A multiset generalization of (set) partition algebra

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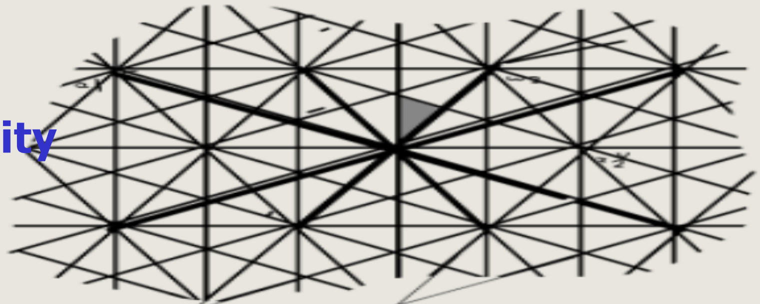
(based on arXiv:1903.10809)

Joint work with Sridhar Narayanan (IMSc Chennai)
and Shraddha Srivastava (Uppsala University)

Plan for the talk

1. Classical Scur--Weyl duality

2.Partition algebra



3.Multiset partition algebra

4. Cellularity and
semi-simplicity

5.Multiset tableaux,
Plethysm and
Restriction problem

Schur--Weyl duality

Schur 1905

'matrix multiplication.'

'place permutation'

$$\boxed{GL_n(\mathbb{C})} \curvearrowright \boxed{(\mathbb{C}^n)^{\otimes k}} \curvearrowleft \boxed{S_k}$$

- Two actions commute.
- $\boxed{\mathbb{C}[S_k]} \twoheadrightarrow \mathrm{End}_{GL_n(\mathbb{C})}\left((\mathbb{C}^n)^{\otimes k}\right)$.

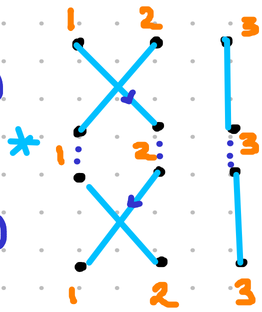
Diagram Algebra

Diagram Algebras

$\mathbb{C}[S_k]$

$\sigma = (12)$

$\tau = (12)$



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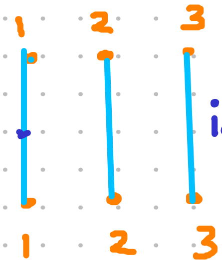
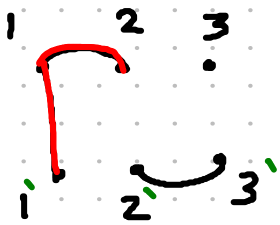


$id = \sigma \circ \tau$

set partition diagram



$= \{\{1, 2, 1'\}, \{3\}, \{2', 3'\}\}$

set partition of

$\{1, 2, 3, 1', 2', 3'\}$

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1905
Schur

$GL_n(\mathbb{C})$

$\cup|$

restriction

S_n

$\boxed{(\mathbb{C}^n)^{\otimes k}}$

$\mathbb{C}[S_k]$

$\cap|$

partition algebra

$P_k(\xi)$

1990

Martin
Jones

$\xi = n$,

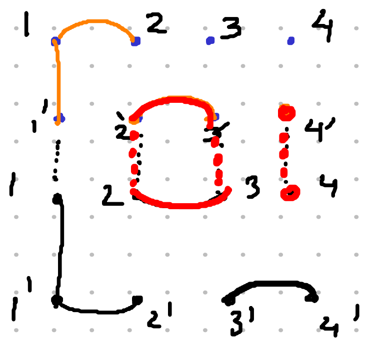
Schur-Weyl duality

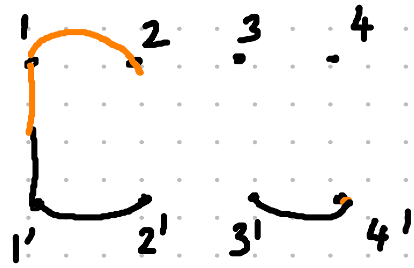
$$P_k(n) \twoheadrightarrow \mathrm{End}_{S_n}\left((\mathbb{C}^n)^{\otimes k}\right)$$

- Isomorphism when $n \geq 2k$.

Partition Algebras

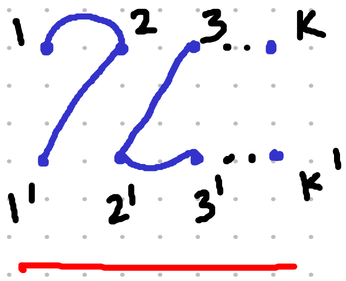
- $P_k(\xi)$ has basis indexed by set partitions of $\{1, 2, \dots, k, 1', 2', \dots, k'\}$.





multiplication in diagram basis

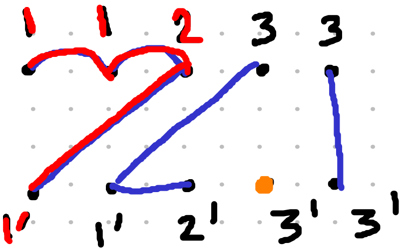
6 **Natural Questions** 4

* 

(set) partition algebra P_k

Q: what happens if some of the vertices are allowed to repeat?

multiset partition

* 

represents $\{\{1,1,2,1'\}, \{3,1',2'\}, \{3'\}, \{3,3'\}\}$

What plays the role of Partition Algebras?

- $P_K(n) \twoheadrightarrow \mathrm{End}_{S_n}\left((\mathbb{C}^n)^{\otimes K}\right)$ $\qquad S_n \curvearrowright (\mathbb{C}^n)^{\otimes K} \curvearrowleft P_K(n)$

- Replace $(\mathbb{C}^n)^{\otimes K}$ by symmetric tensors

- $S_n \curvearrowright \mathrm{Sym}^K \mathbb{C}^n \curvearrowleft ?$

- $? \twoheadrightarrow \mathrm{End}_{S_n}\left(\mathrm{Sym}^K \mathbb{C}^n\right)$

More generaly,

For $\lambda = (\lambda_1, \dots, \lambda_s)$, define $Sym^{\lambda}\mathbb{C}^n := \bigotimes_{i=1}^{s} Sym^{\lambda_i}(\mathbb{C}^n)$.
vector of non-negative
integers

$$\boxed{? \twoheadrightarrow End_{S_n}(Sym^{\lambda}(\mathbb{C}^n))}$$

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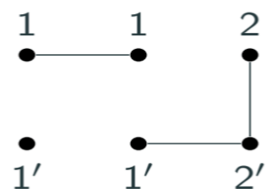
A new diagram algebra based on multiset partitions

of $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, s^{\lambda_s}; 1'^{\lambda_1}, 2'^{\lambda_2}, \dots, s'^{\lambda_s}\}$

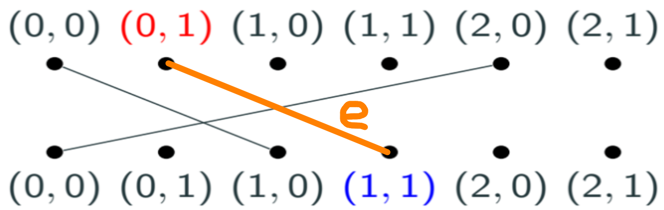
A diagram basis

For $\lambda = (2,1)$, diagram associated with multiset partition $\{\{1'\}, \{2, 1', 2'\}, \{1^2\}\}$ of the multiset $\{1^2, 2, 1'^2, 2'\}$



$(\lambda;\lambda)$ $''$ $wt(\Gamma) = \sum_{e \in E} wt(e)$

$\Gamma =$

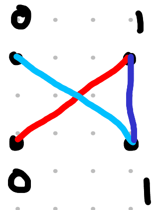


$* \; wt(e) \; '' \; ((0,1);(1,1))$

$\{2, 1', 2'\} = \{1^0, 2^1, 1'^1, 2'^1\}$ corresponds to the edge joining $(0,1)$ and $(1,1)$

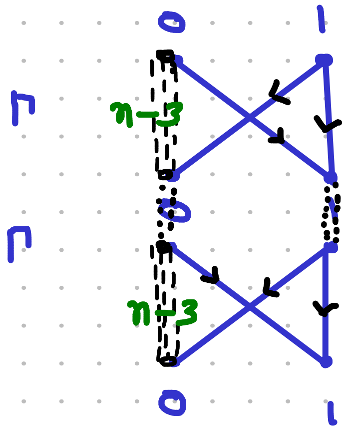
Multiplication of two bipartitite multi-graphs

$\lambda=(2)$, $\Gamma=\{\{1\},\{1'\},\{1,1'\}\}$, $\Gamma=$

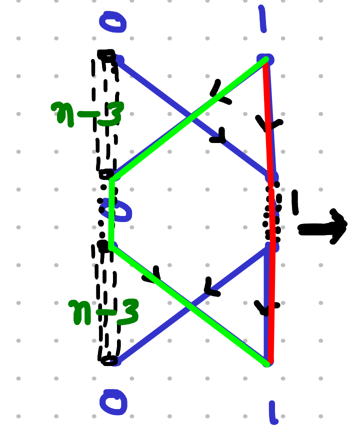


$wt(\Gamma)=(2,2$

To find $\Gamma * \Gamma$



- Add $n-3$ 0-0 edges
- Identify middle vertices
- Look for different paths configuration.



All path configurations

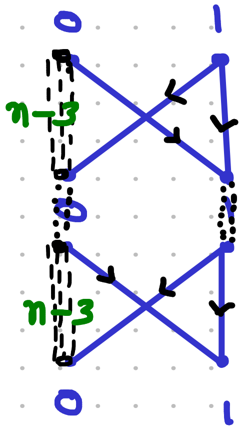
P_1	P_2	P_3	P_4
0 0 ... 0 0 1 1	0 0 0 1 1	0 ... 0 1 1	0 0 0 0 1 1
0 0 0 1 0 1	0 ... 0 1 0 1	0 1 1 0	0 0 0 1 1 0
0 0 0 0 1 1	0 0 0 1 1	0 1 0 1	0 1 1 1 0 0

$\Downarrow$

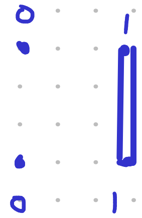
P_1

$$\overset{n-3}{\longleftrightarrow}$$

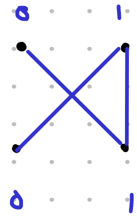
$$\left|\begin{array}{cccccc|cc} 0 & 0 & 0 & \cdots & 0 & & 1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & & 1 & 1 \end{array}\right| = 2\,(n-4) \begin{array}{cccccc} 0 & 0 & \cdots & 0 & 1 & 1 \\ & & \cdot & & & \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{array} = 2(n-4) \left|\begin{array}{cc} 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{array}\right| = 0$$



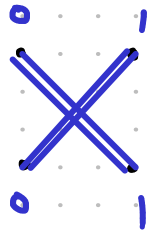
$= 2(n-2)$



$+ (n-2)$



$+ 4$



Define $\mathcal{MP}_\lambda(\xi)$ to be the free module over $\mathbb{C}[\xi]$ with basis $\tilde{\mathcal{B}}_\lambda$.

Theorem (The Multiset partition algebra) (N, P.S)

For $[\Gamma_1], [\Gamma_2]$ in $\tilde{\mathcal{B}}_\lambda$, the linear extension of the following operation

$$[\Gamma_1] * [\Gamma_2] = \sum_{[\Gamma] \in \tilde{\mathcal{B}}_\lambda} \Phi^{[\Gamma]}_{[\Gamma_1][\Gamma_2]}(\xi)[\Gamma],$$

makes $\mathcal{MP}_\lambda(\xi)$ an associative, unital algebra over $\mathbb{C}[\xi]$.

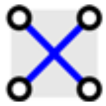
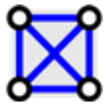
Orbit basis of Partition algebras

$\{d_\pi\}$ —— $\{x_\rho\}$ orbit basis

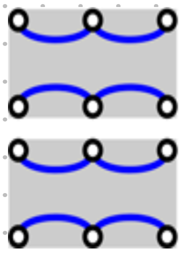
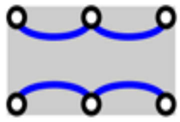
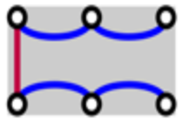
diagram basis

$d_\pi = \sum_{\pi \leq \rho} x_\rho$ ~ Orbit basis

π ρ

 =  +  +  +  + 

* Multiplication in orbit basis :

 $= (n-2)$  $+ (n-1)$  , $\qquad n \geq 2.$

A new Schur--Weyl duality

$\lambda = (\lambda_1, \dots, \lambda_s)$, $|\lambda| = k$

Thm : (N, P, S)

$$\boxed{\mathcal{MP}_\lambda(n) \twoheadrightarrow \mathrm{End}_{S_n}(\mathrm{Sym}^\lambda \mathbb{C}^n)}$$

connection?

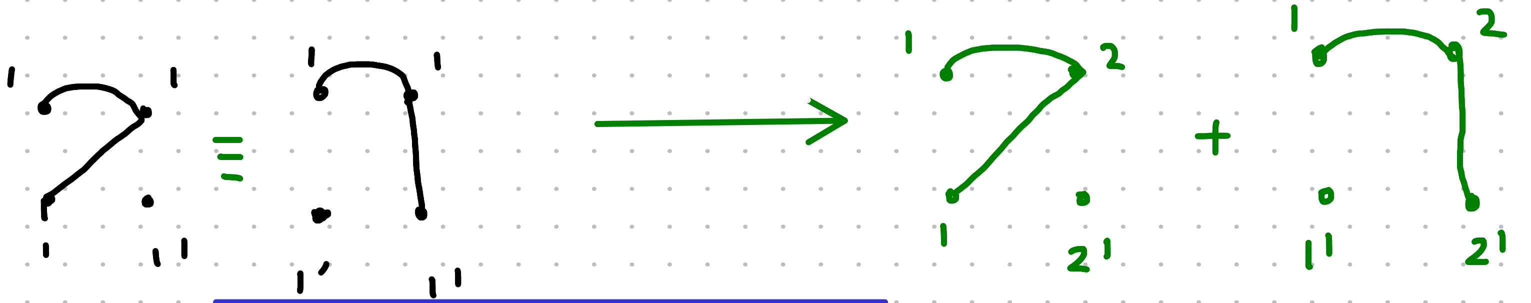
Thm : (V. Jones)

$$\boxed{P_k(n) \twoheadrightarrow \mathrm{End}_{S_n}((\mathbb{C}^n)^{\otimes k})}$$

A bijection turns into an embedding

$$\text{multiset partitions } \Gamma \xrightarrow{\ \phi\ } \text{classes of (set) partitions } \pi$$

$\lambda = (2)$ $\quad \boxed{\{\{1,1,1'\},\{1'\}\} \mapsto \{\{1,2,1'\},\{2'\}\} + \{\{1,2,2'\},\{1'\}\}}$



$$\boxed{\phi([\Gamma]) = \sum_{\Gamma\text{-class}} x_\pi}$$

Bijection $\rightsquigarrow$ Alg. hom.

12 Connection to Partition algebra

Theorem (N,P,S)

There is a canonical embedding

$$\mathcal{MP}_\lambda(\xi) \hookrightarrow \mathcal{P}_k(\xi)$$

which send multiset partitions to certain class of set partitions. Moreover, there exists an idempotent $e \in \mathcal{P}_k(\xi)$

$$\mathcal{MP}_\lambda(\xi) \cong e\mathcal{P}_k(\xi)e.$$

Semisimple and cellular algebra

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Theorem (N, P, S)

$\mathcal{MP}_\lambda(\xi)$ semisimple over $\mathbb{C}$ when ξ is not an integer or ξ is an integer such that $\xi \geq 2k-1$.

- Cellular algebras: Introduced by Graham and Lehrer (Invent. Math 96) motivated by Kazdhan Lusztig's basis of Hecke algebras.
- Example: Partition algebras.[Xi, Compositio Math. 2000]
- **Proposition:** Let A be a cellular algebra with respect to an involution i. Let $e \in A$ be an idempotent such that $i(e) = e$. Then the algebra eAe is also a cellular algebra with respect to the involution i restricted to eAe.

Theorem (N, P, S)

$\mathcal{MP}_\lambda(\xi)$ is cellular over $\mathbb{C}$.

Combinatorial Representation theory

- $(GL_n(\mathbb{C}), S_k)$ - duality

GL$_n$- irr. reps^n.

$$\boxed{(\mathbb{C}^n)^{\otimes k} = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda)\le n}} W_\lambda \otimes V_\lambda}$$

S_k - irr. reps^n

Semi-standard young tableaux

$$\boxed{\dim W_\lambda = SSYT_n(\lambda)}$$

standard young tableaux

$$\boxed{\dim V_\lambda = SYT(\lambda)}$$

Multiset Tableaux

Thm (Orellana-Zabrocki)

$$Sym^{\lambda} \mathbb{C}^n = \bigoplus_{\nu \vdash n} V_\nu^{\oplus m_\nu^\lambda},$$

S_n- mod

m_ν^λ — # semistandard Multiset tableaux of shape $\underline{\nu}$ and content $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, s^{\lambda_s}\}$

— $SSMT(\nu, \lambda)$

(Colmenarejo, Orellana, Saliola, Schlling, Zabrocki)

- Enumerative result:

$$\prod_{i=1}^{s} \binom{n+\lambda_i-1}{\lambda_i} = \sum_{\nu \vdash n} |SYT(\nu)| \times |SSMT(\nu, \{1^{\lambda_1}, \ldots, s^{\lambda_s}\})|$$

- **Representation theory set up (N,P,S)**

$(S_n, MP_\lambda(n))$- $\quad \mathrm{Sym}^\lambda(\mathbb{C}^n) \cong \bigoplus_{\nu \vdash n} V_\nu \otimes M_\nu^\lambda$

Multiset tableaux--Plethysm--Restriction problem

- **Restriction problem**

$$Res_{S_n}^{GL_n(F)} W_\lambda = \bigoplus_{\nu \vdash n} V_\nu^{\oplus r_\nu^\lambda}.$$

$$\boxed{(\text{Littlewood's formulae}) r_\nu^\lambda = \langle s_\lambda, s_\nu[1 + h_1 + h_2 + \cdots] \rangle.}$$

- Schur $s_\nu = \sum_\lambda k_{\nu,\lambda} m_\lambda$ — Gen. func^n. for SSYT.

Generating function for multiset tableaux

Thm: (N,P,S) $\boxed{s_\nu[1+h_1+h_2+\cdots] = \sum_\lambda |SSMT(\nu, \{1^{\lambda_1} \ldots s^{\lambda_s}\})| m_\lambda}$

Thank you for your attention!