Monomial Bases for Combinatorial Hopf Algebras

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based on “Hopf algebras of parking functions and decorated planar trees”,
joint work with Nantel Bergeron, Rafael Gonzalez d’Leon, Shu Xiao Li, Yannic Vargas

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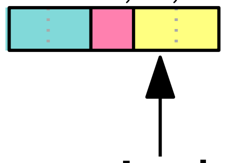
The Monomial basis of QSym

$$M_{2,1,2} = \sum_{i<j<k} x_i^2 x_j^1 x_k^2 = x_1^2 x_2^1 x_3^2 + x_1^2 x_2^1 x_4^2 + \dots x_1^2 x_3^1 x_4^2 + \dots x_2^2 x_4^1 x_7^2 + \dots$$



indexed by compositions

The degree of a composition is the number of squares.

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indexed by compositions

The degree of a composition is the number of squares.

The product (combining of compositions) in the M basis expands positively - it is quasishuffle of blocks:

$$M_{1,2} M_1 = \sum_{i<j} x_i^1 x_j^2 \sum_k x_k^1$$

$$= M_{1,2,1} + M_{1,3} + M_{1,1,2} + M_{2,2} + M_{1,1,2}$$

$i<j<k \quad i<j=k \quad i<k<j \quad i=k<j \quad k<i<j$

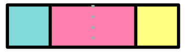
The Monomial basis of QSym

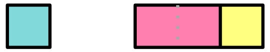
QSym is a Hopf algebra, i.e. it has a coproduct $\Delta : \text{QSym} \to \text{QSym} \otimes \text{QSym}$ (breaking of compositions), compatible with its product.

Given $f(x_1, x_2, \ldots)$, let $f(y_1, y_2, \ldots, z_1, z_2, \ldots) = \sum_i g_i(y_1, y_2, \ldots)h_i(z_1, z_2, \ldots)$.
Let $\Delta(f) = \sum_i g_i \otimes h_i$, and $\Delta_+(f) = \Delta(f) - 1 \otimes f - f \otimes 1$.

The coproduct in the M basis

$$\Delta_+(M_{1,2,1}) = \Delta_+\left(\sum_{i<j<k} x_i^1 x_j^2 x_k^1\right) = \qquad M_1 \otimes M_{2,1} +$$





$$y_i^1 z_j^2 z_k^1$$

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The coproduct in the M basis is deconcatenate between blocks

$$\Delta_+(M_{1,2,1}) = \Delta_+\left(\sum_{i<j<k} x_i^1 x_j^2 x_k^1\right) = M_1 \otimes M_{2,1} + M_{1,2} \otimes M_1$$

$$z_i^1 z_j^2 z_k^1 \qquad y_i^1 z_j^2 z_k^1 \qquad y_i^1 y_j^2 z_k^1 \qquad y_i^1 y_j^2 y_k^1$$

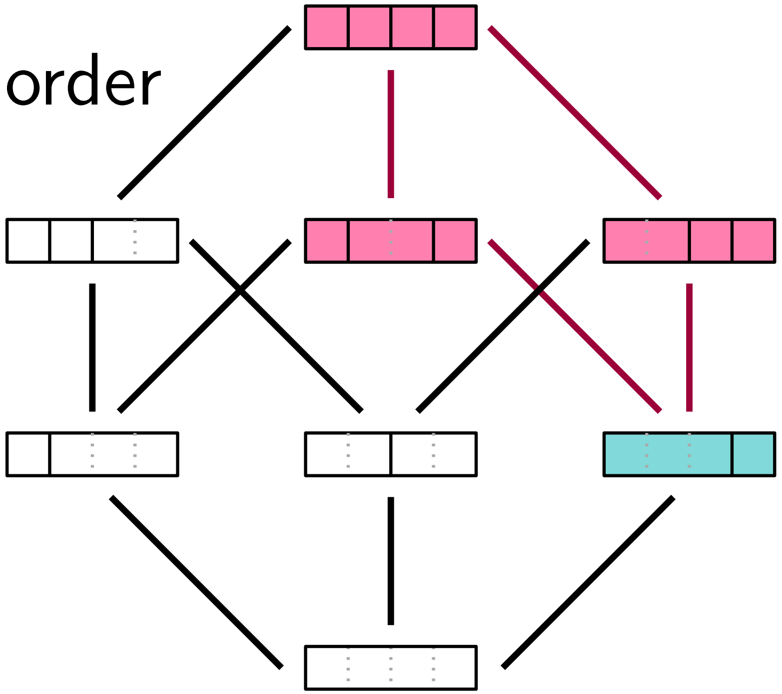
i.e. compositions have a “unique factorisation” and the coproduct deconcatenates the factors – i.e. this coproduct is cofree (i.e. the dual basis in the dual Hopf algebra is free)

Other bases of QSym

M-basis of QSym ⟶ combining compositions by quasishuffle of blocks;
breaking compositions by deconcatenation between blocks.

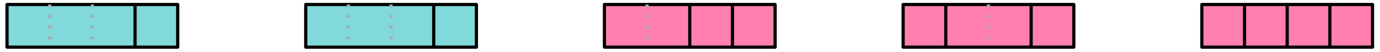
?-basis of QSym ⟶ combining compositions by ???;
breaking compositions by ???.

Fundamental basis: $F_\alpha = \sum_{\beta \geq \alpha} M_\beta$ using the refinement order



e.g.

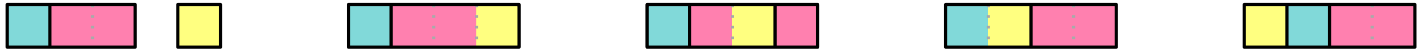
$$F_{3,1} = M_{3,1} + M_{2,1,1} + M_{1,2,1} + M_{1,1,1,1}$$



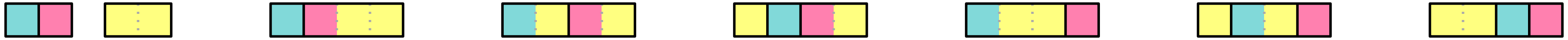
The Fundamental basis of QSym

The product in the F basis is the shuffle of squares:

$$F_{1,2}F_1 = F_{1,3} + F_{1,2,1} + F_{2,2} + F_{1,1,2}$$



$$F_{1,1}F_2 = F_{1,3} + F_{2,2} + F_{1,1,2} + F_{3,1} + F_{1,2,1} + F_{2,1,1}$$



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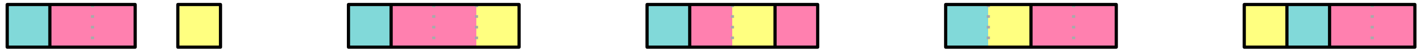
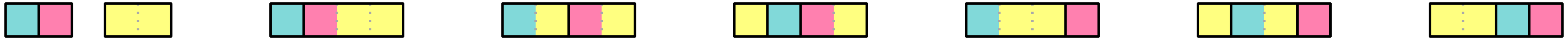


$$F_{1,1}F_2 = F_{1,3} + F_{2,2} + F_{1,1,2} + F_{3,1} + F_{1,2,1} + F_{2,1,1}$$



The coproduct in the F basis is deconcatenate between squares - which produces one term in each degree:

$$\Delta_+(F_{3,1}) = \Delta_+(M_{3,1} + M_{2,1,1} + M_{1,2,1} + M_{1,1,1,1})$$

$$\begin{aligned} &= M_1 \otimes M_{2,1} + M_1 \otimes M_{1,1,1} \\ &\quad + M_2 \otimes M_{1,1} + M_{1,1} \otimes M_{1,1} \\ &\quad + M_3 \otimes M_1 + M_{2,1} \otimes M_1 + M_{1,2} \otimes M_1 + M_{1,1,1} \otimes M_1 \\ &= F_1 \otimes F_{2,1} + F_2 \otimes F_{1,1} + F_3 \otimes F_1 \end{aligned}$$

Other Hopf algebras

- Many other Hopf algebras (on permutations, packed words, binary trees) have a F-like basis:
  - The product is some shuffling of the ground set;
  - The coproduct is deconcatenation of the ground set, producing one term of each degree.

- Often (Bergeron-Zabrocki, Loday-Ronco; Novelli-Thibon, Pilaud-Pons), $\exists$ a poset (weak order, Tamari order) on the underlying objects, and we can define a M-like basis by $F_\alpha = \sum_{\beta \geq \alpha} M_\beta$:
  - The coproduct in the M basis is cofree, given by deconcatenation "between factors" of a unique factorisation - this is proved ad-hoc;
  - The product is ???.

We distill the Aguiar-Sottile approach into axioms: check that shuffling, deconcatenation and the poset interact in these correct ways, and you are guaranteed a M basis with positive product and cofree coproduct.

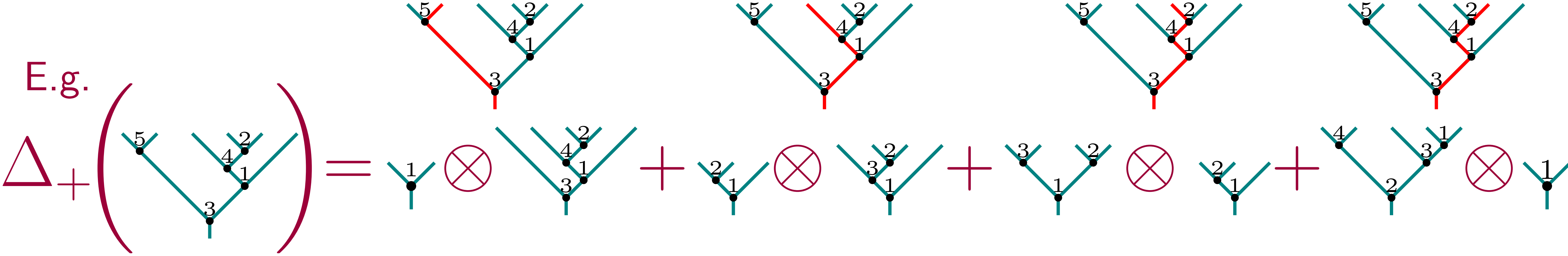
Axioms for coproduct

Example: a new Hopf algebra PSym of parking functions, viewed as binary trees labelled with a permutation satisfying some conditions

$\Delta 1$. Coproduct in fundamental basis is "deconcatenate everywhere"

$$\Delta_+(F_f) = \sum_{i=1}^{\deg f - 1} F_{{}^i f} \otimes F_{f^i}; \quad \deg {}^i f = i$$

E.g.



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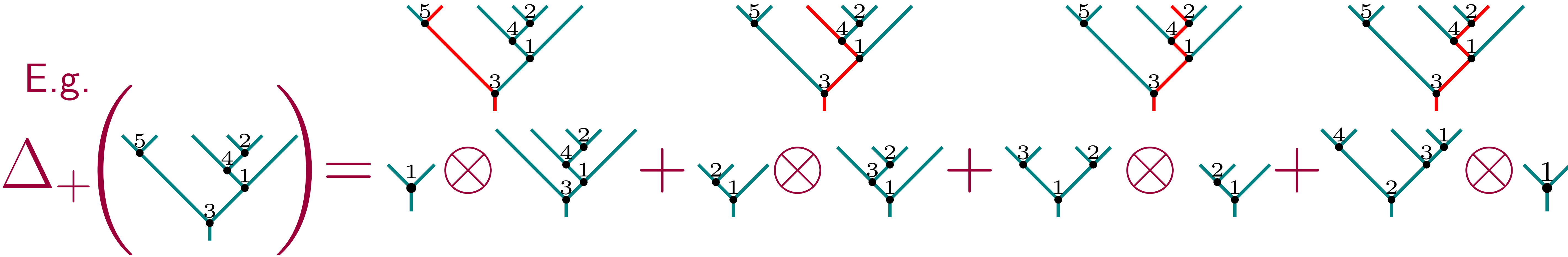


$\Delta 2$. Deconcatenation is order-preserving: if $f \leq f'$, then ${}^i f \leq {}^i f'$ and $f^i \leq f'^i$.

$f \leq f'$ if their trees are comparable in Tamari order and their permutations are comparable in weak order

Axioms for coproduct (cont'd)

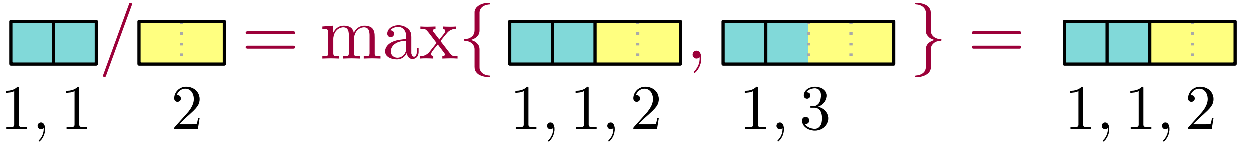
Theorem : If Δ1-3 are satisfied, and we define M basis by $F_f = \sum_{g \geq f} M_g$, then

$$\Delta_+(M_f) = \sum_{i \in \mathrm{GDes}(f)} M_{{}^i f} \otimes M_{f^i}. \quad \text{(deconcatenate "between blocks")}$$

Axioms for coproduct (cont'd)

Δ3. "Maximal concatenation" is well defined:
Given g, h, $\exists$ unique $\max\{f \,|^i f = g, f^i = h\} := g/h$;

e.g.



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$$\Delta_+(M_f) = \sum_{i \in \mathrm{GDes}(f)} M_{{}^i f} \otimes M_{f^i}.$$ (deconcatenate "between blocks")

Axioms for coproduct (cont'd)

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Given g, h, $\exists$ unique $\max\{f \mid {}^i f = g, f^i = h\} := g/h$;

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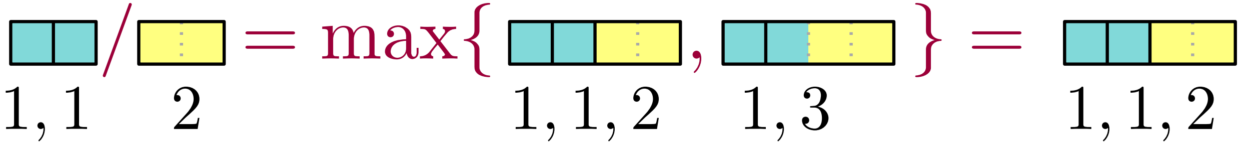

So we can define "between blocks" to be "positions of maximal concatenation", also called global descents $\mathrm{GDes}(f) := \{i \ : f = {}^i f / f^i\}$

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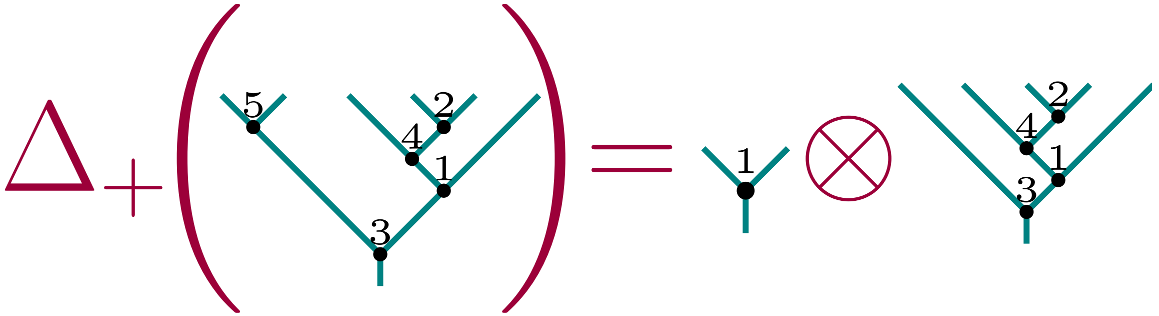
$$\Delta_+(M_f) = \sum_{i \in \mathrm{GDes}(f)} M_{{}^i f} \otimes M_{f^i}. \quad \text{(deconcatenate "between blocks")}$$

E.g. in Monomial Basis:

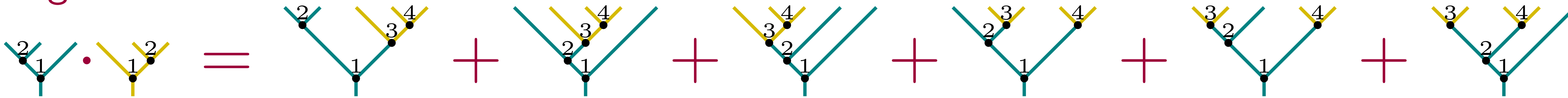


Axioms for product

$m1$. Product in fundamental basis is a sum of shuffles $\zeta(f,g)$:

$$F_f F_g = \sum_{\zeta \in Sh(f,g)} F_{\zeta(f,g)}$$

e.g.



$m2$. Shuffles are order-preserving: if $f \leq f', g \leq g'$, then $\zeta(f,g) \leq \zeta(f',g')$.

$m3$. Shuffles are join-preserving: $\zeta(f_1 \vee f_2, g_1 \vee g_2) \leq \zeta(f_1, g_1) \vee \zeta(f_2, g_2)$.

Theorem : If $m1$-3 are satisfied, then the coefficient of M_h in $M_f M_g$ is the number of shuffles ζ satisfying

- $\zeta(f,g) \leq h$;
- if $f' \geq f, g' \geq g$ satisfy $\zeta(f',g') \leq h$, then $f' = f, g' = g$.

Applications

- To prove that a Hopf algebra is cofree, and have an explicit basis that shows cofreeness, i.e. shows the "unique factorisation" of the combinatorial objects;
- To construct isomorphisms:
  - Vargas's self-duality isomorphism: WQSym $\rightarrow$ WQSym*
    (make a monomial basis for WQSym and for WQSym*, and show their products match)
  - An isomorphism: PSym (our new algebra) $\rightarrow$ PQSym (Novelli-Thibon) ??
    obstacle: known bases on PQSym do not satisfy the axioms, but Hugo Mlodecki has a basis that conjecturally does

Under additional axioms, we can give a cancellation free formula for the antipode in the monomial basis