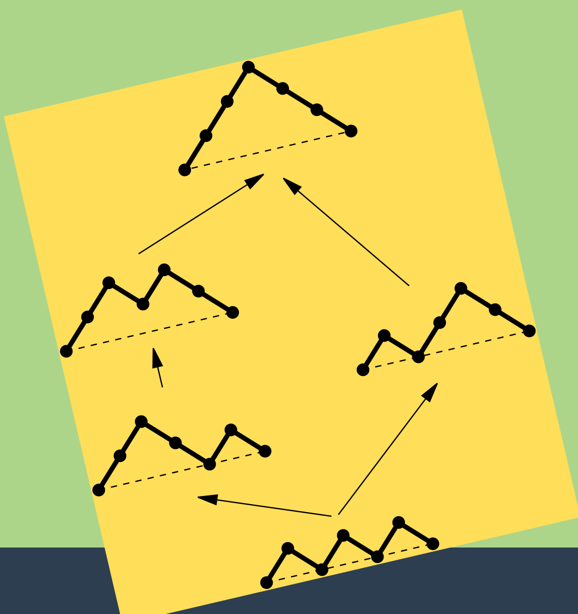
The ascent order on Dyck paths

with Jean-Luc Baril, Sergey Kirgizov (Dijon, F) and Mehdi Naima (Paris, F)



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CNRS, Université de Bordeaux, France

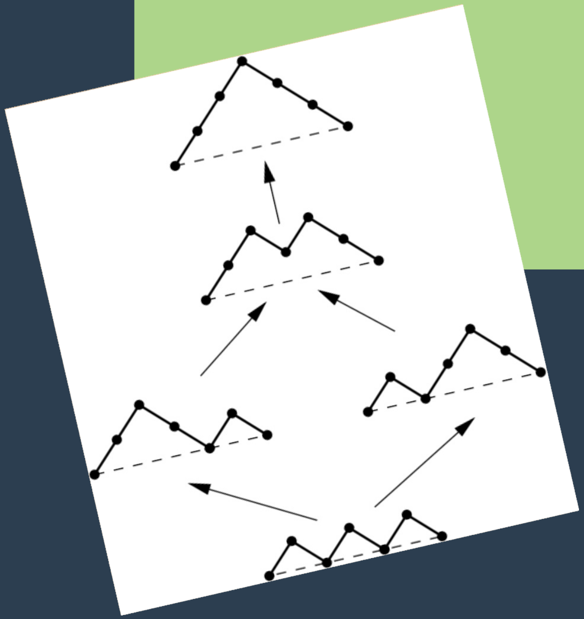
arXiv:2409.15982

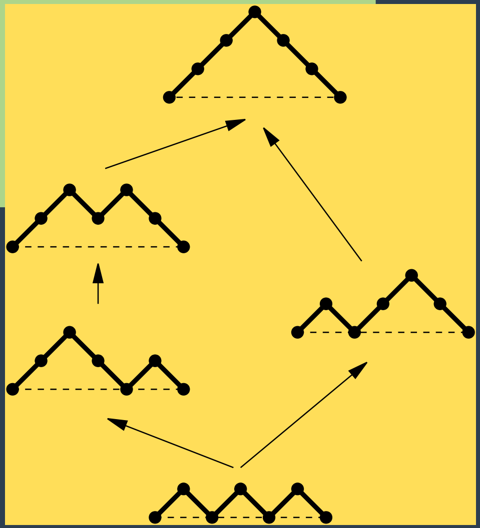
Outline

- A new family of **lattices** A_n
- **Interval** counting
- Interesting **subposets** and their intervals
- Connection with **sylvester congruence classes**

[Hivert, Novelli, Thibon 05]

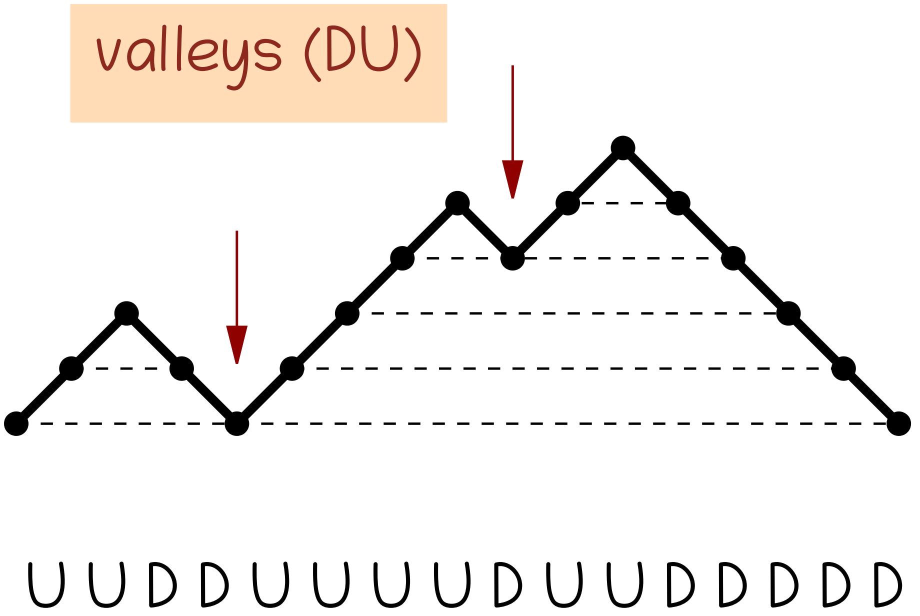
I. Two orders on Dyck paths



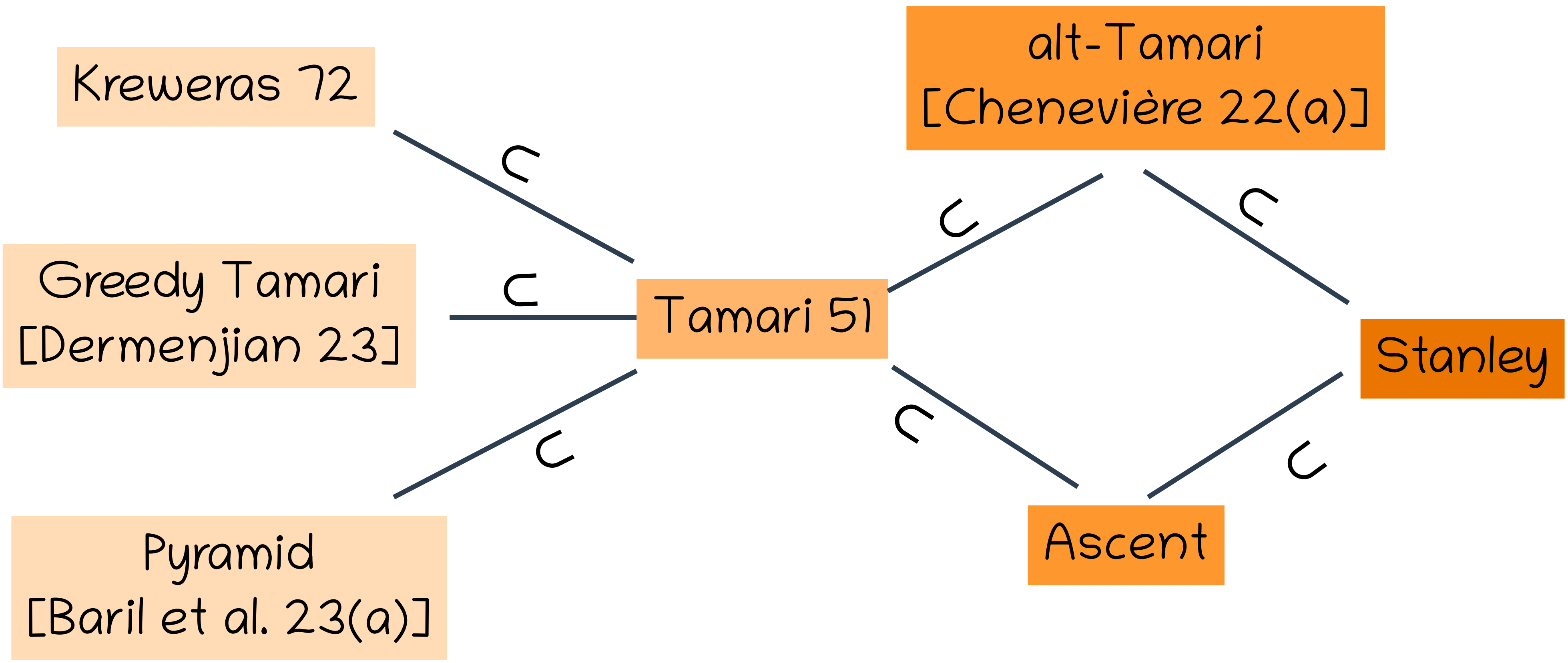


Dyck paths

- A Dyck path of size n=8 (size=number of up steps)



Many posets on Dyck paths of size n



Poset = partially ordered set

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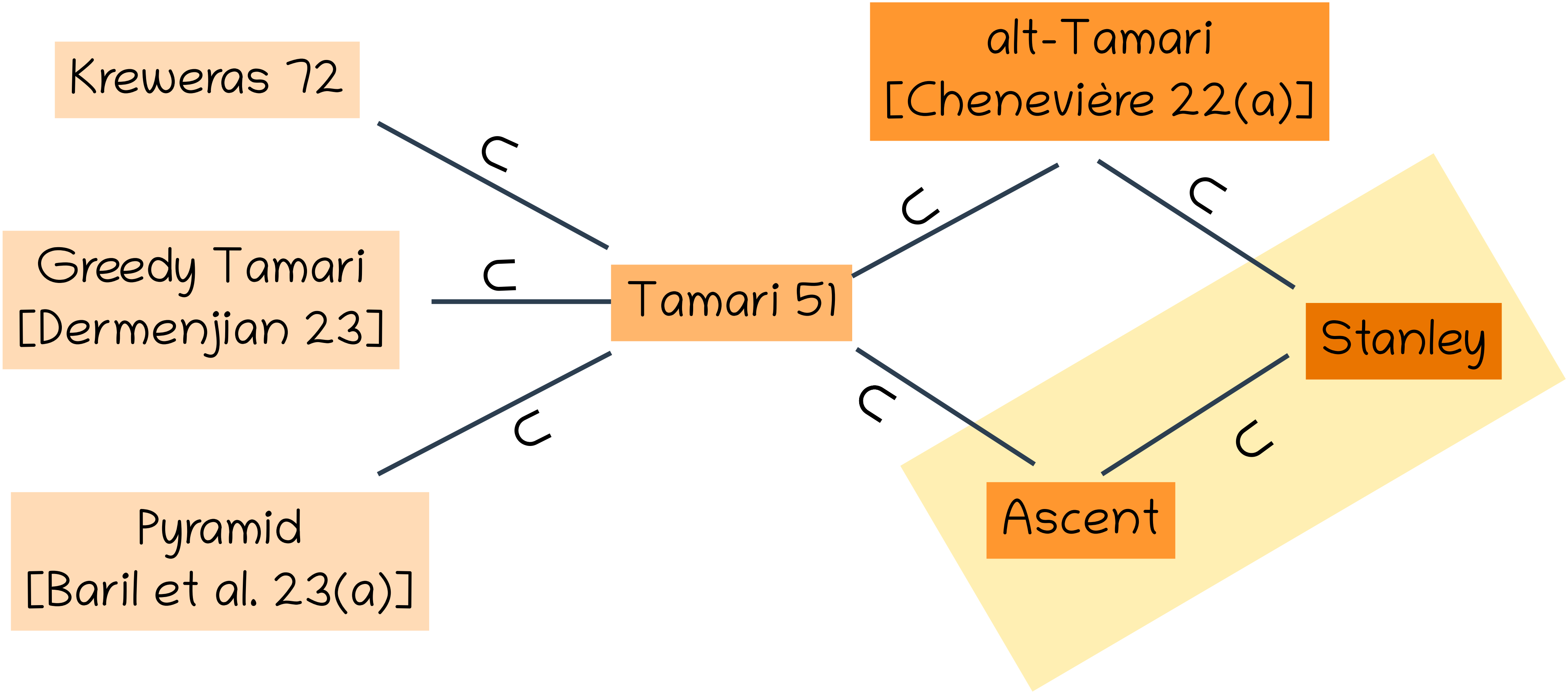

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The simplest poset: Stanley's lattice

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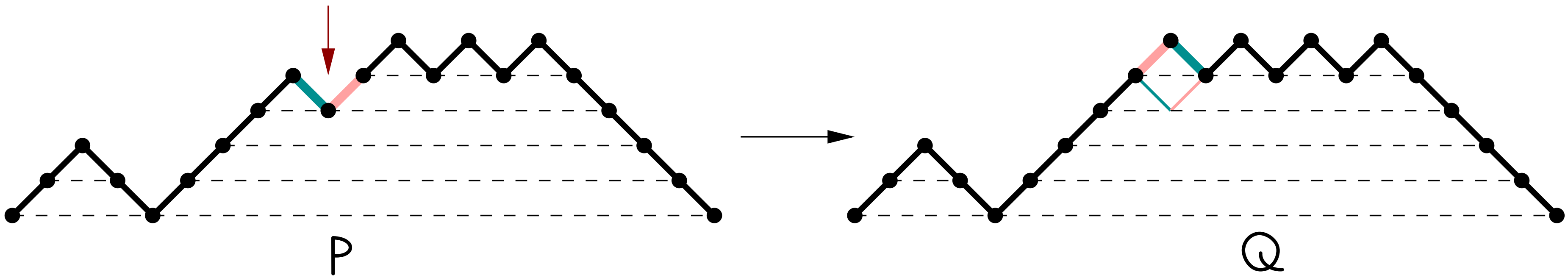

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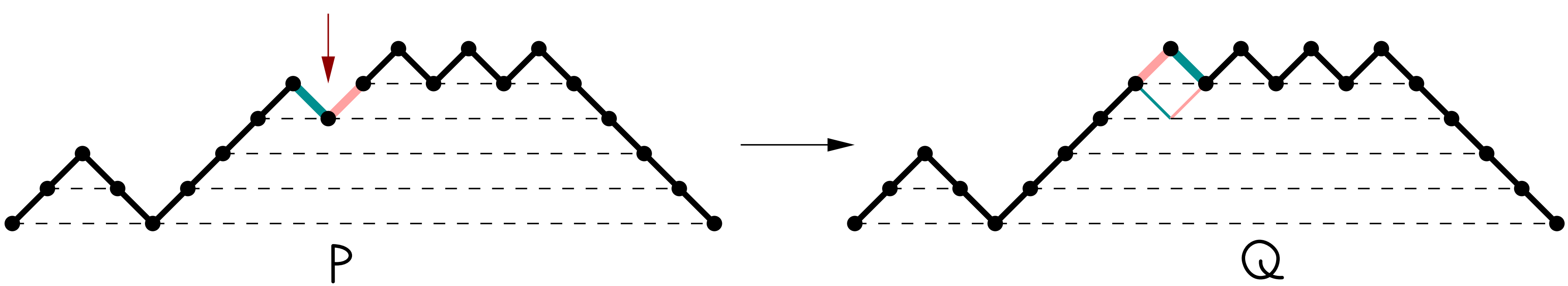
Characterization: $P \leq Q$ iff P fits below



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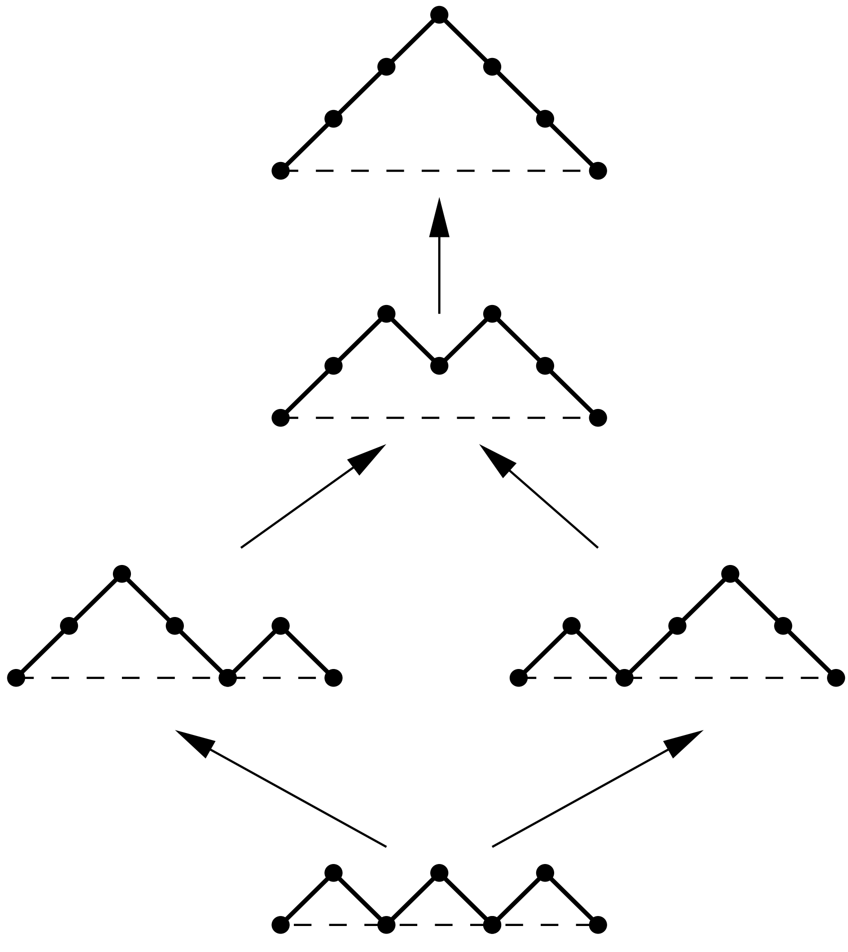

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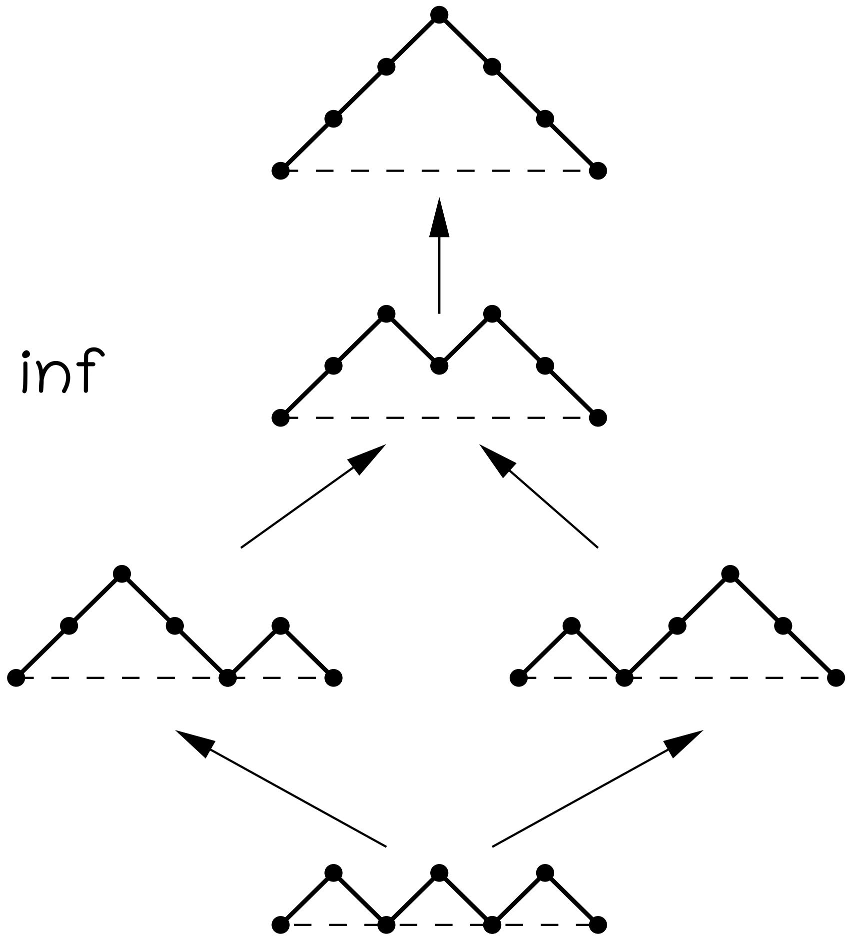
- **Lattice** structure: existence of sup and inf



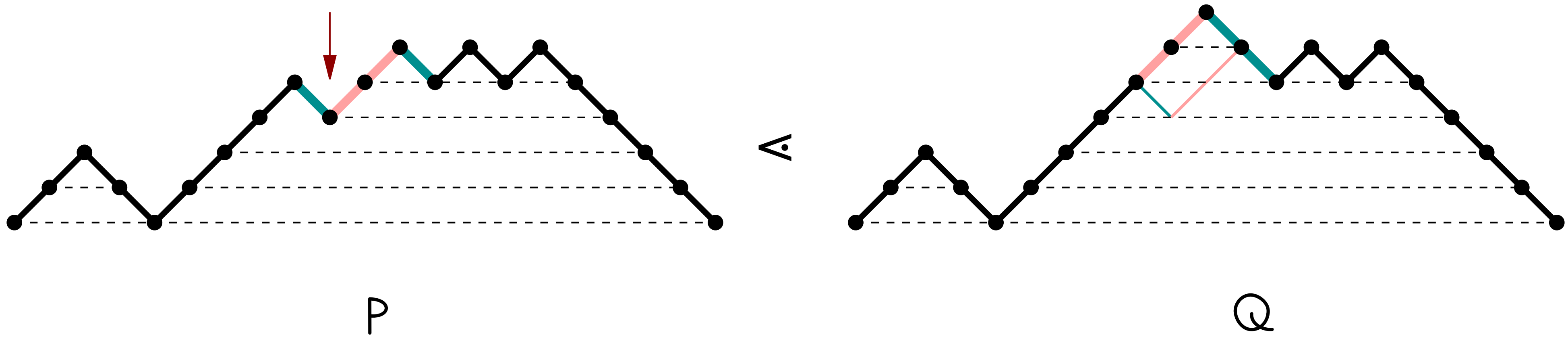
n=3

The ascent poset (or: greedy Stanley lattice?)

- A poset on Dyck paths with n up steps
- **Cover relations:** choose a valley in the path P.

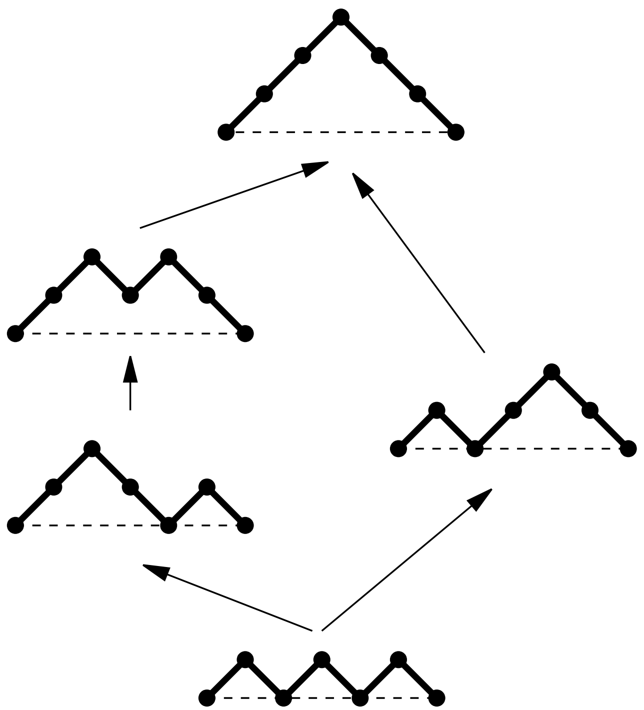
  Swap the down step and the ascent that follows

  (the path moves up).



[Chenevière, Nadeau...]

Ascent posets: n = 3, 4



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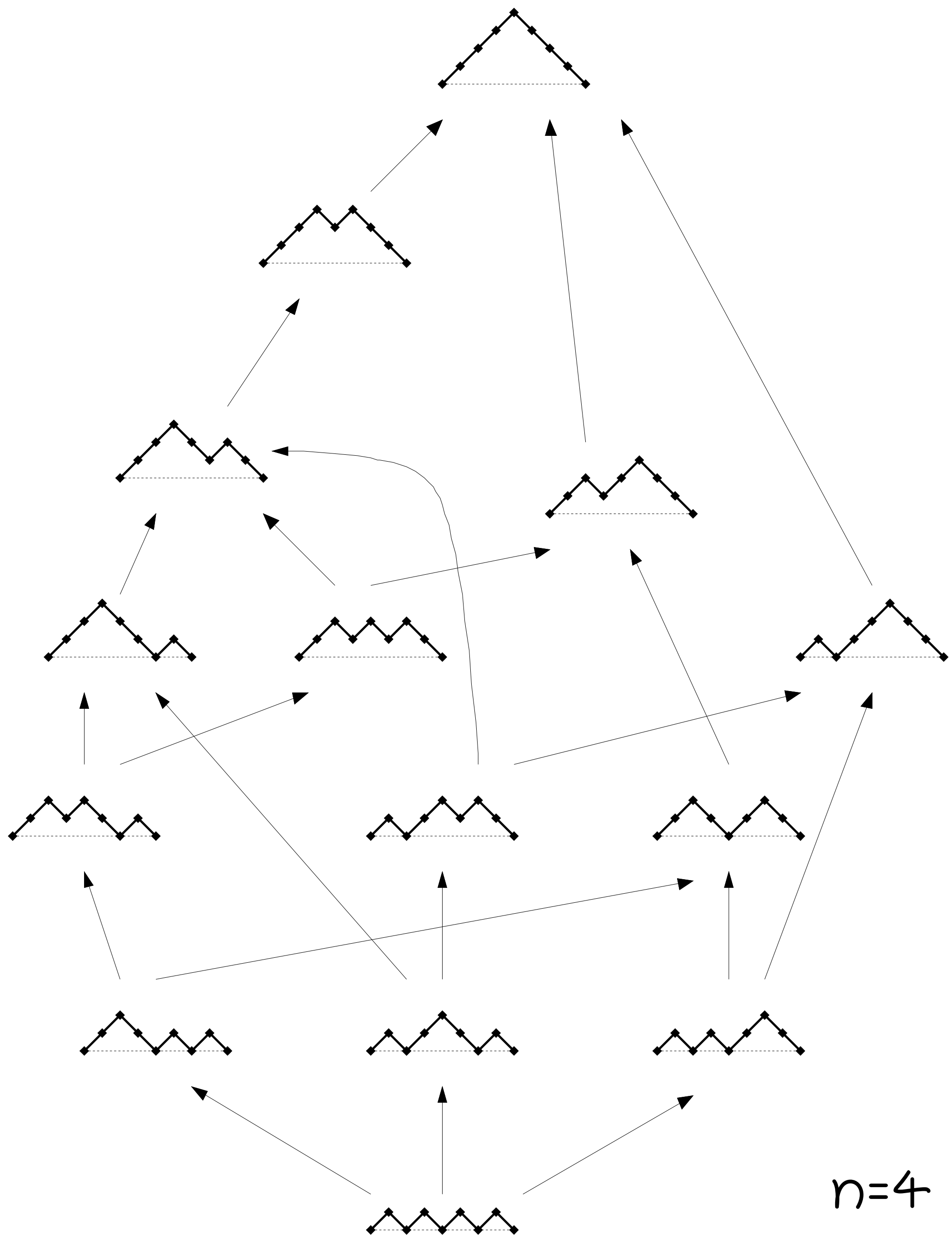

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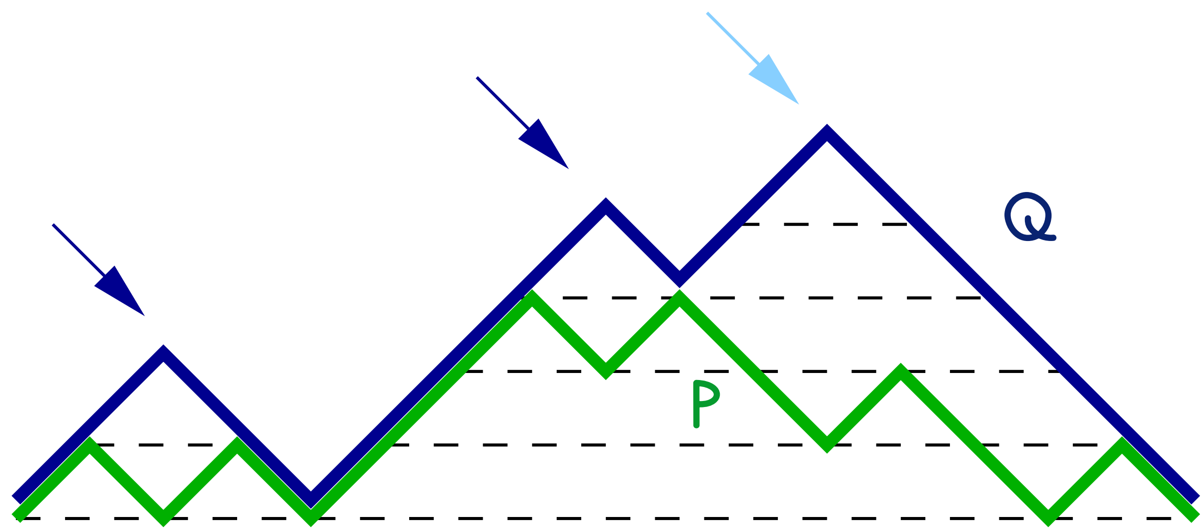
A characterization of the ascent order

Proposition. In the ascent poset, $P \preccurlyeq Q$ iff

- P lies below Q
- every descent of Q **is** (i.e. lies on the same diagonal as) a descent of P.



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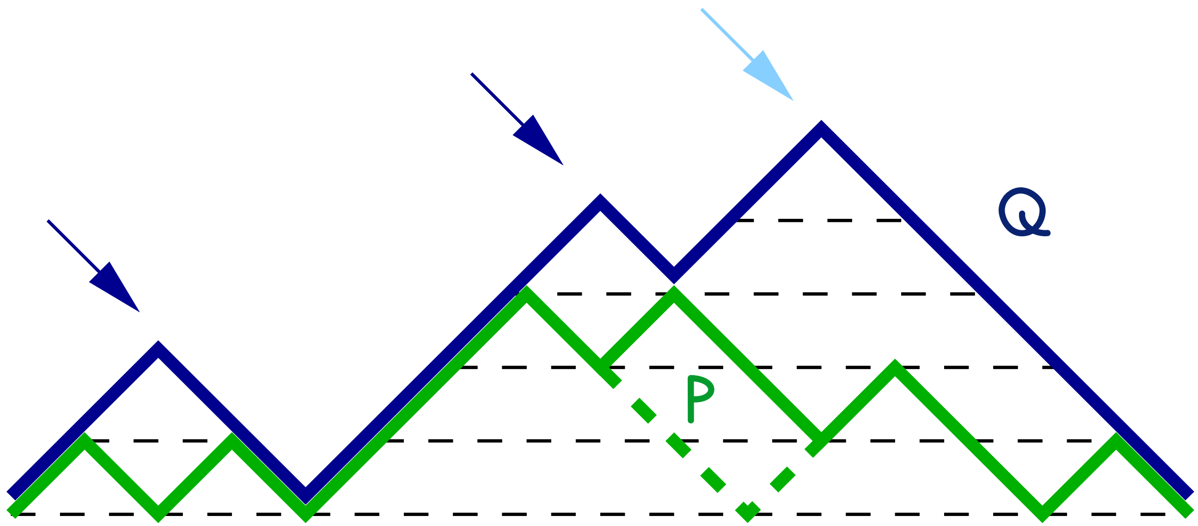

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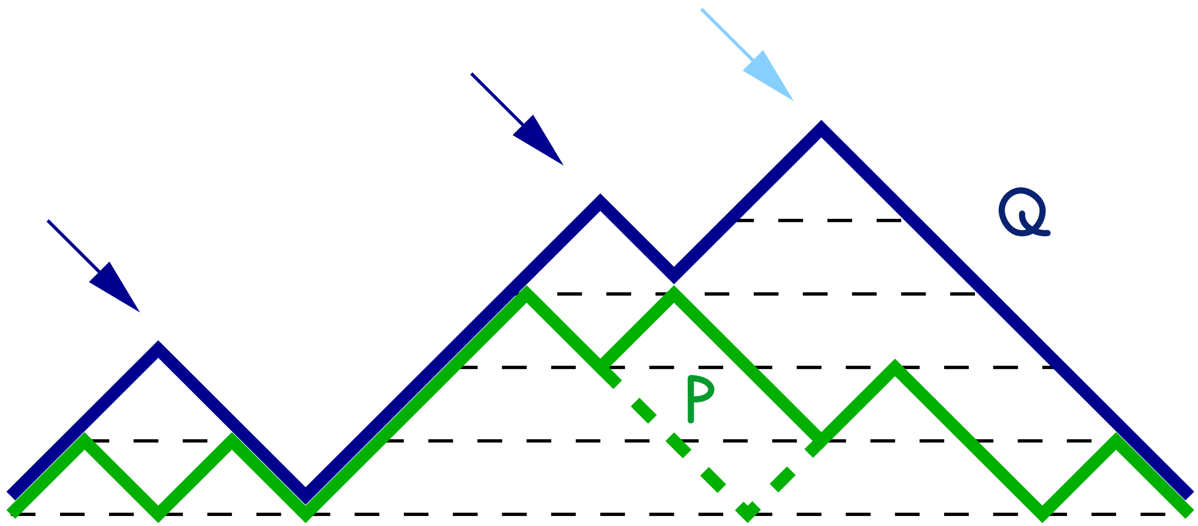

Applications:

- **lattice** structure
- recursive construction of **intervals**

II. The number of intervals

Interval $[P,Q] \sim (P,Q)$ with $P \preccurlyeq Q$

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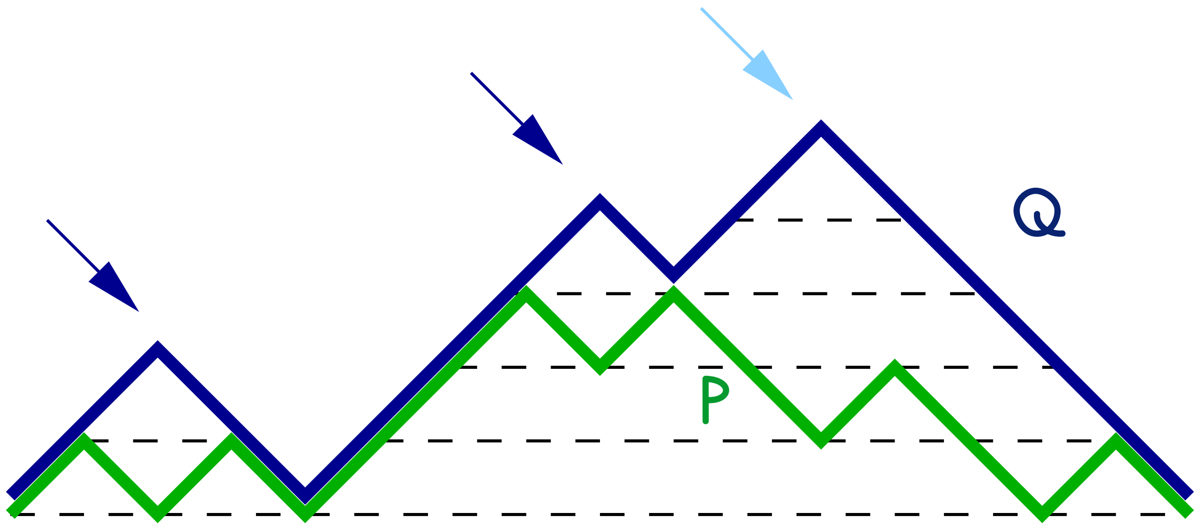

Corollary: if [P,Q] is an interval, deleting the last peak of P and the last peak of Q gives a new interval.

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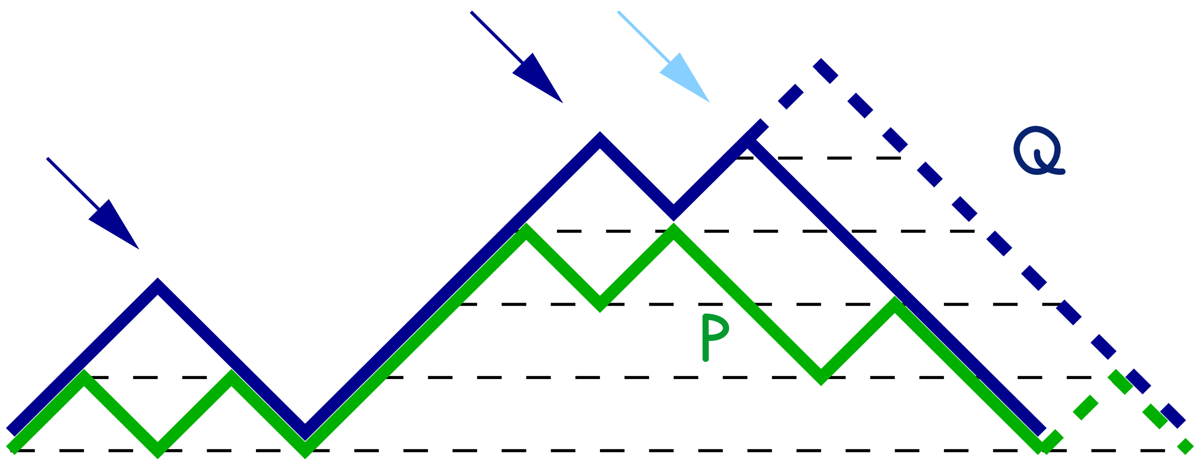

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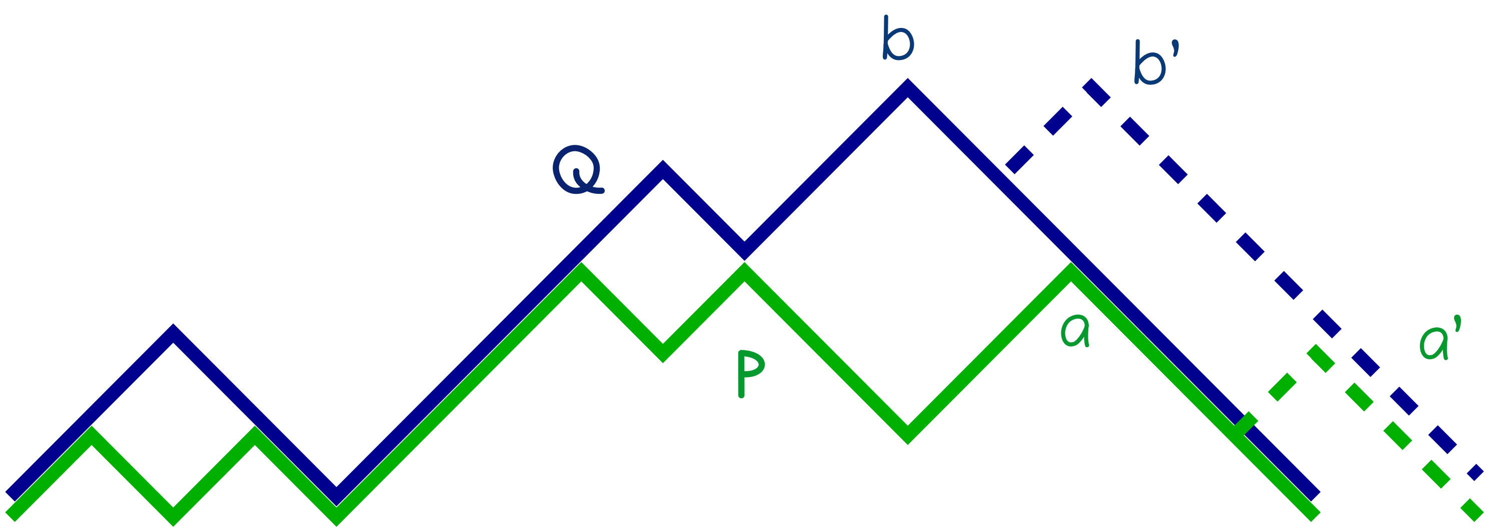
Recursive construction of ascent intervals

Conversely, starting from an interval [P,Q] with final peaks at heights $a \leq b$, adding peaks in P and Q at heights a' and b' gives an interval iff…



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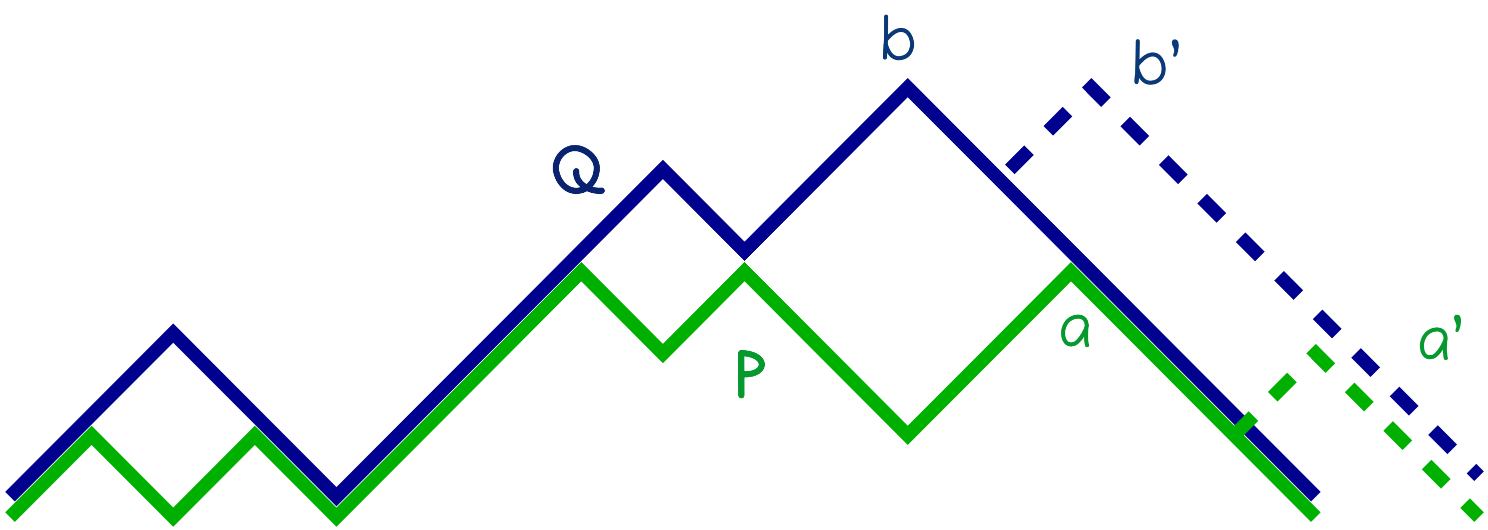

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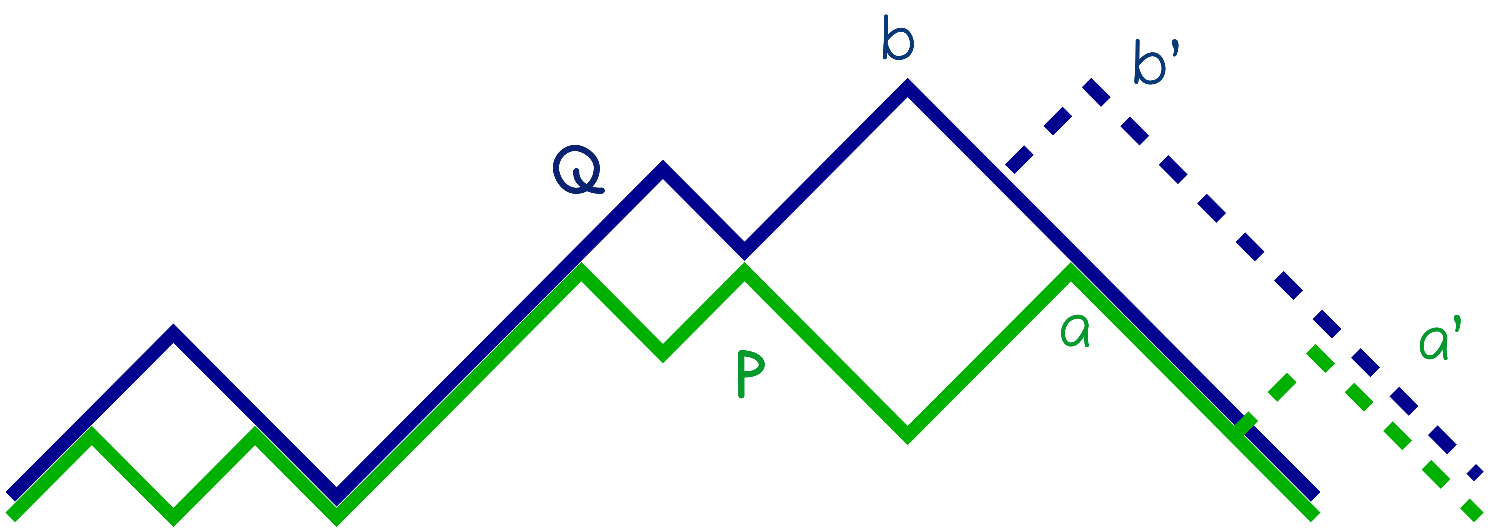

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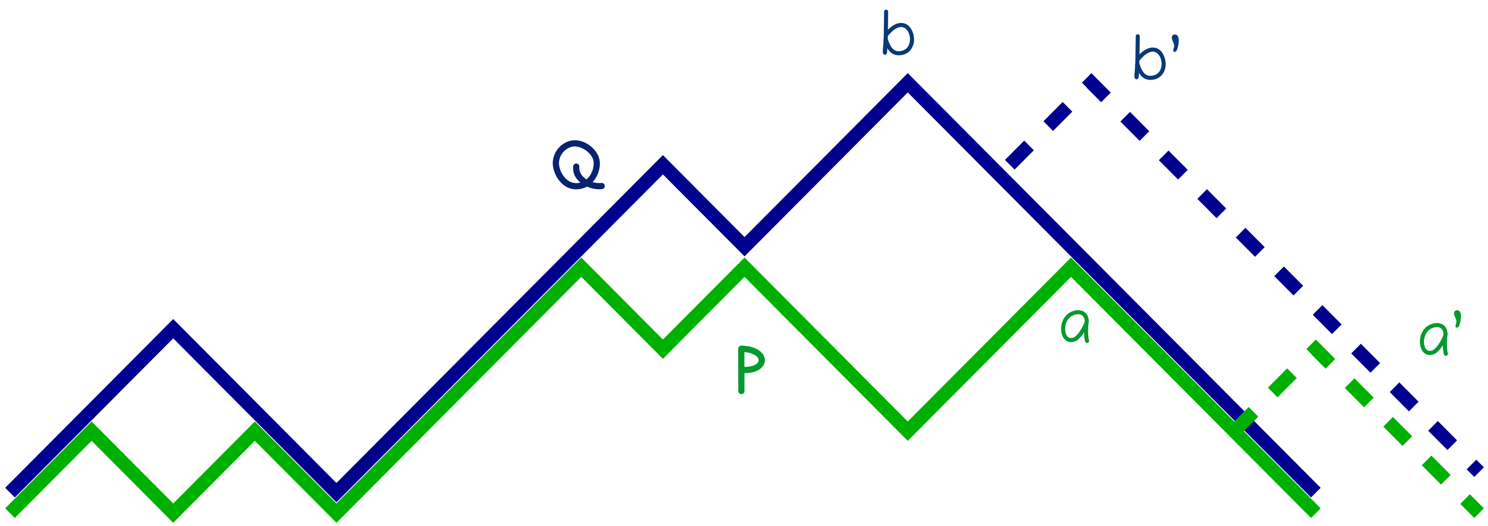

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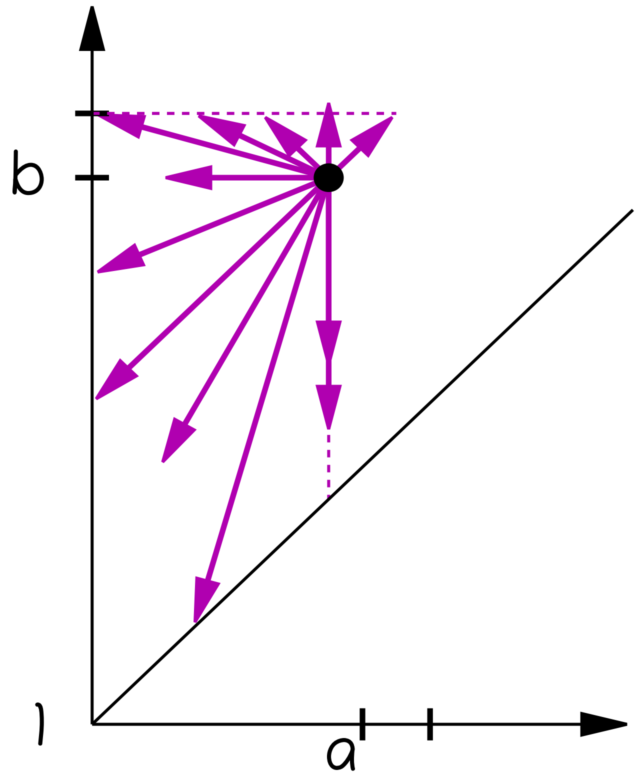

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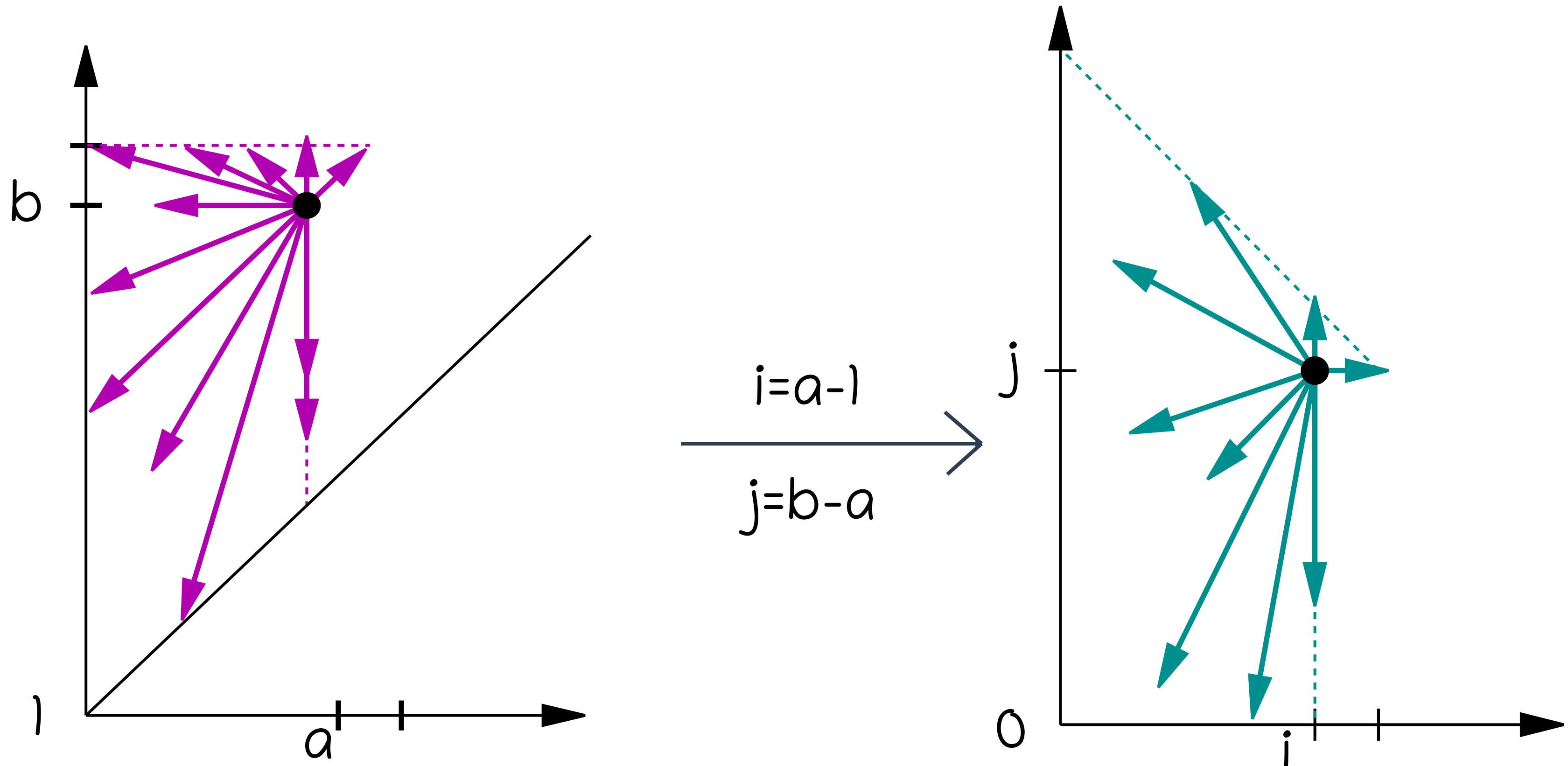

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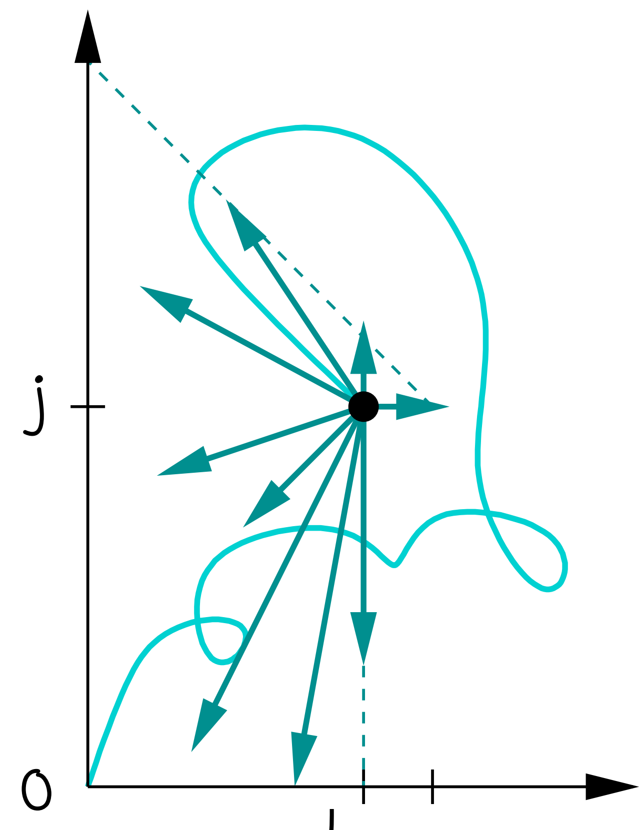
Bijection intervals of size n

$\approx$

quadrant walks of length n-1 starting from (0,0)



Recursive construction of ascent intervals

- Let Q(t;x,y)=Q(x,y) be the GF of the associated quadrant walks:

$$Q(x,y) = \sum_{w} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then the GF of ascent intervals is G=tQ(1,1).

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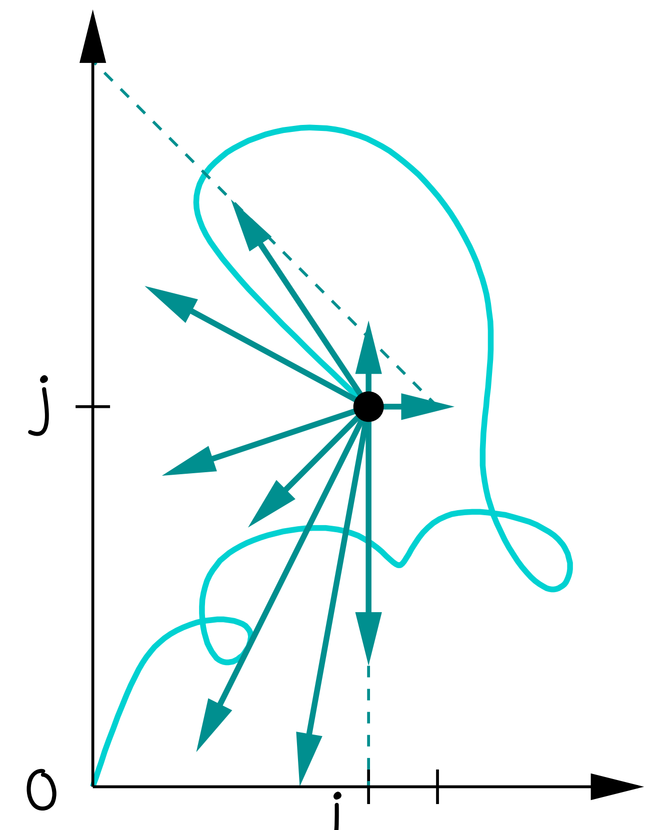

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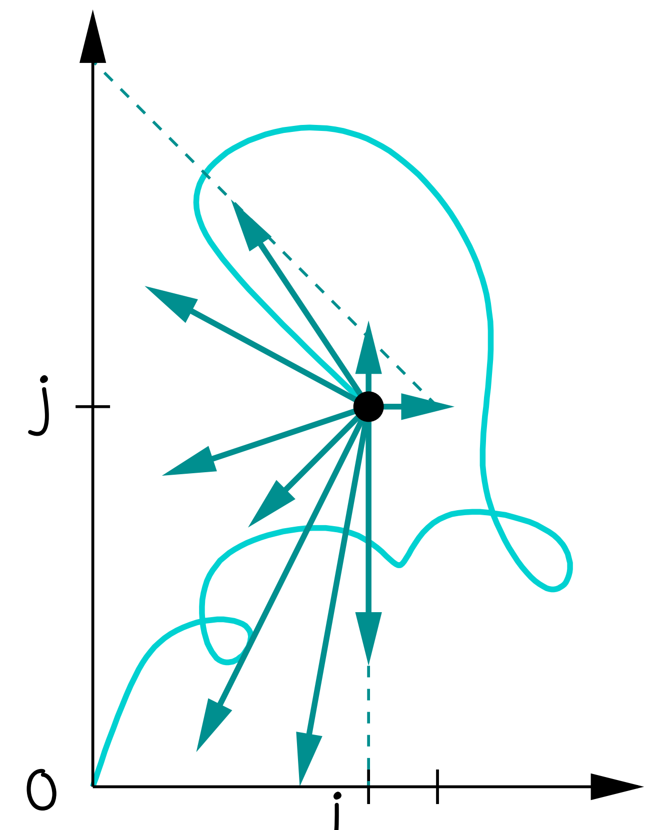

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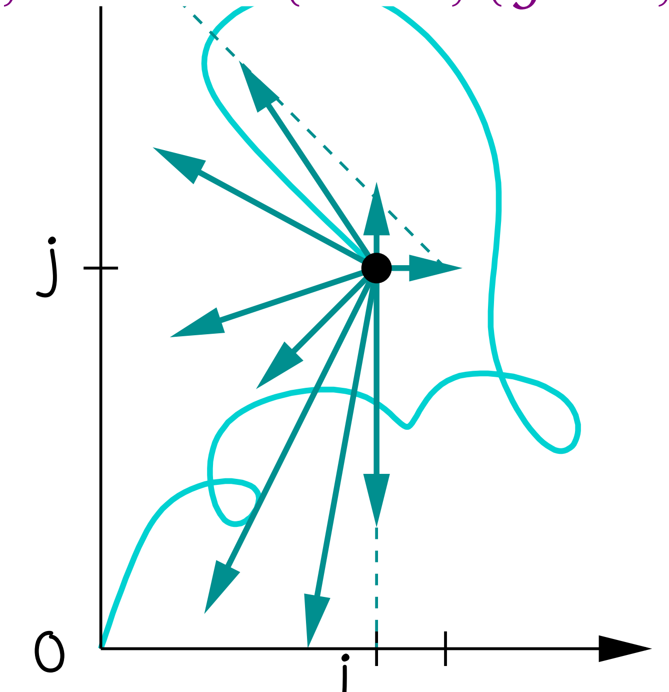

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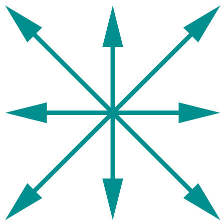
- **Well understood**: **algebraic/differential** properties of quadrant walks **with finitely many small steps** [Bernardi, Bostan, mbm, Raschel, Mishna, Zeilberger, Kauers, Hardouin, Dreyfus, Roques, Singer, Elvey Price...]



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Thm. Ascent intervals have **an algebraic GF**, namely

$$G = tQ(1,1) = Z(1 - 2Z + 2Z^3), \quad \text{where} \quad Z = t(1+Z)(1+2Z)^2.$$

Asymptotics:

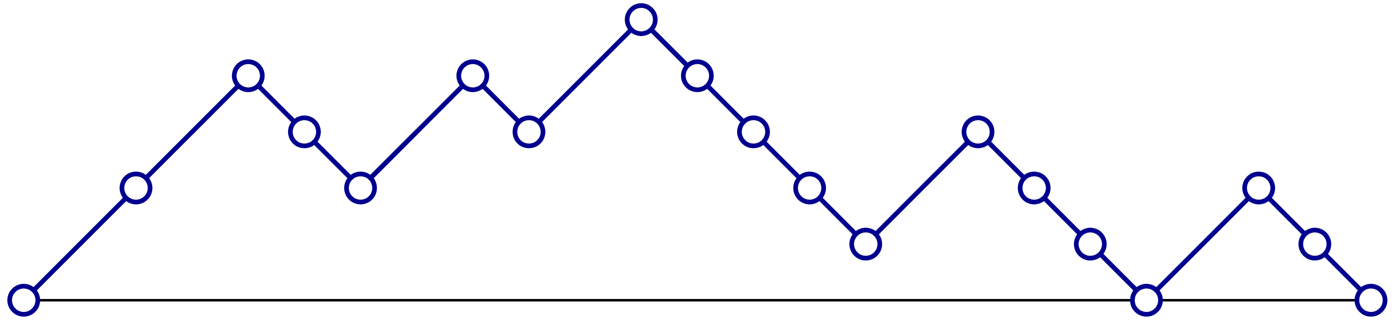
$$g(n) \sim \kappa\, \mu^n n^{-7/2}, \quad \text{with} \quad \mu = \frac{11 + 5\sqrt{5}}{2}.$$

III. m-Dyck paths, and mirrored m-Dyck paths

m-Dyck paths and mirrored m-Dyck paths

In an m-Dyck path, the length of each ascent is a multiple of m.

m=2



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In a mirrored m-Dyck path, the length of each descent is a multiple of m.

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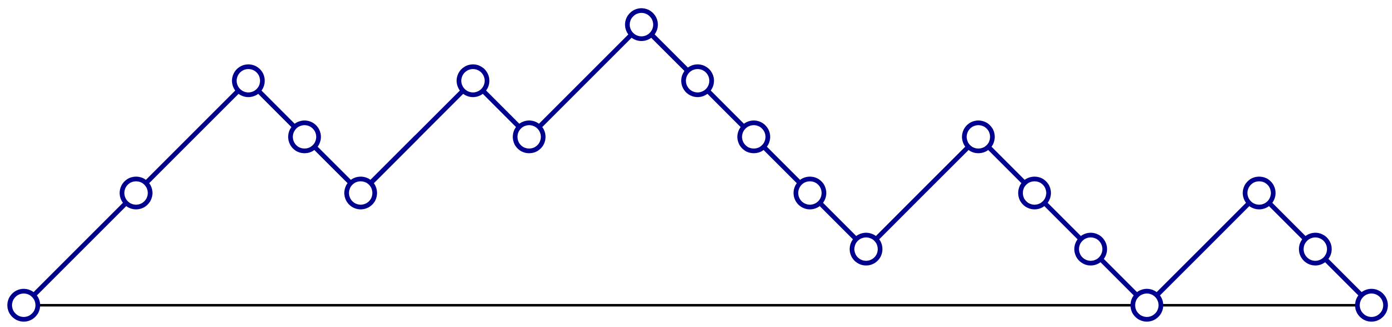

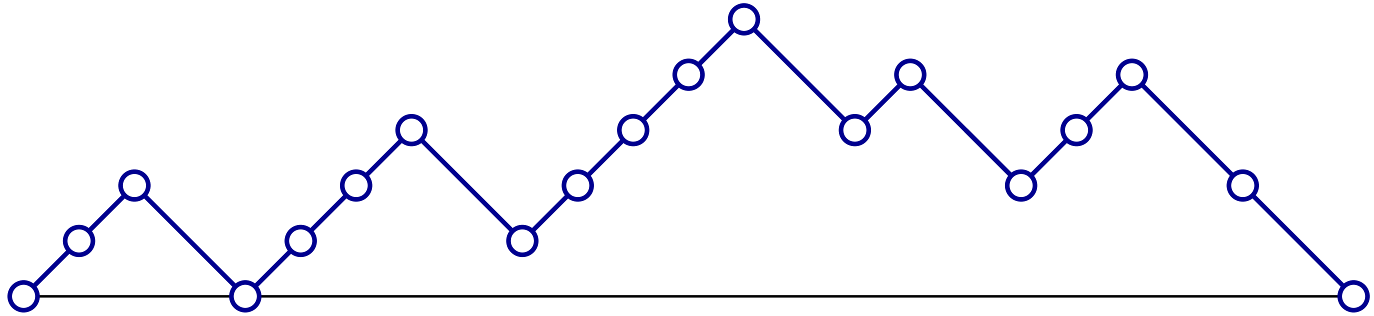

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In an **m-Dyck path**, the length of each **ascent** is a multiple of m.
In a **mirrored m-Dyck path**, the length of each **descent** is a multiple of m.

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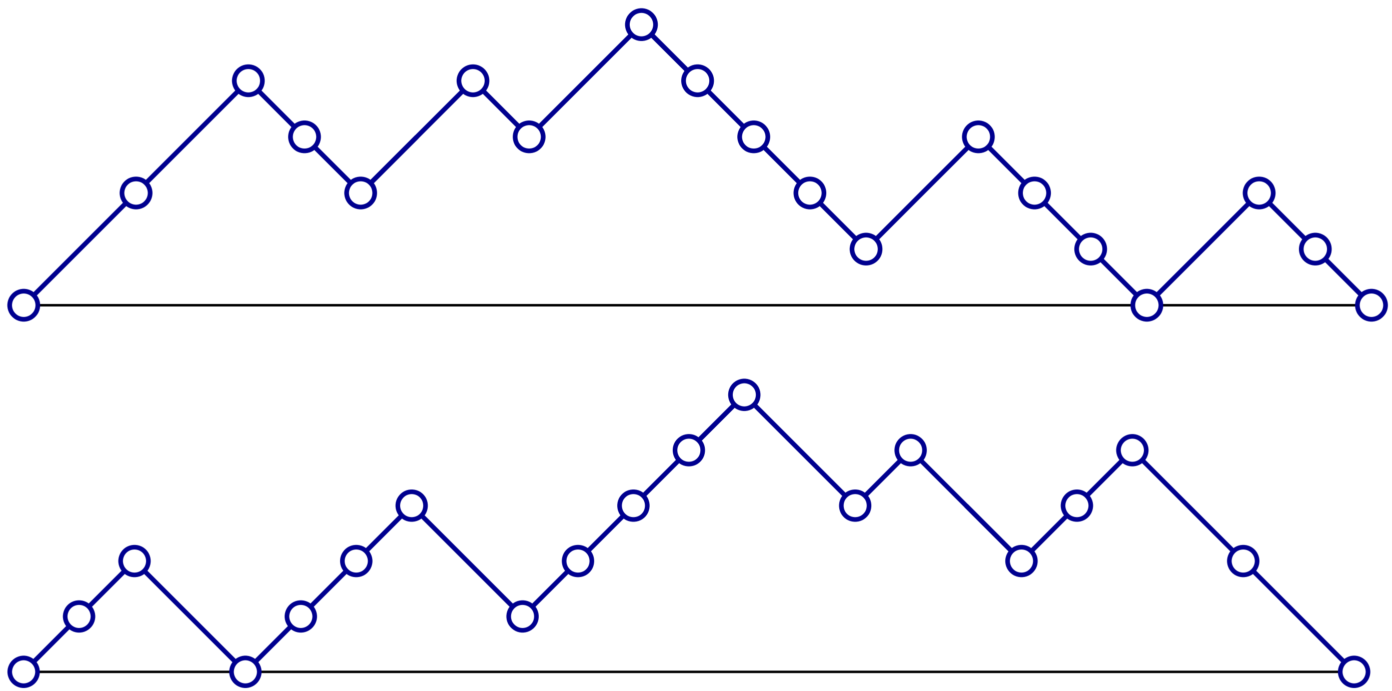
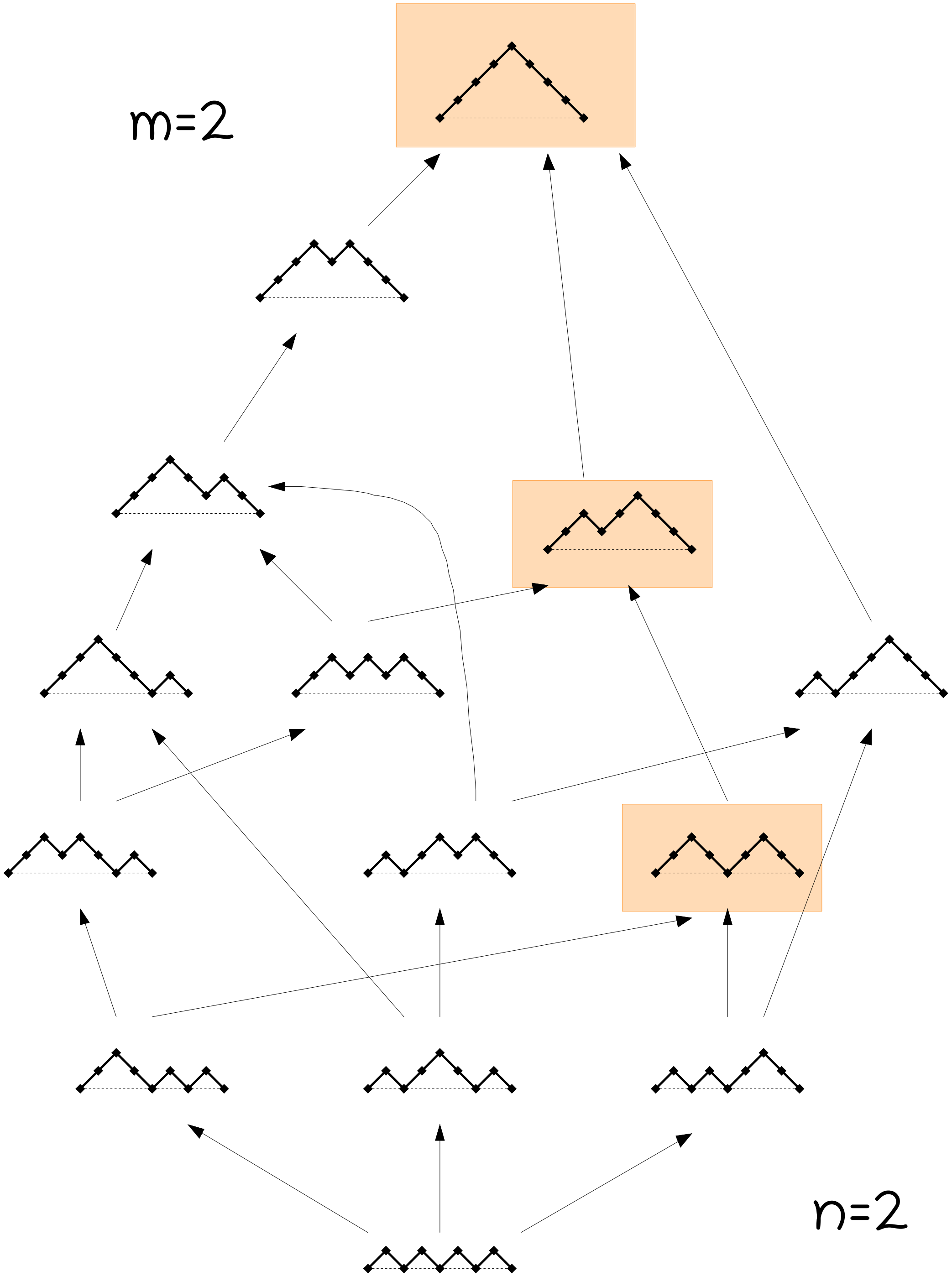


→ Study the order induced by the ascent order on m-Dyck paths and mirrored m-Dyck paths of size mn.

m-Dyck paths



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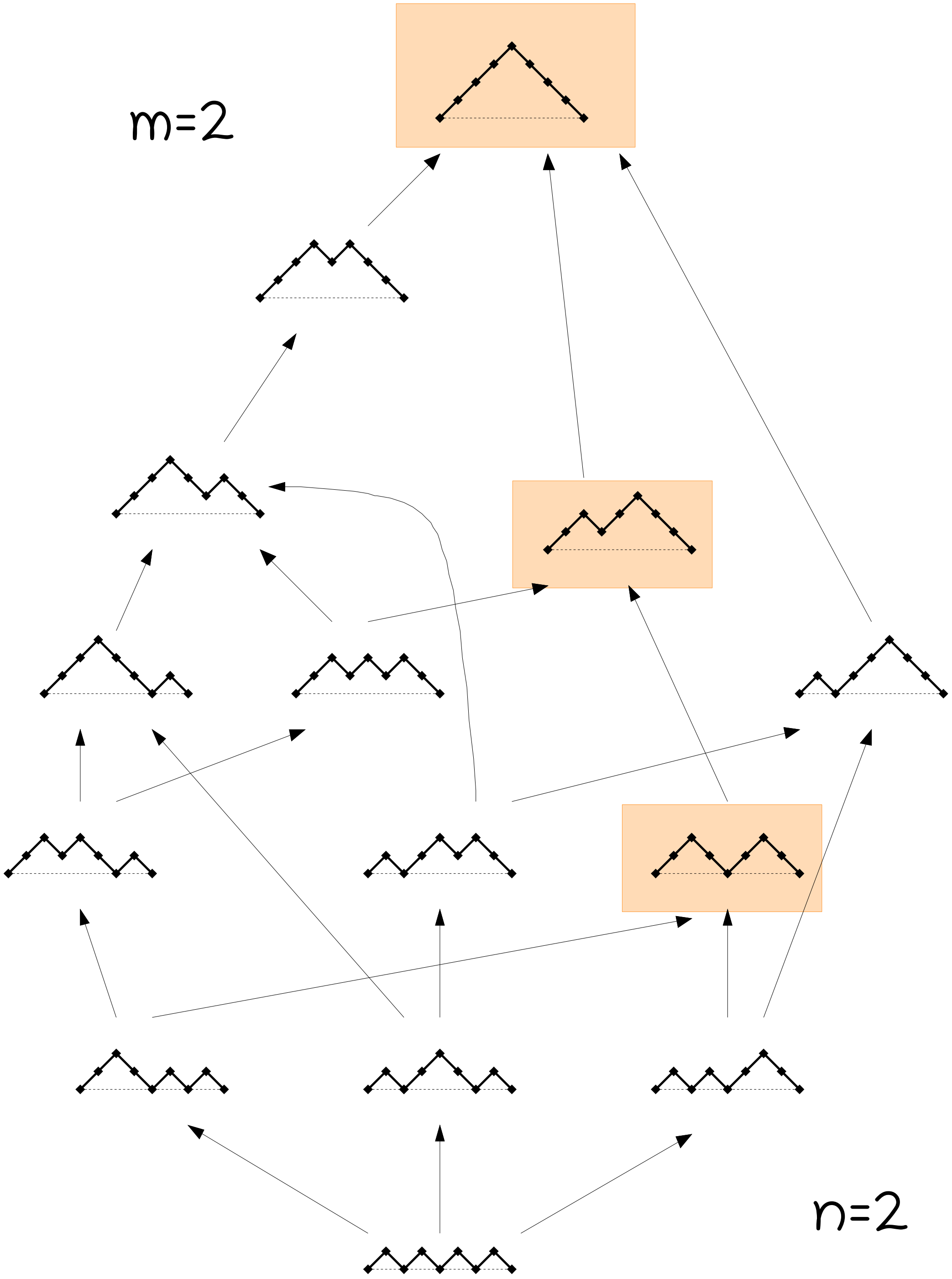
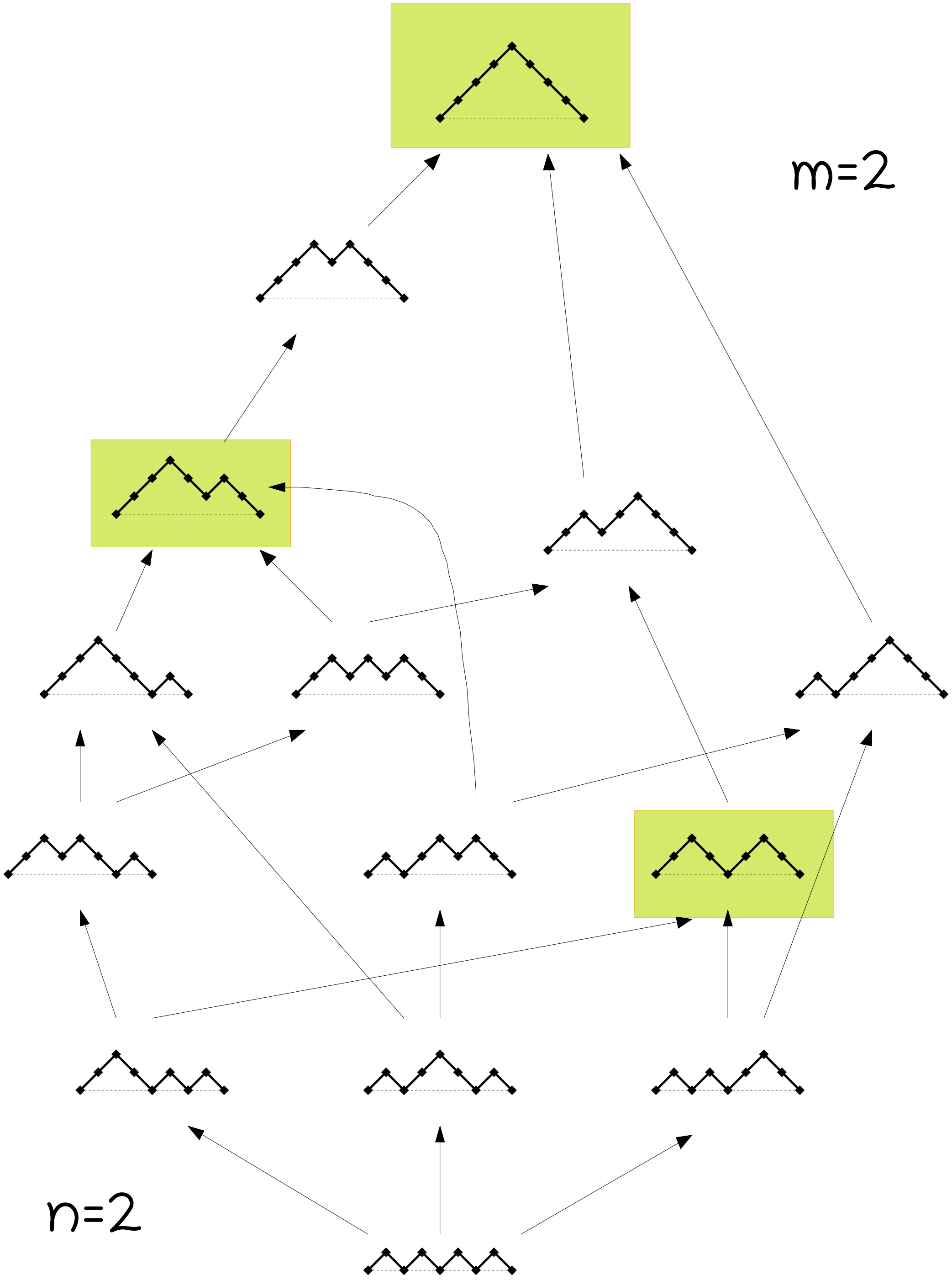


m-Dyck paths form an interval in the ascent lattice A_{mn}.

In particular, it is a lattice.

Mirrored m-Dyck paths

Mirrored m-Dyck paths only form a join semi-lattice.



Intervals of m-Dyck paths and mirrored m-Dyck paths

Two families of functional equations

➔ m-Dyck paths: last peak decomposition

$$Q(x,y) = 1 + tx^m Q(x,y) + ty^2 \frac{x^m Q(x,y) - y^m Q(y,y)}{(x-y)(y-1)} - t \frac{x^m Q(x,1) - Q(1,1)}{(x-1)(y-1)}$$

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➔ **Mirrored m-Dyck paths**: first peak decomposition

$$\overline{Q}(x,y) = 1 + tx^m \frac{y\overline{Q}(x,y) - \overline{Q}(x,1)}{y-1} + ty^2 \frac{x^m \overline{Q}(x,y) - \overline{Q}(1,y)}{(x-1)(y-1)} - t\frac{x^m \overline{Q}(x,1) - \overline{Q}(1,1)}{(x-1)(y-1)}$$

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- **No exact solution, but explicit asymptotic results** ⇒ not algebraic, not D-finite for m>1 (i.e. no linear diff. equation)

Intervals in the m-Dyck ascent lattice

- **Asymptotics** (from random walk results) [Denisov & Wachtel 15]

$$g_m(n) \sim \kappa \mu^n n^\alpha,$$

where

$$\mu = \frac{m\sqrt{m^2+4}+m^2+2}{2} \cdot \left(\frac{2+\sqrt{m^2+4}}{m}\right)^m$$

and

$$\alpha = -1 - \pi/\arccos(c) \qquad \text{with} \qquad c = \sqrt{\frac{m^2+2-\sqrt{m^2+4}}{2m^2+6}}.$$

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For m>1, the exponent α is irrational, and hence the GF of intervals cannot be D-finite.

Uses a deep result of Siegel's G-functions

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Uses a deep result of Siegel's G-functions

Contrast with m-Tamari lattices, where intervals have an algebraic GF

IV. Connection with the sylvester congruence

[Hivert, Novelli, Thibon 05]

In the OEIS...

Observation: for $m=1, 2, \ldots, 5$, the sequence $\bar{g}_m(n)$ that counts intervals of mirrored m-Dyck paths appears in the OEIS.

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- **m=1:** number of sylvester classes of 1-multiparking functions

Search: **seq:1,3,13,69,417,2759 id:243688**

Displaying 1-1 of 1 result found.

Sort: relevance | references | number | modified | created   Format: long | short | data

A243688   Number of Sylvester classes of 1-multiparking functions of length n.

1, 3, 13, 69, 417, 2759

(list; graph; refs; listen; history; text; internal format)

OFFSET	1,2
COMMENTS	See Novelli-Thibon (2014) for precise definition.
LINKS	Table of n, a(n) for n=1..6. J.-C. Novelli, J.-Y. Thibon, Hopf Algebras of m-permutations, (m+1)-ary trees, and m-parking functions, arXiv preprint arXiv:1403.5962, 2014. See Fig. 26.
KEYWORD	nonn,more
AUTHOR	N. J. A. Sloane, Jun 14 2014
STATUS	approved

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- **m=2:** number of sylvester classes of 2-multiparking functions

Search: **seq:1,5,40,407,4797 id:243671**

Displaying 1-1 of 1 result found.

Sort: relevance | references | number | modified | created   Format: long | short | data

A243671   Number of Sylvester classes of 2-parking functions of length n.

1, 5, 40, 407, 4797

(list; graph; refs; listen; history; text; internal format)

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and so on.

Search: **seq:1,5,40,407,4797 id:243671**

Displaying 1-1 of 1 result found.

Sort: relevance | references | number | modified | created   Format: long | short | data

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AUTHOR	N. J. A. Sloane, Jun 14 2014
STATUS	approved

The sylvester congruence

- Defined on words on the alphabet $\mathbb{Z}$
- Generated by commutation relations:

$$\mathrm{ac} \cdots \mathrm{b} \equiv \mathrm{ca} \cdots \mathrm{b}, \quad \mathrm{a} \leq \mathrm{b} < \mathrm{c}.$$

- Class representatives: words avoiding subwords acb with $a \leq b < c$,

called **sylvester words.**

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A general correspondance between sylvester words and intervals of **a larger poset**.

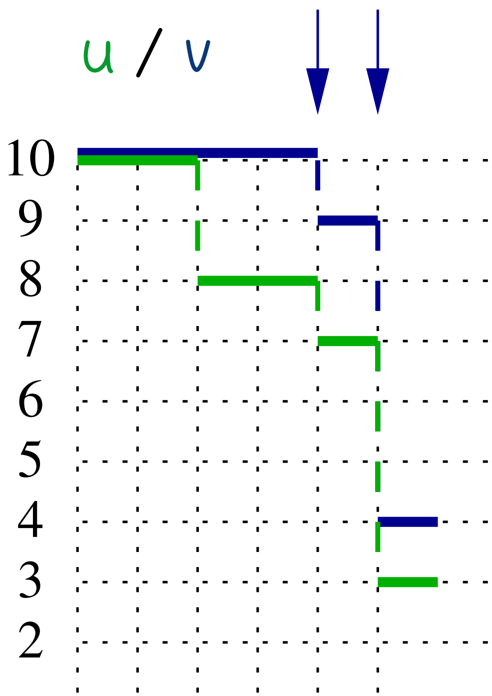
The Nadeau-Tewari poset NT_n [2024]

Def. Let $u=(u_1, \dots, u_n)$ and $v=(v_1, \dots, v_n)$ be two nonincreasing sequences of integers. Then $u \leq v$ for the NT order if

- u lies below v ($u_i \leq v_i$)
- every descent of v is a descent of u.

$v = (10, 10, 10, 10, 9, 4)$

$u = (10, 10, 8, 8, 7, 3)$



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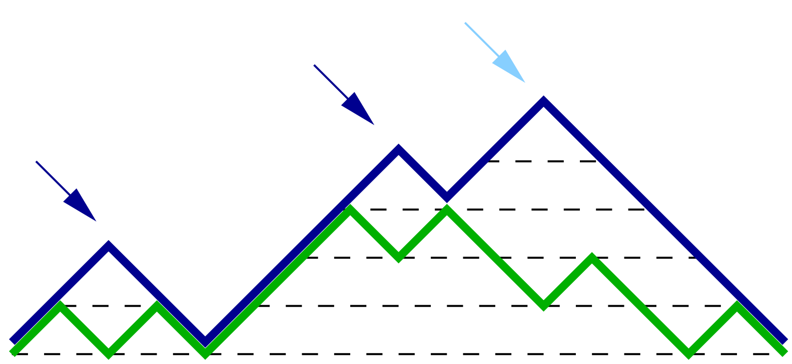

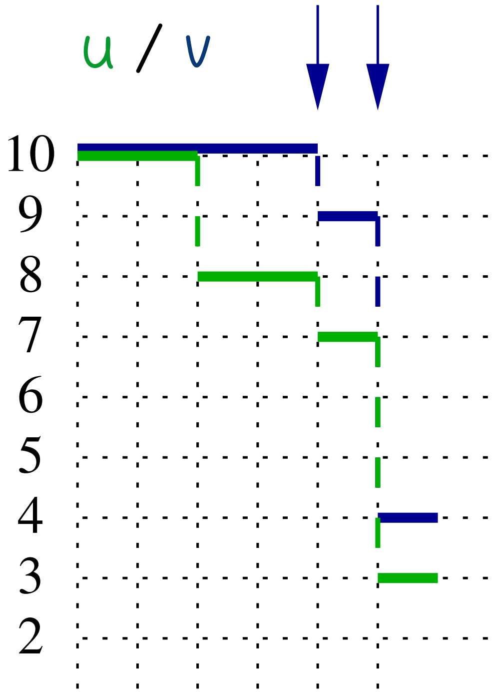

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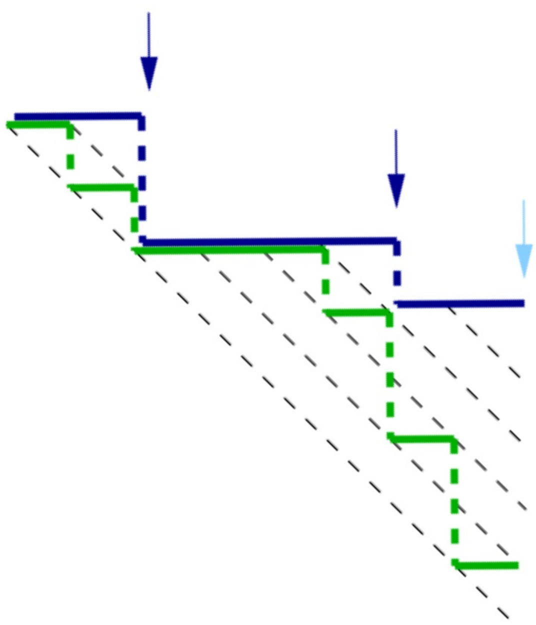

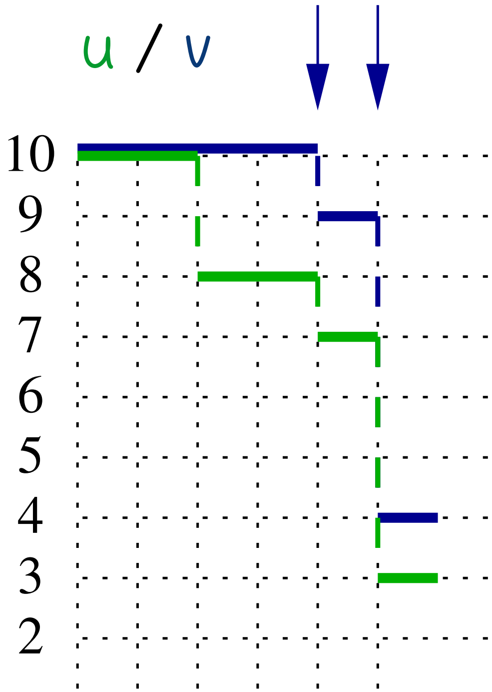

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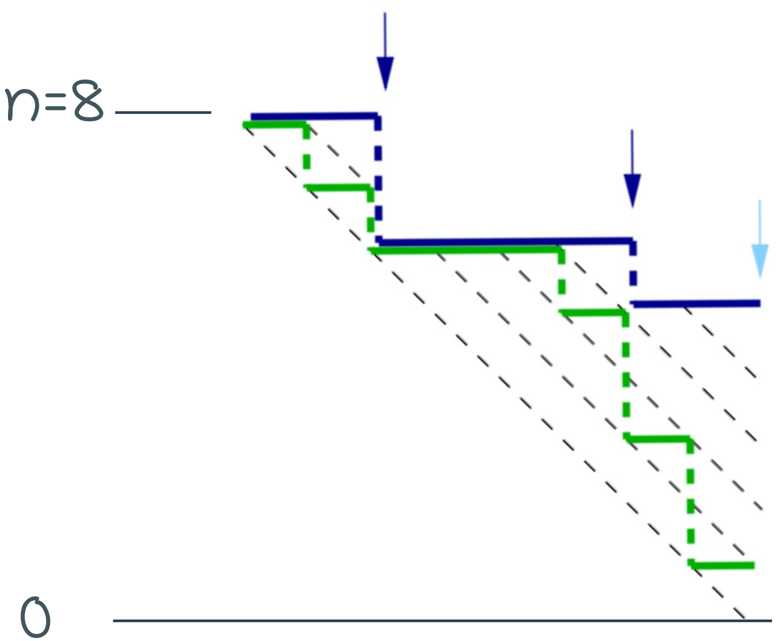
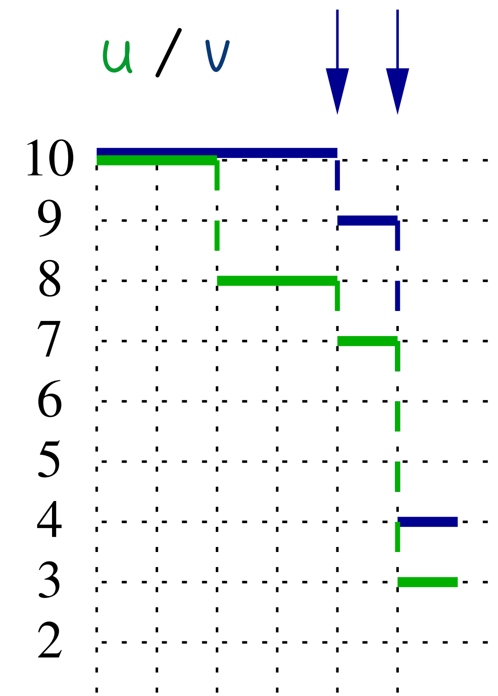


v = (8, 8, 6, 6, 6, 6, 5, 5)

u = (8, 7, 6, 6, 6, 5, 3, 1)



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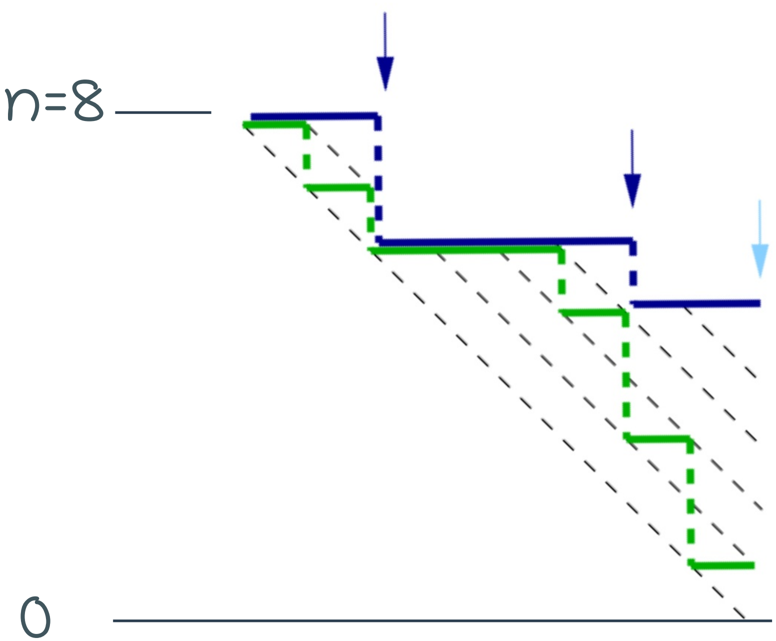
Observation: the ascent lattice A_n is the **interval** in the NT lattice NT_n with min=$(n,n-1,\ldots, 1)$ and max=$(n,n, \ldots, n)$



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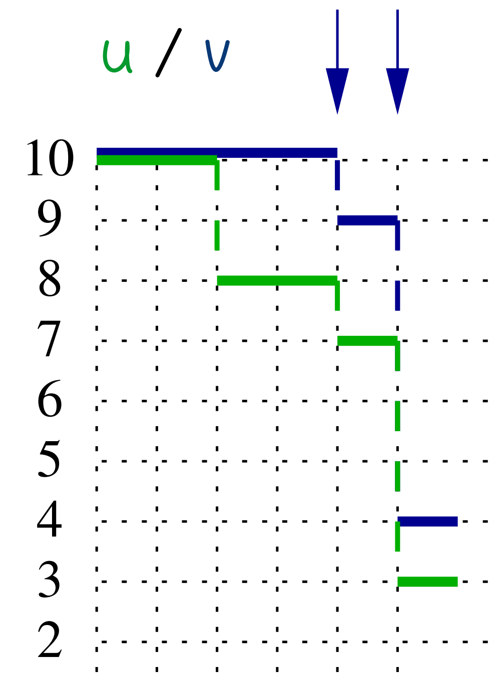


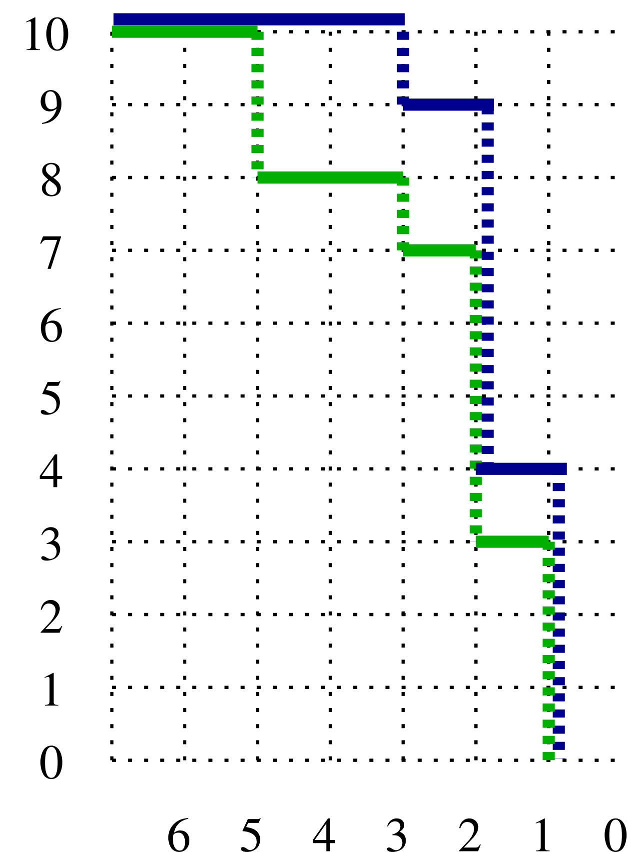
From sylvester words to Nadeau-Tewari intervals

Proposition. For any n, there is an explicit bijection between:

- sylvester words w on the alphabet {1, 2, ..., n} containing the letter 1, and
- intervals [u,v] in the NT lattice of size n, such that u and v have positive entries and the same first letter.

Example

For n=6 and w = 3 2 2 2 2 5 1 1 1 5, we have u= 10 10 8 8 7 3 and v= 10 10 10 10 9 4.



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Specializations: bijections between

- positive sylvester words w of length mn such that $\mathrm{NInc}(w) \leq n^m (n-1)^m \dots 2^m 1^m$ and **ascent intervals of m-Dyck paths** of length mn
- positive sylvester words w of length n such that $\mathrm{NInc}(w) \leq ((n-1)m+1) \dots (2m+1)\,(m+1)\,1$ and **ascent intervals of mirrored m-Dyck paths** of length mn.

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OEIS

Final questions

- **Combinatorial proof** for the number/GF of ascent intervals? (m=1)

$$(n+4)(2n+7)\,g(n+2) = 2\left(11n^2+44n+42\right) g(n+1) + n(2n+1)\,g(n)$$

- **A symmetric joint distribution** on ascent intervals [P,Q] (m=1):

  a(P) = length of the first ascent of P

  r(P,Q) = number of ascents of P before the first descent of Q

- **D-algebraicity** for m-Dyck paths, m>1?

- **Chains of length 3** in the poset? of length d?

- Study **mirrored m-Dyck paths** in other Dyck lattices: intervals?

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Thanks for your attention