A Crystal on Decreasing Factorizations in the 0-Hecke Monoid

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Introduction

Motivation

- Schubert classes for cohomology of flag varieties: Schubert polynomials, $\mathfrak{S}_w$
- Stable limit of $\mathfrak{S}_w$: Stanley symmetric functions, $\mathcal{F}_w$
- **Morse-Schilling '16**: Type A crystal for $\mathcal{F}_w$, with local changes
- Stable limit of Schubert classes for K-theoretic cohomology of flag varieties: stable Grothendieck polynomials, $\mathfrak{G}_w$
- **Fomin-Greene '98**: $\mathfrak{G}_w$'s are Schur positive
- **Monical-Pechenik-Scrimshaw '18**: Induced type A crystal for $\mathfrak{G}_\lambda$ on decreasing factorizations is nonlocal

Question

Is there a type A crystal structure for stable Grothendieck polynomials, $\mathfrak{G}_w$, with local changes?

Background

0-Hecke monoid

Monoid of all finite words in $[n-1] := \{1, 2, \ldots, n-1\}$, where $n \in \mathbb{N}$, such that for all $p, q \in [n-1]$,

$$pp \equiv p, \quad pqp \equiv qpq,$$

if $|p-q| > 1$, we also have $pq \equiv qp$.
Set of all equivalence classes of words: $\mathcal{H}_0(n)$

Example

- $2121 \equiv 1211 \equiv 121 \equiv 212$
- $31312 \equiv 3132 \equiv 312 \equiv 132$

Decreasing factorization of $\mathcal{H}_0$

A decreasing factorization of $w \in \mathcal{H}_0(n)$ is a product of m decreasing **factors** $\mathbf{h} = h^m \ldots h^2 h^1$, with $\mathbf{h} \equiv w$ in $\mathcal{H}_0(n)$.

Example

All decreasing factorizations of $w = 132$, with 5 letters and 3 factors:

$$(31)(31)(2),\ (31)(32)(2),\ (31)(1)(32),$$
$$(31)(3)(32),\ (1)(31)(32),\ (3)(31)(32)$$

Stable Grothendieck polynomial [LS '82, FK '94]

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{h^m \ldots h^2 h^1 \in \mathcal{H}_w} \beta^{\ell(h^1)+\cdots+\ell(h^m)-\ell(w)} x_1^{\ell(h^1)} \ldots x_m^{\ell(h^m)},$$

where $\ell(w)$ is the length of any reduced word of w. Also known as the K-Stanley symmetric function for w.

Example

All decreasing factorizations of $w = 132$ with 3 letters and 3 factors:
$()(31)(2)$, $()(1)(32)$, $(31)()(2)$, $(1)()(32)$, $(31)(2)()$, $(1)(32)()$,
$(3)(1)(2)$, $(1)(3)(2)$

$$[\beta^0]\mathfrak{G}_{132} = (x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2)$$
$$+ \mathbf{2x_1x_2x_3} = s_{21}$$

321-avoiding Hecke words

An element $w \in \mathcal{H}_0(n)$ is 321-**avoiding** if none of the reduced expressions for w contain a consecutive subword of the form $i\ i+1\ i$ for any $i \in [n-1] = \{1, 2, \ldots, n-1\}$.

Example

- $121 \equiv 212$ ✗
- $132 \equiv 312$ ✓
- $22132 \equiv 2132 \equiv 2312$ ✓

$\mathcal{H}^{m,\star}(n)$

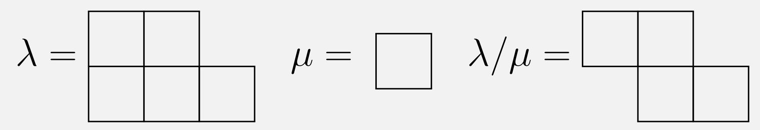
Let $\mathbf{h}$ be a decreasing factorization of $w \in \mathcal{H}_0(n)$, then $\mathbf{h}$ is 321-avoiding if w is 321-avoiding. Denote $\mathcal{H}^{m,\star}(n)$ as the set of all 321-avoiding decreasing factorizations of $\mathcal{H}_0(n)$ with m factors.

Example

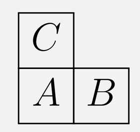
- $(\,)(1)(21) \in \mathcal{H}^3$
- $(\,)(1)(21) \notin \mathcal{H}^{3,\star}$
- $(31)(2) \in \mathcal{H}^{2,\star}$

Semistandard Set-Valued Skew Tableaux (SVT)

Fix partitions λ, μ, with $\mu \subseteq \lambda$. Fill λ/μ with nonempty subsets of $[m]$.



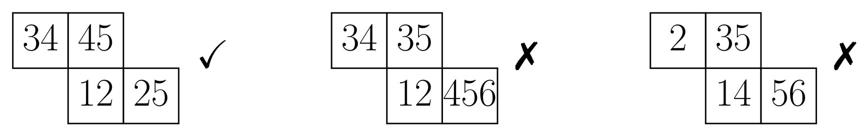
Rule:



$\max(A) \leqslant \min(B)$, $\max(A) < \min(C)$

Set of all such tableaux of shape λ/μ and maximum entry m is denoted $\mathsf{SVT}^m(\lambda/\mu)$

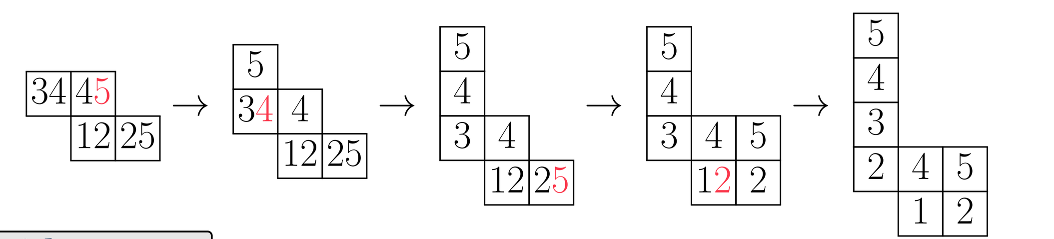
Example: Which one is a valid filling?



Uncrowding Operator on SVT [Buch '02, BM '12, RTY '18]

- Identify the topmost row in T containing a multicell.
- Let x be the largest letter in that row which lies in a multicell.
- Delete this x and perform RSK algorithm into the rows above.
- Yields a semistandard skew tableau.

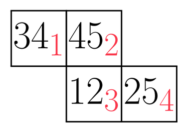
Example



Residue map

- $\mathsf{res} : \mathsf{SVT}^m(\lambda/\mu) \to \mathcal{H}^{m,\star}$
- Associate cell (i, j) with $\ell(\lambda) + j - i$, where $\ell(\lambda)$ is the number parts in λ.
- Form ith factor h^i by taking the labels of all cells in T containing i in decreasing order.

Example



$\xrightarrow{\mathsf{res}} (42)(21)(1)(43)(3) \in \mathcal{H}^{5,\star}$

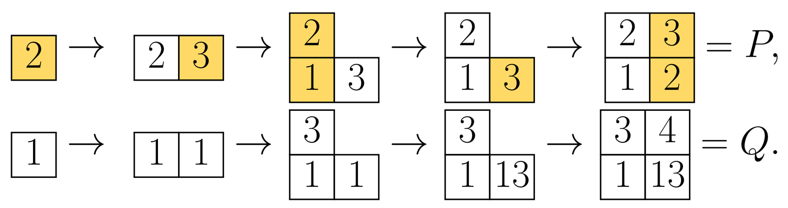
Hecke insertion [Buch et al. '08, Patrias-Pylyavskyy '16]

Read $\mathbf{h}$ from right to left, insert x to row R of an **increasing tableau**:

- Try to append x to the right of R (record and terminate).
- Try to bump the minimal $y > x$ (proceed to the next row).

Example

$$\mathbf{h} = (2)(31)(\,)(32) = \begin{bmatrix} 4 & 3 & 3 & 1 & 1 \\ 2 & 3 & 1 & 3 & 2 \end{bmatrix}$$

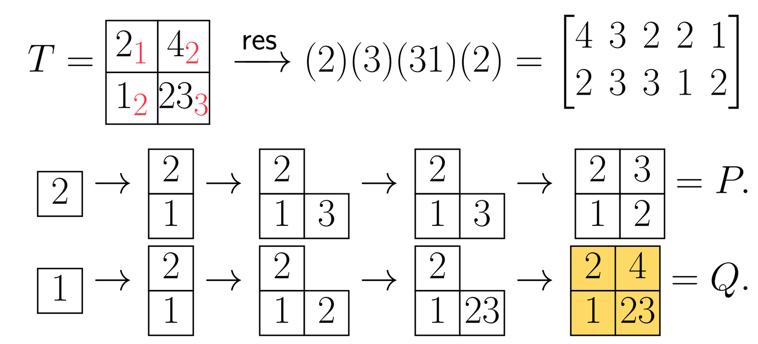


Key Results

Theorem 1: Hecke insertion and the residue map

Let $T \in \mathsf{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \mathsf{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q). Then $Q = T$.

Example



⋆-Crystal Structure on $\mathcal{H}^{m,\star}$

Bracketing Rule on $\mathbf{h} = h^m \ldots h^{i+1} h^i \ldots h^1$

- Start with the **largest** letter b in h^{i+1}, pair it with the smallest $a \geqslant b$ in h^i. If there is no such a, then b is unpaired.
- Proceed in decreasing order in h^{i+1}, ignore previously paired letters.

Crystal Operator $f_i^\star$

Denote largest unpaired letter in h^i by x.

- If x does not exist, $f_i^\star(\mathbf{h}) = 0$.
- Else, if $x + 1 \in h^i \cap h^{i+1}$, remove $x + 1$ from h^i, add x to h^{i+1}.
- Otherwise, remove x from h^i and add x to h^{i+1}.

Example

- $(1)(32) \xrightarrow{f_1} (31)(2)$
- $(7532)(621)(4) \xrightarrow{f_2} (75321)(61)(4)$

$w = 132$, β^1 $\quad \mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \boldsymbol{\beta(2s_{211} + s_{22})} + \beta^2(3s_{2111} + 2s_{221}) + \cdots$

$(3,1)(3,2)(\,)$
$(3,1)(1)(2)$
$(3,1)(2)(2)$
$(3,1)(3)(2)$
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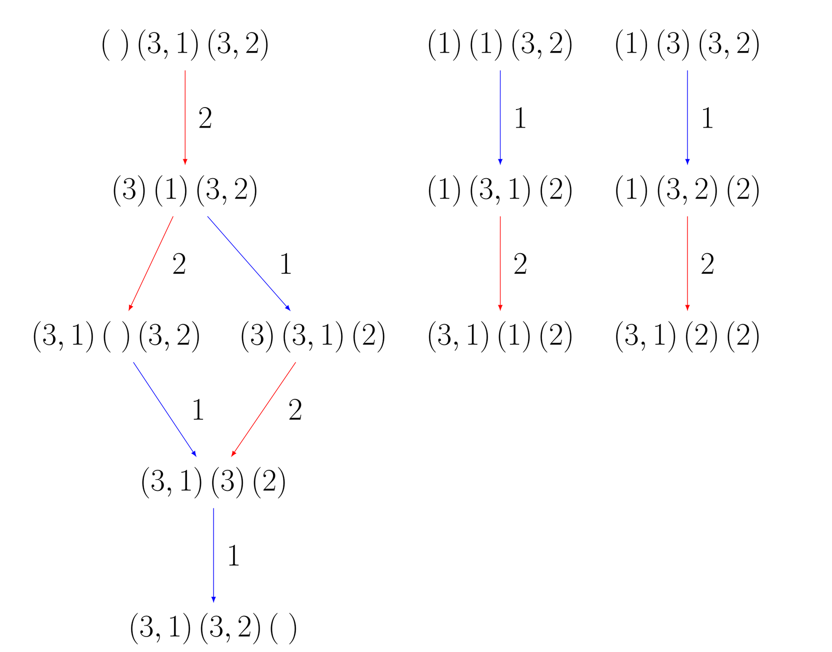


Theorem 2: The residue map is a crystal isomorphism

The crystal on semistandard set-valued skew tableaux and the crystal on $\mathcal{H}^{m,\star}$ intertwine under the residue map.

$$\begin{array}{ccc} \mathsf{SVT}^m(\lambda/\mu) & \xrightarrow{\mathsf{res}} & \mathcal{H}^{m,\star} \\ \downarrow f_k & & \downarrow f_k^\star \\ \mathsf{SVT}^m(\lambda/\mu) & \xrightarrow{\mathsf{res}} & \mathcal{H}^{m,\star}. \end{array}$$

Example

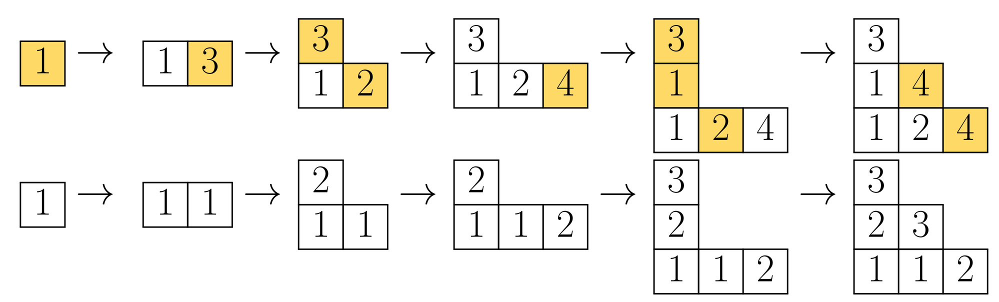
⋆-insertion

Insert x into row R of a transpose of a semistandard tableau:

- Try to append x to the right of R (terminate and record).
- If $x \notin R$, bump the minimal $z > x$ (proceed to the next row).
- If $x \in R$, no change to R, proceed to next row with y minimal such that $[y, x] \subseteq R$.

Example

$$\mathbf{h} = (42)(42)(31) = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 & 1 \\ 4 & 2 & 4 & 2 & 3 & 1 \end{bmatrix}$$



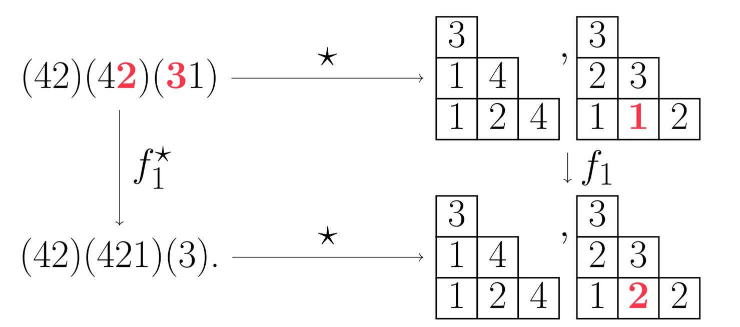
Theorem 3: Association with ⋆-crystal

Let $\mathbf{h} \in \mathcal{H}^{m,\star}$ and $(P^\star(\mathbf{h}), Q^\star(\mathbf{h})) = \star(\mathbf{h})$, then we have:

1. $f_i^\star(\mathbf{h}) \neq 0$ if and only if $f_i(Q^\star(\mathbf{h})) \neq 0$.
2. If $f_i^\star(\mathbf{h}) \neq 0$, then $Q^\star(f_i^\star(\mathbf{h})) = f_i Q^\star(\mathbf{h})$.

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathsf{SSYT}^m \\ \downarrow f_i^\star & & \downarrow f_i \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathsf{SSYT}^m. \end{array}$$

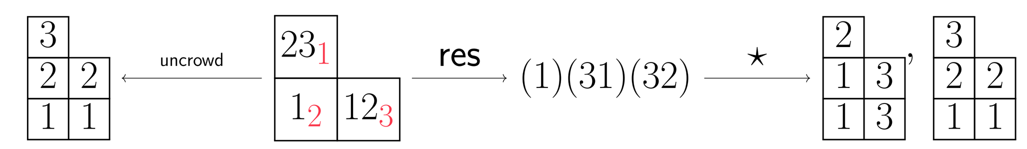
Example



Theorem 4: ⋆-insertion and the uncrowding map

Let $T \in \mathsf{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \mathsf{uncrowd}(T)$, and $(P, Q) = \star \circ \mathsf{res}(T)$. Then $Q = \tilde{P}$.

Example



[1] Jason Bandlow and Jennifer Morse. Combinatorial expansions in K-theoretic bases. *Electron. J. Combin.*, 19(4):Paper 39, 27, 2012.

[2] Anders Skovsted Buch. A Littlewood-Richardson rule for the K-theory of Grassmannians. *Acta Math.*, 189(1):37–78, 2002.

[3] Anders Skovsted Buch, Andrew Kresch, Mark Shimozono, Harry Tamvakis, and Alexander Yong. Stable Grothendieck polynomials and K-theoretic factor sequences. *Math. Ann.*, 340(2):359–382, 2008.

[4] Daniel Bump and Anne Schilling. *Crystal bases.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. Representations and combinatorics.

[5] Sergey Fomin and Curtis Greene. Noncommutative Schur functions and their applications. volume 193, pages 179–200. 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[6] Sergey Fomin and Anatol N. Kirillov. Grothendieck polynomials and the Yang-Baxter equation. In *Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique*, pages 183–189. DIMACS, Piscataway, NJ, 1994.

[7] Alain Lascoux and Marcel-Paul Schützenberger. Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(11):629–633, 1982.

[8] Cara Monical, Oliver Pechenik, and Travis Scrimshaw. Crystal structures for symmetric Grothendieck polynomials. *arXiv preprint arXiv:1807.03294*, 2018.

[9] Jennifer Morse and Anne Schilling. Crystal approach to affine Schubert calculus. *Int. Math. Res. Not. IMRN*, (8):2239–2294, 2016.

[10] Rebecca Patrias and Pavlo Pylyavskyy. Combinatorics of K-theory via a K-theoretic Poirier-Reutenauer bialgebra. *Discrete Math.*, 339(3):1095–1115, 2016.

[11] Vic Reiner, Bridget E. Tenner, and Alexander Yong. Poset edge densities, nearly reduced words, and barely set-valued tableaux. *J. Combin. Theory, Ser. A*, 158:66–125, 2018.