Quotients of positroids and lattice path matroids

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Universidad de los Andes

with K. Knauer (≥ 20)

AICoVE I
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BLACK LIVES
MATTER


LAS VIDAS NEGRAS
IMPORTAN

Outline

Matroids and Grassmannians

Positroids and LPMs

Quotients of positroids

Matroids and Grassmannian

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Positroids - Postnikov '05-

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- Grassmann necklace $I_P = (13, 34, 34, 45, 51)$
- Decorated permutation $\pi = 4\underline{2}513$
- and many more combinatorial objects...

Lattice path matroids LPMs

Fix $0 \leq k \leq n$ and let $U, L \in \binom{[n]}{k}$.
The **lattice path matroid** $M[U, L]$ is the matroid on $[n]$ whose bases are those $B \in \binom{[n]}{k}$ such that $U \leq B \leq L$.

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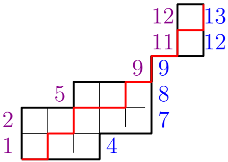

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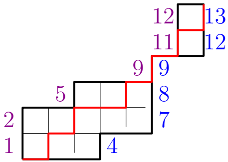

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◦ Every LPM is a positroid.

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

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Problems:

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(2) Is every flag $P_1 \subset \cdots \subset P_n$ of positroids a point in $\mathcal{F}\ell_n^{\geq 0}$?

(3) What can we say about flags $L_1 \subset \cdots \subset L_n$ of LPMs?

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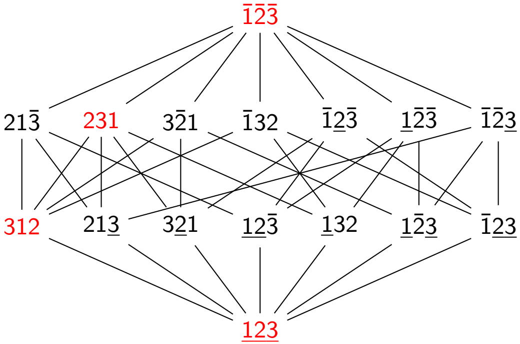
A. Chavez
UC Davis

D. Tamayo
U. Paris-Saclay



Quotients of uniform positroids. arXiv:1912.06873

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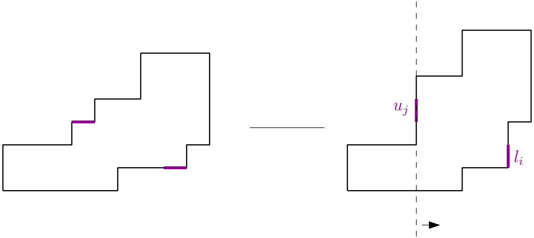
K. Knauer
U. of Barcelona

Theorem [B-Knauer'20]: Let $M = M[U, L]$ be an LPM of rank k on $[n]$ and let $i, j \in [n]$. Then $M[U/j, L/i]$ is a quotient of M if and only if $\max(0, u_j - \ell_i) \leq j - i$.



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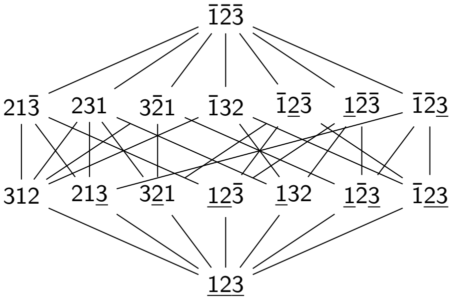

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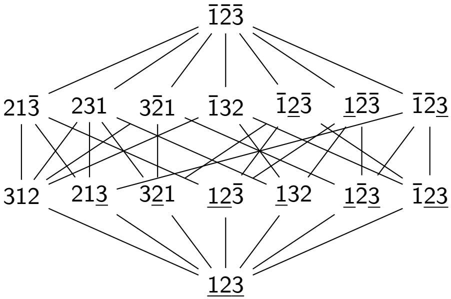

NO.
The flag $3\underline{2}1 \lessdot 231$ is not a point in $\mathcal{F}\ell_n^{\geq 0}$:
$3\underline{2}1 : \{1, 3\}$
$231 : \{12, 23, 13\}$

$$\begin{pmatrix} a & 0 & b \\ c & d & e \end{pmatrix}$$

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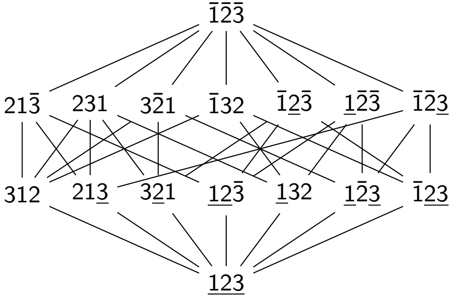

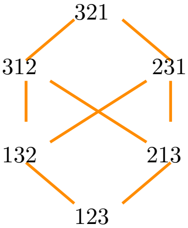

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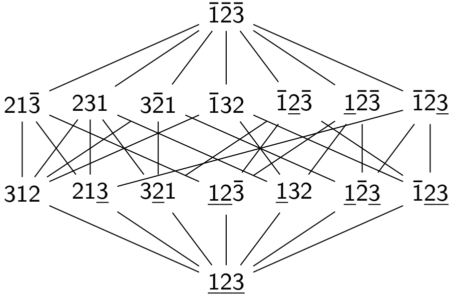

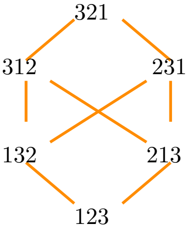

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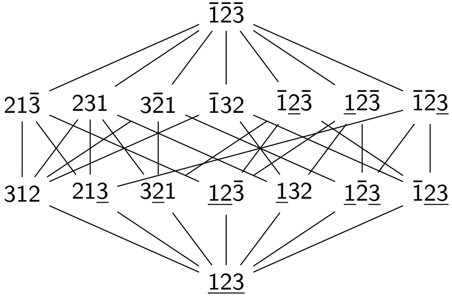

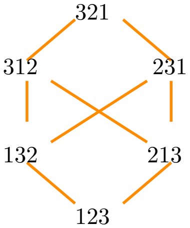

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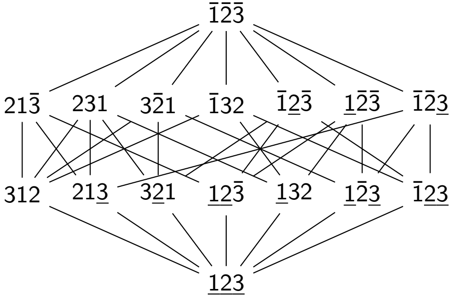

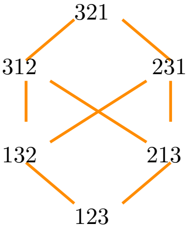

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Theorem: [B-Knauer'20]
Every flag $L_1 \subset \cdots \subset L_n$ of LPMs is an interval in the Bruhat order.

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Proposition: [B-Knauer'20]
If an interval $[u, v]$ in the (right weak) Bruhat order is a hypercube then it is a flag of LPMs.

(3') What faces of the permutahedra are flags of LPMs?

Thank you!