Schedules and the Delta Conjecture

Jim Haglund, Univ. of Pennsylvania

$$R_n = \mathbb{C}[x_1,\dots,x_n] \big/ \langle \mathbb{C}[x_1,\dots,x_n]_+^{S_n} \rangle$$

$$= \mathbb{C}[x_1,\dots,x_n] \big/ \langle e_1, e_2, \dots, e_n \rangle$$

$\sigma \in S_n$: $\sigma f(x_1,\dots,x_n) = f(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n})$

Thm (Artin)

$$\{ x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid 0 \le \alpha_i < i, \quad 1 \le i \le n \}$$

is a monomial basis for R_n as a $\mathbb{C}$-vector space.

Def'n $\sigma \in S_n$ $\displaystyle\prod_{\substack{i \\ \sigma_i > \sigma_{i+1}}} x_{\sigma_1} \cdots x_{\sigma_i} = GS(\sigma)$

ex. $\sigma = 51423$ $GS(\sigma) = x_5 \, x_5 x_1 x_4$

Thm (Garsia-Stanton)

$\{ GS(\sigma) \mid \sigma \in S_n \}$ forms another monomial basis for R_n

ex. $n=3$

	Artin	GS
123	1	1
132	x_3	$x_1 x_3$
213	x_2	x_2
231	$x_2 x_3$	$x_2 x_3$

312	x_3^2	x_3
321	$x_2 x_3^2$	$x_3 \, x_3 \, x_2$

<u>Def'n</u> Garsia and Haiman pioneered the study of

$$DR_n = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \Big/ \langle \mathbb{C}[x_1,\dots,y_1,\dots]_+^{S_n} \rangle$$

where for $\sigma \in S_n$,

$$\sigma f(x_1,\dots,x_n,y_1,\dots,y_n) = f(x_{\sigma_1},\dots,x_{\sigma_n},y_{\sigma_1},\dots,y_{\sigma_n})$$

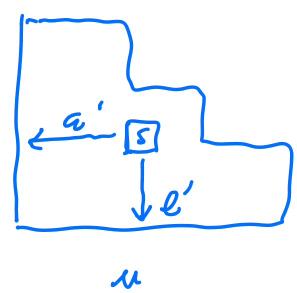
(the "diagonal action". DR_n is called the diagonal coinvariant ring)

<u>Def'n</u> For $f \in \Lambda$, define linear operators Δ'_f and Δ_f:

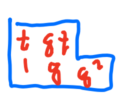
$$\Delta'_f \tilde{H}_\mu(X;q,t) = f[B_\mu(q,t)-1]\, \tilde{H}_\mu(X;q,t)$$

$$\Delta_f \tilde{H}_\mu(X;q,t) = f[B_\mu(q,t)]\, \tilde{H}_\mu(X;q,t)$$

where $B_\mu(q,t) = \sum_{s \in \mu} q^{a'} t^{e'}$



ex. $B_{3,2} = 1 + q + q^2 + t + qt$



<u>Thm</u> (Haiman, 2002. Conjectured by Garsia + Haiman in the early 1990's)

The bigraded Frobenius series $\mathrm{Frob}(DR_n;q,t)$ of DR_n is given by Δ'_{e} $e_n(X)$

ex. DR_2 has basis $\{1, x_1, y_1\}$

$(12)\, x_1 = x_2 = -x_1 \qquad (\text{since } x_1 + x_2 = e_1 = 0)$

$(1)(2)\, x_1 = x_1$

so $\sigma x_1 = \operatorname{sgn}(\sigma)\, x_1$

$$\{1, \quad x_1, \quad y_1\}$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$s_2 \qquad s_{11} q \qquad s_{11} t$$

q: homogeneous x-degree

t: homogeneous y-degree

$$\operatorname{Frob}(DR_2; q,t) = s_2 + s_{11}(q+t)$$

$C_2(q,t)$

$$\operatorname{Hilb}(DR_2; q,t) = 1 + q + t$$

<u>Corollary of Haiman's Thm</u>

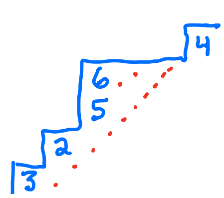
$$\dim(DR_n) = (n+1)^{n-1}$$

$$\langle DR_n, s_{1^n} \rangle = \frac{1}{n+1}\binom{2n}{n} = C_n$$

Thm (Carlsson - Mellit 2015; Shuffle Conjec.)

$$\Delta'_{e_{n-1}} e_n(x) = \sum_{W \in WPF(n)} t^{\text{area}} q^{\text{dinv}} x^W$$

ex.



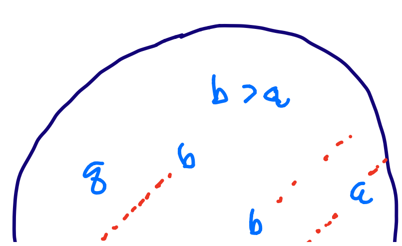
$0 = a_6$

$2 = a_5$

$1 = a_4$

$1 = a_3$

$1 = a_2$

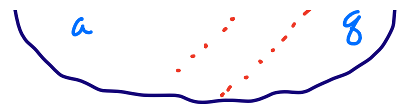


12

$t^5 q^4 x_2^2 x_3 x_4 x_5 x_6$

$\underline{0 = a_1}$

$5 : \text{area}$



dinv pairs $(5,2)$ $(5,3)$ $(5,4)$ $(4,2)$ so $\text{dinv} = 4$

Lemma (H, Loehr; schedule formula)

$$\sum_{P \in PF(n)} q^{\text{dinv}} t^{\text{area}} = \sum_{\sigma \in S_n} t^{\text{maj}(\sigma)} \prod_{i=1}^{n} [w_i(\sigma)]_q$$

ex. $\sigma = 3\ 1\ 4\ 6\ 2\ 5\ 7$

$3 \mid 1\ 4\ 6 \mid 2\ 5\ 7 \mid 0$

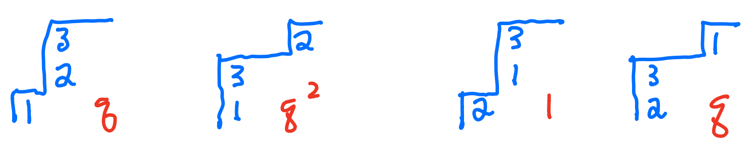
w_i $1\ 2\ 2\ 2\ 3\ 2\ 1$

$w_i :=$ # elements $> \sigma_i$ in same run as σ_i + # elements $< \sigma_i$ in next run to right

pf. say $\sigma = 3\ 1\ 2$

$3 \mid 1\ 2 \mid 0$

w_i $2\ 2\ 1$ $\rightarrow (1+q)(1+q)$



dinv pairs $(2,1)$ $(3,2)$ $(2,1)$ $\emptyset$ $(3,1)$

$q + q^2 + 1 + q = (1+q)(1+q)$

Thm (Carlsson-Oblomkov 2019)

A monomial basis for OR_n is obtained by taking the union of all the X, Y monomials corresponding to the terms in the schedule formula, as described below.

$\sigma = \; 3 | 1\;4\;6 | 2\;5\;7 | 0$

$w_i: \; 1\;2\;2\;2\;3\;2\;1$

$(1+x_1)(1+x_4)(1+x_6)(1+x_2+x_2^2)(1+x_5)(1+x_7)\,y_3^2 y_1 y_4 y_6$

multiply this all out; each resulting monomial is a basis element

GS term for $\sigma = 3146257$

ex. $n=3$

$1\;2\;3|0 \qquad (1+x_1+x_1^2)(1+x_2)$

$w_i: \; 3\;2\;1$

$1\;3|2|0 \qquad y_1 y_3$

$w_i: \; 1\;1\;1$

$2|1\;3|0 \qquad (1+x_1)\,y_2$

$2\;3|1|0 \qquad (1+x_2)\,y_2 y_3$

$3|1\;2|0 \qquad (1+x_3)(1+x_1)\,y_3$

$3|2|1|0 \qquad y_3^2 y_2$

Note: $y_i \equiv 0$ gives Artin basis (under $x_j \to x_{n-j+1}$) for $R_n(x_1, x_2, \dots, x_n)$

$x_i \equiv 0$ gives GS basis for $R_n(y_1, y_2, \dots, y_n)$

Long-term goal: Find a combinatorial model for $\langle \Delta_{s_\beta} s_\gamma(x), s_\mu(x) \rangle$ for all β, γ, μ

The Delta Universe

Delta Conjecture: For $0 \le k \le n-1$,

$$\Delta'_{e_k} e_n(x) = \sum_{w \in WPF(n)} q^{\mathrm{dinv}} x^w t^{\mathrm{area}} \prod_{\substack{2 \le i \le n \\ a_i = a_{i-1}+1}} \left(1 + \frac{z}{t^{a_i}}\right) \Bigg|_{z^{n-1-k}}$$

rise version

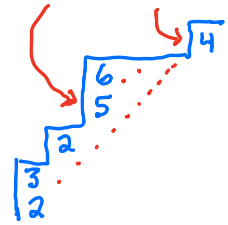
$$\Delta'_{e_k} e_n(x) = \sum_{w \in WPF(n)} t^{\mathrm{area}} x^w q^{\mathrm{dinv}} \prod_{\substack{\text{moveable} \\ \text{valleys}}} \left(1 + \frac{z}{q^{d_i+1}}\right) \Bigg|_{z^{n-1-k}}$$

valley version

Here d_i = # of dinv pairs involving the car in row i and cars in rows above it.

ex.

moveable valleys



a_i	d_i
0	0
2	0
1	1
1	1
1	1
0	1
5	4

$t^5 q^4 x_2^2 x_3 x_4 x_5 x_6$

rise version factor

$t^5 q^4 x_2^2 x_3 x_4 x_5 x_6 \left(1 + \frac{z}{t}\right)\left(1 + \frac{z}{t^2}\right)$

valley version factor

$t^5 q^4 x_2^2 x_3 x_4 x_5 x_6 \left(1 + \frac{z}{q}\right)\left(1 + \frac{z}{q^2}\right)$

dinv pairs (5,2) (5,3) (5,4) (4,2) so dinv = 4

- 2019 D'Adderio, Iraci and Wyngaerd introduce compositional refinement of rise version of Delta Conjecture
- 2020 D'Adderio and Mellit prove this Compositional Delta Conjecture, which implies the rise version of the Delta Conjecture
- 2020 Blasiak, Haiman, Morse, Pun and Seelinger prove the Extended Delta Conjecture.

which also implies the rise version of the Delta Conjecture

Extended Delta

$$\Delta_{h_\ell} \Delta'_{e_k} e_n(X) = \sum_{\substack{W \in WPF(n+\ell) \\ \ell \text{ } 0\text{'s}}} x^W t^{\text{area}} q^{\text{dinv}} \prod_{a_i = a_{i-1}+1} \left(1 + \frac{z}{t^{a_i}}\right) \Bigg|_{z^{n-1-k}}$$

- Valley version is still open (there is not even a proof that the combinatorial side is a symmetric function)

- 2019 Zabrocki conjectures that

$$\sum_{k=0}^{n-1} \Delta'_{e_k} e_n(X) z^{n-1-k} = \text{Frobenius characteristic}$$

of the "super diagonal coinvariant ring"

$$SDR_n(X, Y, \Theta) = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n] / I$$

where I = ideal generated by polynomials without constant term invariant under the diagonal action

$$\sigma f(X, Y, \Theta) = f(x_{\sigma_1}, \dots, x_{\sigma_n}, y_{\sigma_1}, \dots, y_{\sigma_n}, \theta_{\sigma_1}, \dots, \theta_{\sigma_n})$$

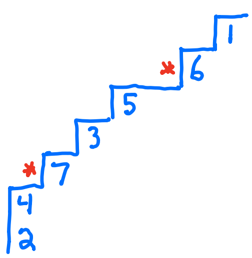
Here $\theta_i \theta_j = -\theta_j \theta_i$ all $1 \le i \le j \le n$ and the θ_i commute with the x and y variables

Schedule Formula for the Valley Version

$$\sum_{\substack{T\in PF(n)\\ n-1-k \text{ marked} \\ \text{valleys}}} t^{\text{area}}\, q^{\text{dinv} - \sum_{*} d_i + 1}$$

$$= \sum_{L\in OP_{n,k+1}} t^{\text{maj}(\tau(L))} \prod_{i=1}^{n} \left[w_i(\tau^*(L)) \right]_q$$

$n-(n-1-k)$
$= k+1$ blocks
$=$ # unmarked cars



$\underline{d_i}$ = # dinv pairs with cars in rows above

0
0
1
2
2
3
1

$q^{9-1-3=5}$

numbers between green |'s are blocks of L

ex. $L = 14 \mid 25 \mid 3\ 10\ 11 \mid 78 \mid 69$

numbers between red |'s are runs of $\tau(L)$ ("minimaj" algorithm)

$\tau(L) = 4 \mid 1\ 2\ 5\ 10\ 11 \mid 3\ 7\ 8 \mid 6\ 9$

Next * each element of τ not leftmost in τ

from its block in L

$$\tau^*(L) = 4 \mid 1^* \; 2 \; 5^* \; 10 \; 11^* \mid 3^* \; 7 \; 8^* \mid 6 \; 9^*$$

(# unmarked = # blocks)

Now compute w_i by rules below

$$\begin{array}{r|c|ccccc|ccc|cc|c} \tau^*(L) = & 4 & 1^* & 2 & 5^* & 10 & 11^* & 3^* & 7 & 8^* & 6 & 9^* & 0 \\ w_i & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & \end{array}$$

- to compute w_i of unmarked element c, count # of unmarked to right in its run, plus # unmarked $< c$ in next run to the right
- to compute w_i of marked element c, count # of unmarked to left in its run, plus # unmarked $> c$ in next run to the left

- removing marked elements from $\tau^*(L)$ gives partial permutation with the same run dividers

$$\tau^*(L) = 4 \mid 1^* \; 2 \; 5^* \; 10 \; 11^* \mid 3^* \; 7 \; 8^* \mid 6 \; 9^*$$
$$4 \mid \; 2 \quad 10 \mid 7 \mid 6$$

- removing $|$'s from $\tau^*(L)$ and then placing a $|$ before each unmarked element recovers L

$$\tau^*(L) = 4 \mid 1^* \; 2 \; 5^* \; 10 \; 11^* \mid 3^* \; 7 \; 8^* \mid 6 \; 9^*$$

$\to \quad 4\ 1^*|2\ 5^*|10\ 11^*\ 3^*|7\ 8^*|6\ 9^*$

$L = 14\,|\,25\,|\,3\ 10\ 11\,|\,78\,|\,69$

<u>Candidate Basis for SDR$_n$</u>

ex. $\tau^*(L) = 4\,|\,1^*\ 2\ 5^*\ 10\ 11^*\,|\,3^*\ 7\ 8^*\,|\,6\ 9^*\,|\,0$

$w_i \quad 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 1$

$\theta_1\theta_5\theta_{11}\theta_3\theta_8\theta_9\ y_4^3 y_1^2 y_2^2 y_5^2 y_{10}^2 y_{11}^2 y_3 y_7 y_8\ (1+x_{11})(1+x_8)$

ex. $\tau^*(L) = 8\,|\,1^*\ 3\ 6^*\ 7\,|\,2\ 4\ 5\,|\,0$

$w_i \quad 2\ 1\ 2\ 2\ 3\ 3\ 2\ 1$

$\theta_1\theta_6\ y_8\ y_8 y_1 y_3 y_6 y_7\ (1+x_8)(1+x_3)(1+x_6)(1+x_7+x_7^2)$
$(1+x_2+x_2^2)(1+x_4)$

weight

all $y_i \equiv t \to t^{maj(\tau)} = t^{minimaj(L)}$ $\quad z$

$x_i \equiv q \to \prod [w_i(\tau^*)]$ $\quad$ Schedule Formula

$z_i \equiv z \to z^{\#\text{ marked valleys}}$ $\quad 1+q$

Candidate Basis works for $n \leq 3$ by direct computation and for $n=4$ using Maple $\quad t$

ex. $n=2$

tri-graded character

$= z\,\Delta'_{e_0}\,e_2(x) + \Delta'_{e_1}\,e_2(x)$

$= s_2 + (q+t+z)\,s_{11}$

L	$\tau(L)$	$\tau^*(L)$	with w_i	basis element
12	12	12^*	$12^* \mid 0$ w_i: $1\ 1$	Θ_2
$1 \mid 2$	12	12	$12 \mid 0$ w_i: $2\ 1$	$1+x_1$
$2 \mid 1$	21	21	$2 \mid 1 \mid 0$ w_i: $1\ 1$	y_1

Proof of Delta Schedule Formula

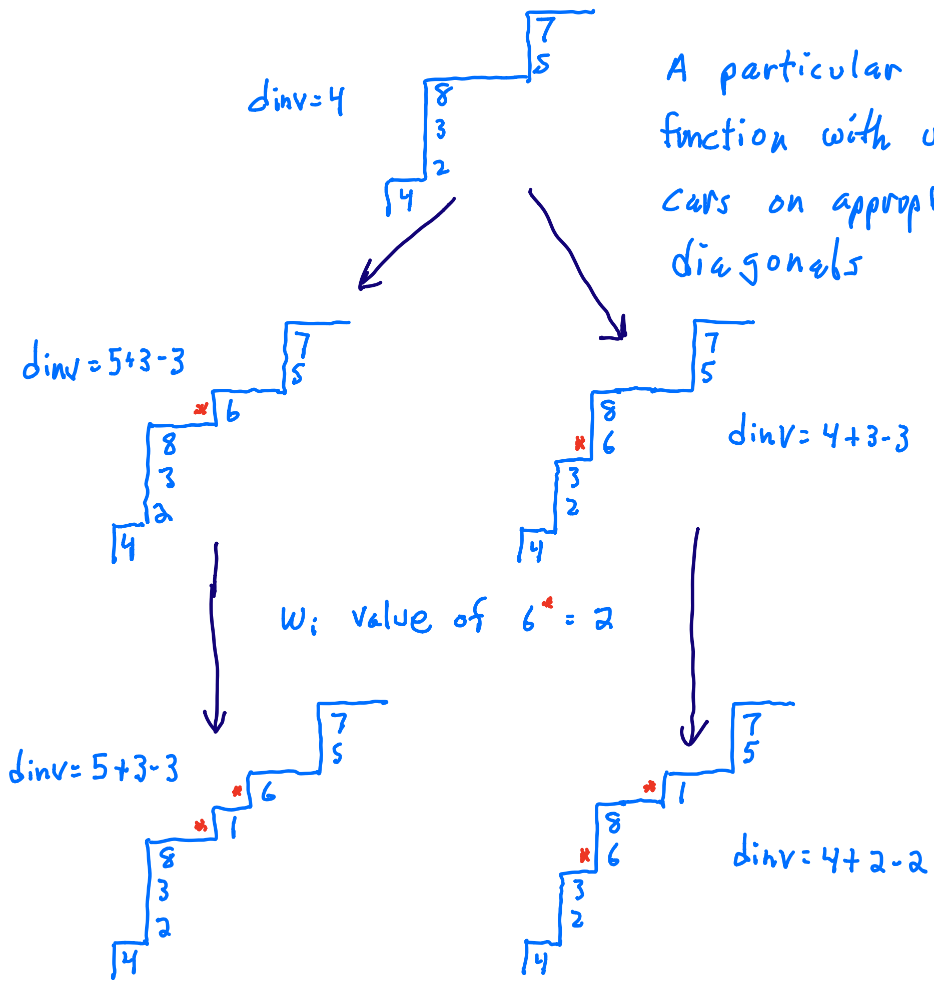
- Given a marked permutation $\tau^*(L)$ corresponding to an ordered set partition L, first insert all unmarked elements, right to left as usual, then insert all marked elements, also right to left. The resulting q-powers factor as $\prod [w_i(\tau^*(L))]_q$. The area of all the resulting marked parking functions is $\mathrm{maj}(\tau(L))$

$= \mathrm{minimaj}(L)$

ex. $\tau^*(L) = 8 \mid 1^* 3 \; 6^* 7 \mid 2 4 5$

$L = 1 8 \mid 3 6 \mid 7 \mid 2 \mid 4 \mid 5$

A particular parking function with unmarked cars on appropriate diagonals



w; value of 1^* = 1