subword complexes are polytopal.

Special cases include:

- Cyclic polytopes
- Duals of associahedra
- Cluster complexes of cluster algebras of finite type
- Duals of pointed-pseudotriangulation polytopes
- Simplicial multi-associahedra (conjectured)

Woo, Pilaud–Pocchiola, Serrano–Stump, Stump, C.-Labbé–Stump, Rote–Santos–Streinu, Jonsson, ...

Our main theorem

Theorem (C.–Doolittle)

Spherical subword complexes of finite type are completely determined by their facet-ridge graph. In other words, they satisfy Kalai's Conjecture.

Our current proof is <u>not</u> constructive.

It is based on the topological tools developed by Blind and Mani.

Thank you!