Gröbner Geometry of Schubert Polynomials... Through Ice

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Based on joint work with Zachary Hamaker (Florida) and Oliver Pechenik (Michigan)

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Schubert Varieties

Schubert Classes in Cohomology

The **complete flag variety** is the quotient $\mathcal{F}\ell(n) = \mathrm{GL}(n)/B$.

There's a natural action of B on $\mathcal{F}\ell(n)$ by left multiplication. The orbits Ω_w are called **Schubert cells** and give rise to the **Bruhat decomposition**:

$$\mathcal{F}\ell(n) = \coprod_{w \in \mathcal{S}_n} \Omega_w.$$

The **Schubert varieties** are the closures of these orbits: $\mathfrak{X}_w = \overline{\Omega_w}$.

Schubert varieties give rise to **Schubert classes** σ_w in the cohomology ring $H^*(\mathcal{F}\ell(n))$. The Schubert classes form a linear basis for $H^*(\mathcal{F}\ell(n))$.

The Borel Isomorphism

Thanks to Borel, there is an isomorphism

$$\Phi : H^*(\mathcal{F}\ell(n)) \to \mathbb{Z}[x_1, x_2, \ldots, x_n]/I$$

where I is the ideal generated by the (non-constant) elementary symmetric polynomials.

Question: What is a "good" polynomial representative for the coset $\Phi(\sigma_w)$?

One Answer: Schubert polynomials (Lascoux-Schützenberger 1982).

The Definition of $\mathfrak{S}_w(\mathbf{x})$

Start with the **longest** permutation in $\mathcal{S}_n$

$$w_0 = n\, n-1 \ldots 1 \qquad\qquad \mathfrak{S}_{w_0}(\mathbf{x}) := x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

The rest are defined recursively by **divided difference operators:**

$$\partial_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}} \quad \text{and} \quad \mathfrak{S}_{ws_i}(\mathbf{x}) := \partial_i \mathfrak{S}_w(\mathbf{x}) \ \text{ if } \ w(i) > w(i+1).$$

There are also **double Schubert polynomials**, which are defined by the same operators, with the initial condition

$$\mathfrak{S}_{w_0}(\mathbf{x}; \mathbf{y}) := \prod_{i+j \leq n} (x_i - y_j).$$

But how natural is this choice?

Monomial positivity:

$$\mathfrak{S}_{14523} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2$$

Stability: For $w \in \mathcal{S}_n$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$ where $\iota : \mathcal{S}_n \to \mathcal{S}_{n+1}$ is the natural inclusion.

No cleanup:

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w \quad \Rightarrow \quad \sigma \cup \sigma_v = \sum_w c_{u,v}^w \sigma_w$$

Lift of the Schur polynomials: Each Schur polynomial is itself a Schubert polynomial.

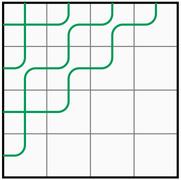
Plus, there's many combinatorial formulas

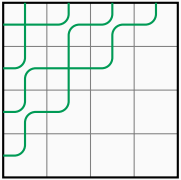
Theorem (Fomin-Kirillov 1996 / Bergeron-Billey 1993)

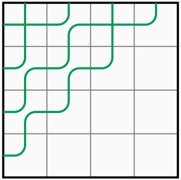
The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ *is the weighted sum*

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{\mathcal{P}\in \mathrm{Pipes}(w)} \mathtt{wt}(\mathcal{P}).$$

Example:







$$\mathfrak{S}_{2143} = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3).$$

Yes, but what about geometric naturality?

Degeneracy Loci: Fulton (1992) expressed Chern class formulas for degeneracy loci of maps of flagged vector bundles in terms of double Schubert polynomials.

Gröbner Geometry: Knutson and Miller (2005) used a specific geometric setup and explicit Gröbner degenerations to uniquely identify Schubert polynomials as the “right” representatives for Schubert classes.

Gröbner Geometry

Matrix Schubert Varieties

There's a natural projection

$$\pi : \mathrm{GL}(n) \to \mathcal{F}\ell(n)$$

and inclusion

$$\iota : \mathrm{GL}(n) \to \mathrm{Mat}(n).$$

Using these, Fulton (1982) defined the **matrix Schubert variety**:

$$X_w := \overline{\iota\,(\pi^{-1}(\mathfrak{X}_w))}.$$

Fulton also introduced the **Schubert determinantal ideal** I_w, and gave explicit generators. Fulton showed I_w is prime and $X_w = V(I_w)$.

Roughly, $X_w \subseteq \mathrm{Mat}(n)$ is defined by rank conditions on maximal northwest submatrices.

Example: For $w = 2143$, the matrix Schubert variety X_w looks like this:

$$\{(m_{ij}) \in \mathrm{Mat}(4) : \mathrm{rk}(m_{11}) = 0 \quad \text{and} \quad \mathrm{rk}\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \leq 2\}.$$

It is cut out by the Schubert determinantal ideal:

$$I_w = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle.$$

Multidegrees

The group $\mathrm{T} \times \mathrm{T}$ acts on the space of $n \times n$ matrices $\mathrm{Mat}(n)$ by

$$(t, u) \cdot M := tMu^{-1}$$

and endows its coordinate ring with a $\mathbb{Z}^{2n}$ grading.

The multidegree $\mathcal{C}(\mathcal{M}; \mathbf{x}; \mathbf{y})$ is a function from $\mathbb{Z}^{2n}$ graded modules $\mathcal{M}$ to polynomials in $\mathbb{Z}[x_1, \dots, x_n; y_1, \dots, y_n]$.

Whenever a subvariety $X \subseteq \mathrm{Mat}(n)$ is **stable** under the action of $\mathrm{T} \times \mathrm{T}$, its coordinate ring is a $\mathbb{Z}^{2n}$ graded module. In this case, we write $\mathcal{C}(X; \mathbf{x}; \mathbf{y})$.

Theorem (Knutson-Miller 2005)

$$\mathcal{C}(X_w; \mathbf{x}; \mathbf{y}) = \mathfrak{S}_w(\mathbf{x}; \mathbf{y}).$$

Computing $\mathcal{C}(X;\mathbf{x};\mathbf{y})$

Let L_D be the coordinate subspace defined by setting the coordinate $z_{ij} = 0$ whenever $(i,j) \in D$. By **normalization**

$$\mathcal{C}(L_D;\mathbf{x};\mathbf{y}) = \prod_{(i,j)\in D} (x_i - y_j).$$

By **additivity**, if $X = \bigcup_{i=1}^{n} X_i$, is a (possibly scheme theoretic) union then

$$\mathcal{C}(X;\mathbf{x};\mathbf{y}) = \sum_{i\in I} \mathrm{mult}_{X_i}(X) C(X_i;\mathbf{x};\mathbf{y}),$$

where the sum is over X_i so that $\texttt{codim}(X_i) = \texttt{codim}(X)$.

Finally, multidegrees are preserved by "nice enough" **degenerations**.

Gröbner Degenerations

Let $Z = (z_{ij})$ be a matrix of generic variables. Fix a monomial term order $\prec$ on $\mathbb{C}[Z]$. Write $\texttt{init}_\prec(f)$ for the lead term of f.

The **initial ideal** of I is $\texttt{init}_\prec(I) = \langle \texttt{init}_\prec(f) : f \in I \rangle$.

If $G \subseteq I$ and $\texttt{init}_\prec(I) = \langle \texttt{init}_\prec(g) : g \in G \rangle$ then G is a Gröbner basis for I.

It's a fact that finite Gröbner bases always exist and Buchberger's algorithm gives an explict way to find them.

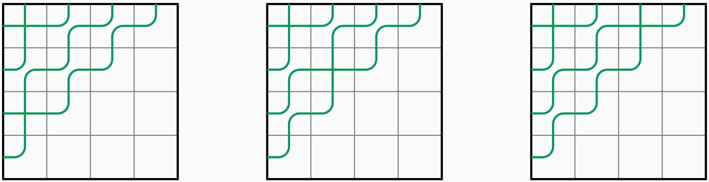
A Toy Example

Take the Schubert determinantal ideal

$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

and degenerate with respect to an **antidiagonal** term order.

$$\texttt{init}_{\prec_a}(I_{2143}) = \langle z_{11}, z_{13}z_{22}z_{31} \rangle = \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle.$$



Pipe dreams are natural

Theorem (Knutson-Miller 2005)

1. *With respect to any antidiagonal term order, Fulton's generators for I_w are a Gröbner basis.*
2. *The initial scheme of X_w with respect to an antidiagonal term order is a union of coordinate subspaces indexed by* $\mathsf{Pipes}(w)$.

$$\mathtt{init}_{\prec_a}(X_w) = \bigcup_{\mathcal{P} \in \mathsf{Pipes}(w)} L_{C(\mathcal{P})}.$$

In particular,

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{\mathcal{P} \in \mathsf{Pipes}(w)} \mathcal{C}(L_{C(\mathcal{P})};\mathbf{x};\mathbf{y}) = \sum_{\mathcal{P} \in \mathsf{Pipes}(w)} \mathtt{wt}(\mathcal{P}).$$

What about diagonal term orders?

Knutson-Miller-Yong (2005) showed Fulton's generators are Gröbner for a diagonal term order if and only if w is vexillary (2143 avoiding).

KMY also showed how to label the components of degenerations of vexillary matrix Schubert varieties with flagged tableaux and diagonal pipe dreams.

Outside of the vexillary setting, limits might fail to be reduced. This was discouraging.

Bumpless Pipe Dreams

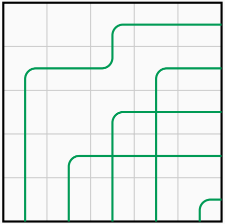
Bumpless Pipe Dreams

A **bumpless pipe dream** is a tiling of the $n \times n$ grid with the six tiles pictured above so that there are n pipes which

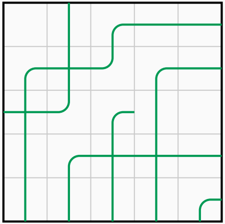
1. start at the right edge of the grid,
2. end at the bottom edge, and
3. pairwise cross at most one time.

The **diagram** of a BPD, denoted $D(\mathcal{P})$, is the set of blank tiles.

Bumpless Pipe Dreams

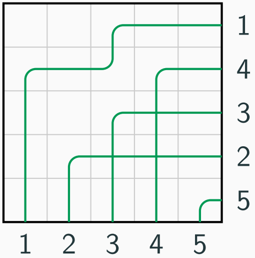


Example



Not an Example

The Permutation of a Bumpless Pipe Dream



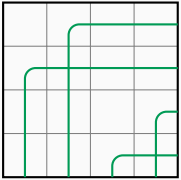
Write BPipes(w) for the set of bumpless pipe dreams which trace out the permutation w.

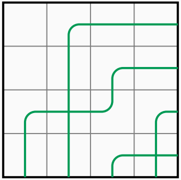
Theorem (Lam-Lee-Shimozono 2018)

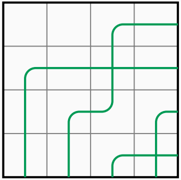
The double Schubert polynomial $\mathfrak{S}_w(\mathbf{x};\mathbf{y})$ *is the weighted sum*

$$\mathfrak{S}_w(\mathbf{x};\mathbf{y}) = \sum_{\mathcal{P}\in \mathsf{BPipes}(w)} \mathtt{wt}(\mathcal{P}).$$

$$\mathfrak{S}_{2143} = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2).$$

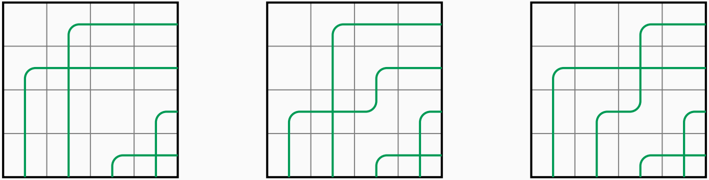




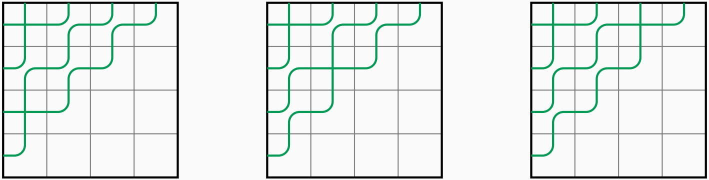


Example: $w = 2143$

$$\mathfrak{S}_w = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2).$$



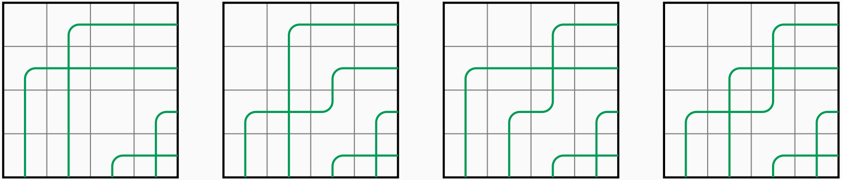
$$\mathfrak{S}_w = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3).$$



Lascoux's ASM Formula

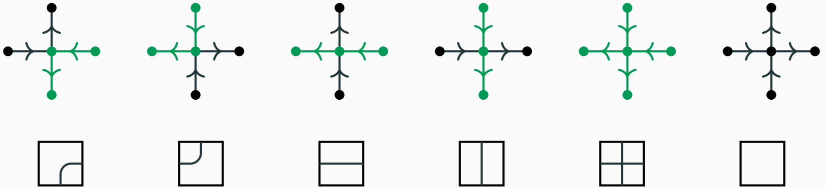
Lascoux (2002) gave a formula for double Grothendieck polynomials as a weighted sum over alternating sign matrices.

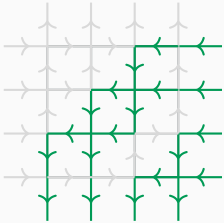
You can reformulate Lascoux's work in terms of (possibly non-reduced) bumpless pipe dreams (W- 2020).

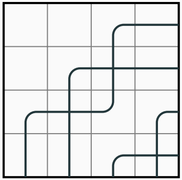


Using this perspective, you can recover the formula of LLS.

Why Ice?



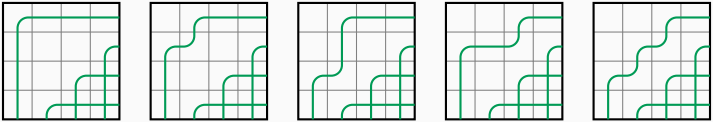




From vexillary BPD to tableaux

Theorem (W- 2020)

There's a bijection from vexillary BPDs to flagged tableaux.



2	2
3	

1	2
3	

1	2
2	

1	1
3	

1	1
2	

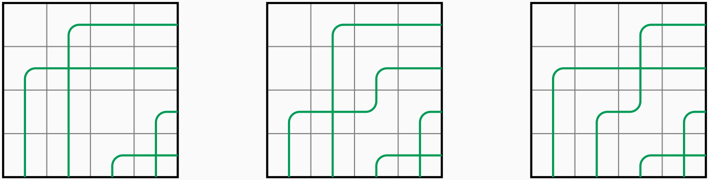
Is it possible BPDs are the "right" combinatorial object to describe diagonal Gröbner degenerations?

Take the Schubert determinantal ideal

$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

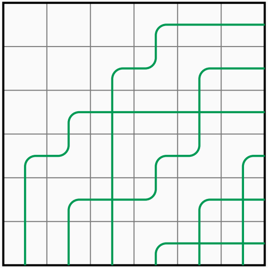
and degenerate with respect to a **diagonal** term order.

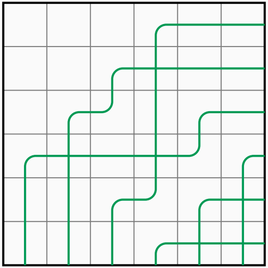
$$\texttt{init}_{\prec_d}(I_{2143}) = \langle z_{11}, z_{12}z_{21}z_{33} \rangle = \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle.$$



Permutations with Duplicate BPD Diagrams

Write Dup(n) for the set of permutations in $\mathcal{S}_n$ which have multiple BPDs with the same diagram.





$$\mathrm{Dup}(6) = \{214365, 321654\}.$$

A Conjecture

Conjecture (Hamaker-Pechenik-W- 2020)

For all (set theoretic) components of $\texttt{init}_{\prec_d}(X_w)$,

$$\mathrm{mult}_{L_D}(\texttt{init}_{\prec_d}(X_w)) = \#\{\mathcal{P} \in \mathsf{BPipes}(w) : D(\mathcal{P}) = D\}.$$

This would mean bumpless pipe dreams label irreducible components of of the diagonal Gröbner degeneration of X_w with the correct multiplicity.

Different Generators

Take the "obvious" defining equations for I_w. Some of them may be single variables z_{ij}. Throw away all other terms that contain these variables. These are the **CDG generators** (Conca-De Negri-Gorla 2015).

Fulton Generators:

$$I_{2143} = \left\langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

CDG Generators:

$$I_{2143} = \langle z_{11}, -z_{12}z_{21}z_{33} + z_{12}z_{23}z_{31} + z_{13}z_{21}z_{32} - z_{13}z_{22}z_{31} \rangle$$

We call w **CDG** if its CDG generators are a diagonal Gröbner basis for I_w.

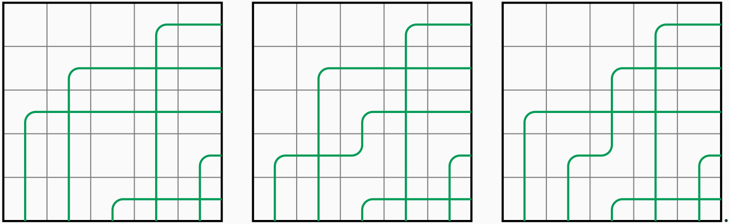
Another Class of Permutations

A permutation is **predominant** if its Lehmer code is of the form $\lambda 0^m \ell$ for some partition λ and $m, \ell \in \mathbb{N}$.

Theorem (Hamaker-Pechenik-W- 2020)

If w is predominant, then $\mathtt{in}_{\prec_d}(X_w)$ is reduced and CDG and the main conjecture holds for w.

$w = 42153$:

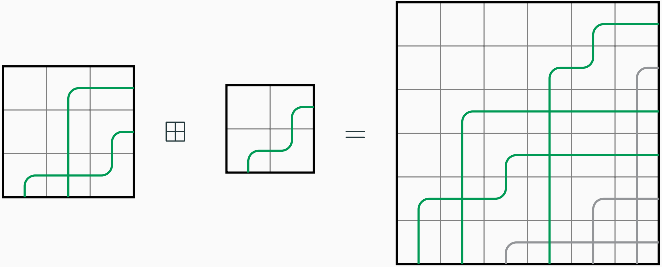


A block sum trick

Let $u = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and w be the permutation associated to the partial permutation $u \boxplus v$. The permutation matrix for w is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Block sums of BPDs



Lemma

If $w = u \boxplus v$*, then there is a bijection from* $\mathrm{BPipes}(u) \times \mathrm{BPipes}(v)$ *to* $\mathrm{BPipes}(w)$.

Main Theorem

We say a permutation is **banner** if it is a block sum of predominant, copredominant, and vexillary partial permutations.

Theorem (Hamaker-Pechenik-W- 2020)

Let w be a banner permutation. Then

1. *w is CDG, and*
2. *$\mathtt{init}_{\prec_d}(I_w)$ is radical; in particular*

$$\mathtt{init}_{\prec_d}(X_w) = \bigcup_{\mathcal{P} \in \mathrm{BPipes}(w)} L_{D(\mathcal{P})}.$$

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Questions?