A tale of two polytopes 1: the bipermutahedron

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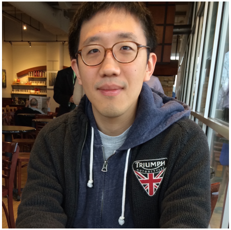
AlCoVe
The Internet, June 15, 2020

Part 1 is joint work with Graham Denham + June Huh (15-20).



Part 2 is joint work with Laura Escobar (20).

The plan

1. What is the bipermutahedral fan?
2. What is the bipermutahedron?
3. Why study them? An origin story.

The permutahedral fan as a moduli space

Permutahedral fan Σ_n in $N_n = \mathbb{R}^n/\mathbb{R}$:

Hyperplane arrangement $x_i = x_j$ for $i \neq j$ in N_n.

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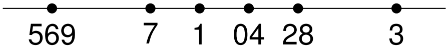

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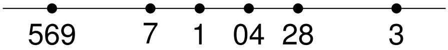

Stratification: relative order

Strata: ordered set partitions 3|28|04|1|7|569

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Bipermutahedral fan $\Sigma_{n,n}$ in $N_n \times N_n$:

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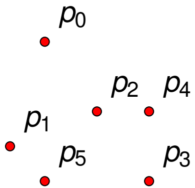

Stratification: • draw lowest supporting -45° diagonal ℓ
• record relative order of x and y projections onto ℓ

Strata: **bisequences** 34|2|035|1|24|0

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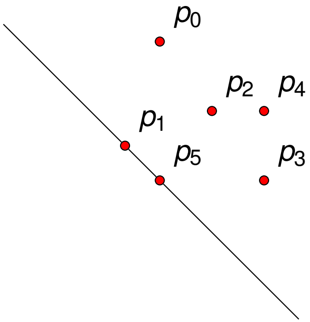

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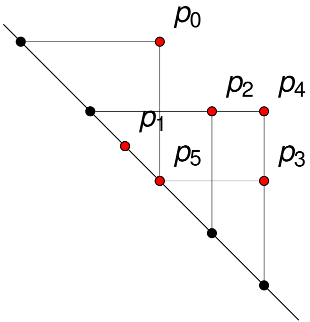

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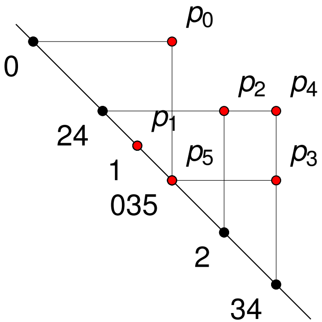

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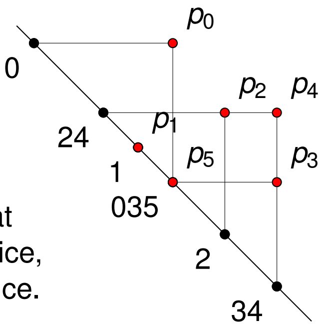


Strata: **bisequences** on $[n]$

Sequences $\mathcal{B} = B_1 | \cdots | B_m$ such that
- each number appears once or twice,
- some number appears exactly once.

Ex: 34|2|035|1|24|0

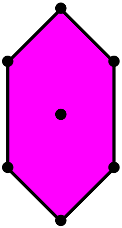
The bipermutahedron

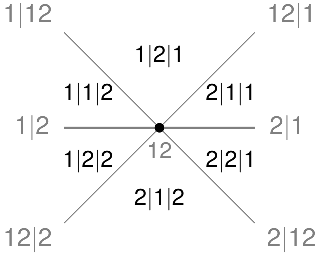
Permutahedral fan Σ_n:

Normal fan of the permutahedron Π_n.

Bipermutahedral fan $\Sigma_{n,n}$:

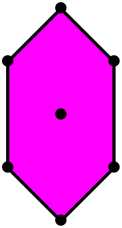
Normal fan of the **bipermutahedron** $\Pi_{n,n}$.

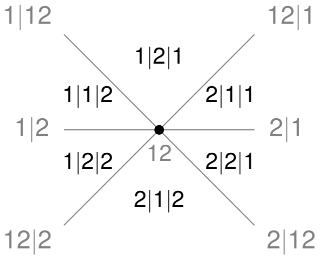




For the precise construction, see [Ardila-Denham-Huh 2020].

Combinatorial structure of the bipermutahedron





- faces: **bisequences** 12|45|4|235
- vertices: **bipermutations** 1|5|4|1|3|4|2|5|3. $(2n)!/2^n$
  (one number appears once, others twice)
- facets: **bisubsets** 1245|235 $3^n - 3$
  ($S, T \neq \emptyset$, not both $[n]$, with $S \cup T = [n]$)

The h-vector of the bipermutahedron

The bipermutahedron is simple; consider its h-polynomial:

$$h_n(x) = h_0(\Pi_{n,n}) + h_1(\Pi_{n,n})x + \cdots + h_{2n-2}(\Pi_{n,n})x^{2n-2}$$

We call it the **biEulerian polynomial.**

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- All roots of the biEulerian polynomial are real and negative.
- The *h*-vector of the bipermutahedron is log-concave.

Origin story: the geometry of matroids

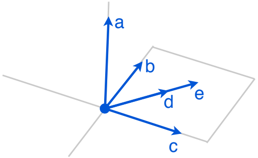
Matroids capture the combinatorial essence of independence.

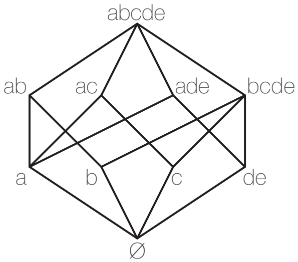
Prototypical example:

E= set of vectors in a vector space V

$\mathcal{F} = \{$spans of subsets of $E\}$

(a poset under $\subseteq$)

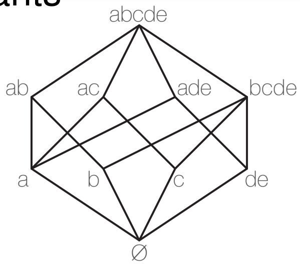




Definition. A matroid $(E, \mathcal{F})$ is
- a set E and
- a collection $\mathcal{F}$ of subsets of E

satisfying some axioms.

Numerical invariants



Given a matroid M,

n = number of elements
r = rank = height of poset
f**-vector** = |coeffs| of $\chi_M(q)$
h**-vector** = |coeffs| of $\chi_M(q+1)$

Ex: $n = 5$ $r = 3$ $f = (1,4,5,2)$ $h = (1,1,0,0)$

Theorem.

1. [Adiprasito-Huh-Katz '15] $f_0, f_1, \ldots, f_r$ **is log-concave.**
Conjectured by Rota 71, Welsh 71, 76, Heron 72, Mason 72.

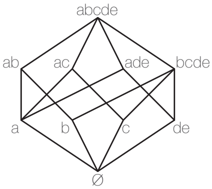
2. [Ardila-Denham-Huh '20] $h_0, h_1, \ldots, h_r$ **is log-concave.**
Conjectured by Brylawski 82, Dawson 83, Hibi 89.

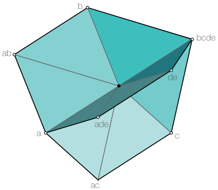
Log-concavity of f-vector: geometry of matroids

[Adiprasito–Huh–Katz 15]

1. Use the **Bergman fan** Σ_M as a geometric model for M.

   $(r-1)$-dim fan in N_n, $\text{Supp}(\Sigma_M) = \mathit{Trop}(M)$ [FA-Klivans 06]





2. Find classes α, β in the Chow ring $A^\bullet(\Sigma_M)$ with

$$\alpha^{r-i}\beta^i = f_i \qquad (1 \leq i \leq r)$$

3. Prove the Hodge-Riemann relations for the fan Σ_M.

   They imply $(\alpha^{r-i}\beta^i : 0 \leq i \leq r)$ is log-concave.

Log-conc of h-vector: Lagrangian geom of matroids

[Ardila–Denham–Huh 20]

1. Use the **conormal fan** $\Sigma_{M,M^\perp}$ as a geometric model for M.
   $(n-2)$-dim fan in $N_n \times N_n$

2. Find classes γ, δ in the Chow ring $A^\bullet(\Sigma_{M,M^\perp})$ with

$$\gamma^i \delta^{n-2-i} = h_{r-i} \qquad (1 \leq i \leq r)$$

3. Prove the Hodge-Riemann relations for the fan $\Sigma_{M,M^\perp}$.
   They imply $(\gamma^i \delta^{n-2-i} : 0 \leq i \leq r)$ is log-concave.

How to define the conormal fan $\Sigma_{M,M^\perp}$?

Varchenko's **critical set varieties** offer hints/requirements:

1. Support($\Sigma_{M,M^\perp}$) "should be" $Trop(M) \times Trop(M^\perp)$.
   Tropical analog of conormal bundle.
2. $\Sigma_{M,M^\perp}$ "should be" simplicial, so the Chow ring is tractable.
   Try: $\Sigma_{M,M^\perp} = \Sigma_M \times \Sigma_{M^\perp}$?

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3. There "should be" classes γ and δ with $\gamma^i \delta^{n-2-i} = h_{r-i}$ $(*)$
   - γ "should be" the pullback of α along
     $$\pi : \Sigma_M \times \Sigma_{M^\perp} \to \Sigma_M, \quad \pi(x,y) = x$$
   - δ "should be" the pullback of α along
     $$\sigma : \Sigma_M \times \Sigma_{M^\perp} \to \Delta_n, \quad \sigma(x,y) = x + y$$
     where Δ_n is the normal fan of the standard simplex.
   - Geometry predicts $(*)$, prove it algebro-combinatorially.

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Idea: Do it simultaneously for all matroids on E.

[FA – Klivans 06]

Permutahedral fan Σ_E resolved this issue for all Bergman fans:

$$\Sigma_M := \text{Trop}(M) \cap \Sigma_E$$

[FA – Denham – Huh 20]

Bipermutahedral fan $\Sigma_{E,E}$ resolves this for all conormal fans:

$$\Sigma_{M,M^\perp} := (\text{Trop}(M) \times \text{Trop}(M^\perp)) \cap \Sigma_{E,E}$$

How to define the bipermutahedral fan?

What do we want?

A **nice** complete fan Σ in $N_n \times N_n$ such that:

a. $\pi_1 : \Sigma \to \Sigma_n, \quad \pi(x,y) = x$ is a map of fans

b. $\pi_2 : \Sigma \to \Sigma_n, \quad \pi(x,y) = y$ is a map of fans

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where Σ_n = braid fan and Δ_n = fan of $\mathbb{P}^{n-1}$.

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This is the **harmonic fan/polytope** $H_{n,n}$ of Laura Escobar's talk.

Good news: It has all these properties + beautiful combinatorics.
Bad news: It is not simplicial. How to compute in its Chow ring?

How to define the bipermutahedral fan?

We want a **nice, polytopal, simplicial** fan with these properties.

Try 1: $H_{n,n}$ = coarsest refinement of $\Sigma_n \times \Sigma_n$ and $\sigma^{-1}(\Delta_n)$.

Try 2: $\Sigma_{n,n}$ = **nice polytopal simplicial** refinement of $H_{n,n}$.

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How do we find it? It's more of an art than a science...
The **bipermutohedral fan** $\Sigma_{n,n}$ is the nicest one we could find.

muchas gracias

(part 1 of) [ADH20]: `https://arxiv.org/abs/2004.13116`

[AE20]: `https://arxiv.org/abs/2006.03078`