An RSK correspondence in type $\widetilde{A}_n$

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AlCoVE

http://www-personal.umich.edu/~pechenik/thomasslides.pdf
www.lacim.uqam.ca/~hugh/AlCoVE/slides.pdf

Outline

1. RSK for A_n
2. RSK for $\widetilde{A}_n$
3. Supplementary material: Quivers

Robinson–Schensted–Knuth correspondence

Let ν be a partition (which we draw in English notation).

We think of ν as defining a poset structure on its boxes, with the top at the top/left.

Fill(ν) consists of arbitrary maps from the boxes of ν to $\mathbb{N}$.

RPP(ν) consists of maps from the boxes of ν to $\mathbb{N}$ which are order-reversing (so they weakly increase down and to the right).

We are going to view RSK as a bijection from Fill(ν) to RPP(ν).

	2	
1	1	

$\xrightarrow{\text{RSK}}$

1	2	2
1	3	

RSK II

Classical RSK is obtained by taking ν to be a square partition, and iterpreting the output RPP as a pair of Gelfand–Tsetlin patterns.

We will give two definitions of RSK, one in terms of GK-invariants, and the other in terms of toggles (both of which I will recall).

The main theorem (going back to Kirillov-Berenstein) of this part will be that they define the same map, and that it is a bijection.

Greene-Kleitman invariants

Given a finite poset P and a map $f : P \to \mathbb{N}$, define

$$\mu_1(f) = \max_{\substack{r \in \mathbb{N}, \\ x_1 < x_2 < \cdots < x_r \subseteq P}} \sum_i f(x_i)$$

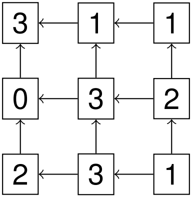
Similarly, define

$$\mu_m(f) = \max_{\substack{r_1, \dots, r_m \in \mathbb{N} \\ x_{i1} < x_{i1} < \cdots < x_{ir_i} \text{ for } 1 \leq i \leq m \\ x_{ij} \neq x_{i'j'} \text{ for } (i,j) \neq (i',j')}} \sum_{i,j} f(x_{ij})$$

Then define $\lambda_m(f) = \mu_m(f) - \mu_{m-1}(f)$.
We have $\lambda_1(f) \geq \lambda_2(f) \geq \dots$.

Example of GK invariants

Consider this poset corresponding to the partition (3,3,3), with f the given labelling.



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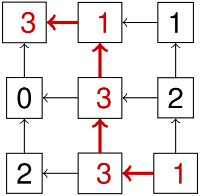

$\mu_1(f) = 3 + 1 + 3 + 3 + 1 = 11.$

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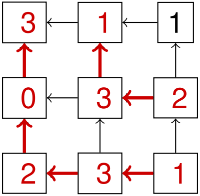

$\mu_1(f) = 11,\ \mu_2(f) = 3 + 1 + 0 + 3 + 2 + 3 + 2 + 1 = 15.$

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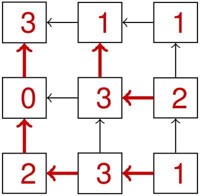

$\mu_1(f) = 11, \ \mu_2(f) = 15$
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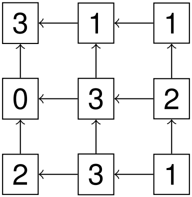

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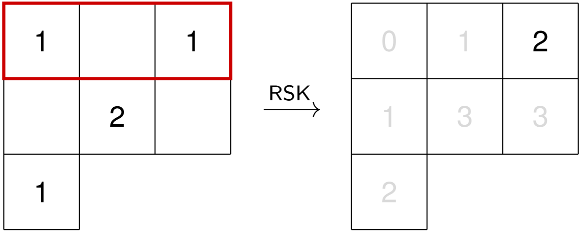
$\lambda(f) = (11, 4, 1).$

Definition of RSK in terms of GK-invariants

We want to define the RPP corresponding to $f \in \text{Fill}(\nu)$.

For each box in the bottom/right border of ν, calculate the GK invariants of the part of the filling above that box.

Insert the parts into the corresponding diagonal of the RPP.



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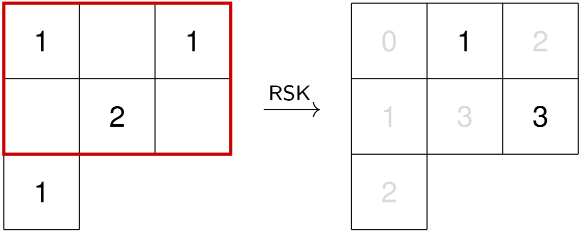

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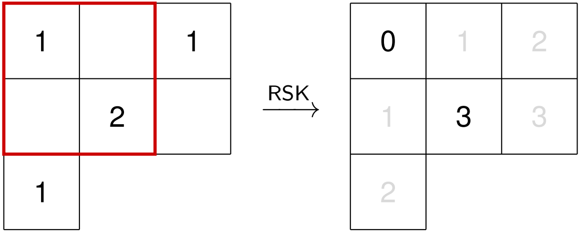

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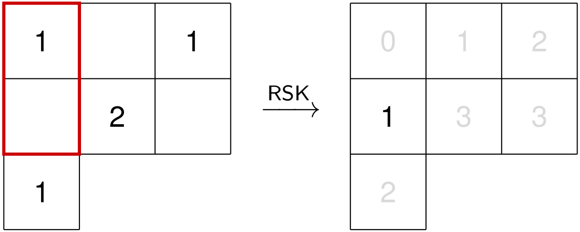

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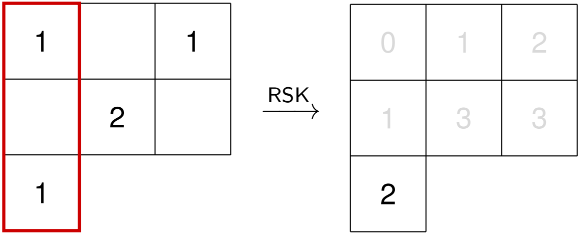

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Insert the parts into the corresponding diagonal of the RPP.

1		1
	2	
1		

$\xrightarrow{\text{RSK}}$

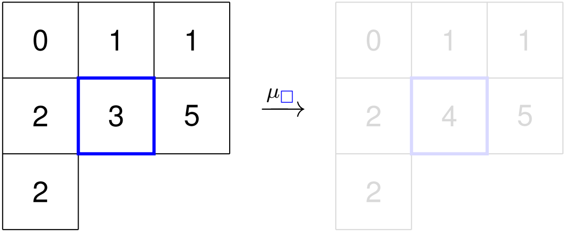
0	1	2
1	3	3
2		

Toggling an RPP at a box

Given an RPP, and a specified box x, there is an operation called the (piecewise linear) toggle at that box, which we denote μ_x.
It replaces $f(x)$ by

$$\max_{y \gtrdot x} f(y) + \min_{z \lessdot x} f(z) - f(x).$$

For example, we can toggle at the box marked in blue:



The new entry is 4 since $4 = \max(1, 2) + \min(5) - 3$.

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0	1	1
2	**3**	5
2		

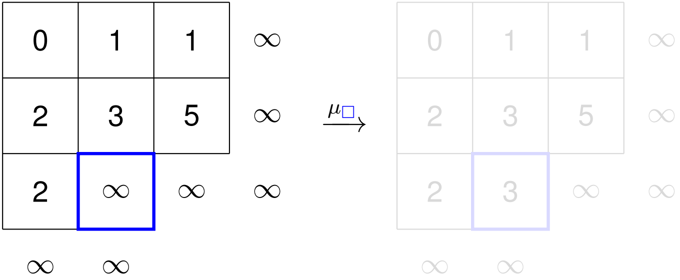
$\xrightarrow{\mu_\square}$

0	1	1
2	**4**	5
2		

The new entry is 4 since $4 = \max(1,2) + \min(5) - 3$.

Border cases: ∞ below

Entries below the RPP are filled with ∞.
Toggling at a box immediately below the RPP replaces the ∞ there by the value of the larger of its two neighbours.



This is correctly calculated by the formula

$$f(\square) = \min(\infty, \infty) + \max(2, 3) - \infty$$

if we take $\infty - \infty = 0$.

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0	1	1	∞
2	3	5	∞
2	∞	∞	∞
∞	∞		

$\xrightarrow{\mu_\square}$

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2	3	5	∞
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∞	∞		

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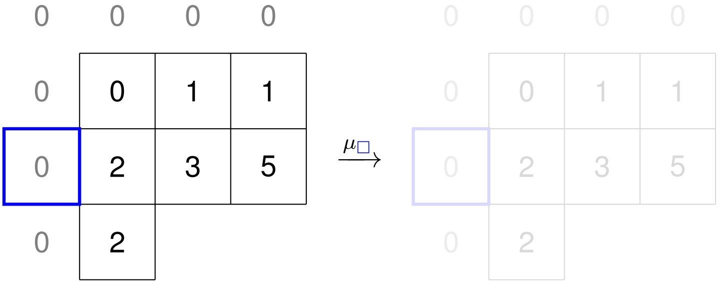
We think of the other entries outside the RPP as being filled with 0. Therefore, toggling at a square above the RPP does nothing.



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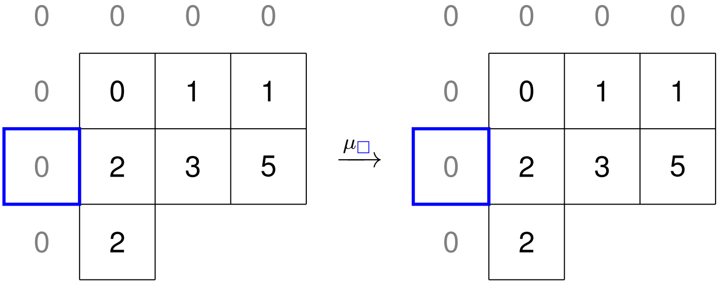

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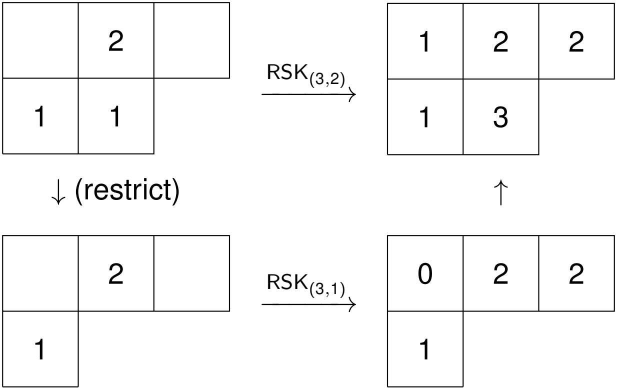
Definition of RSK in terms of toggles: induction step

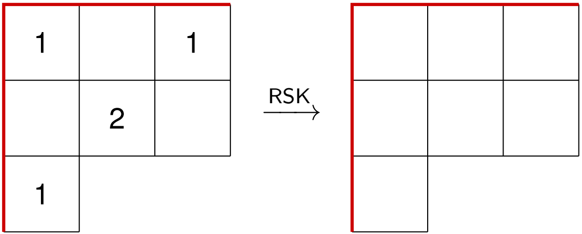
Suppose RSK is defined for ν. We will define it for $\nu' \gtrdot \nu$.



The step corresponding to the ↑: toggle the diagonal of ν'/ν, then add $f(\nu'/\nu)$ to ν'/ν.

Inductive definition of RSK in terms of toggles

Start with the trivial map from fillings of the empty partition to RPPs of the empty partition, and then build up RSK using the previous prescription.



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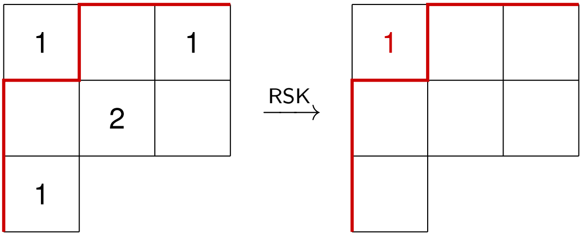

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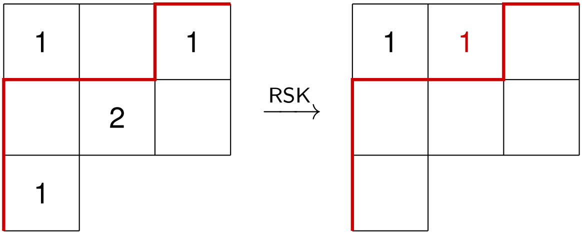

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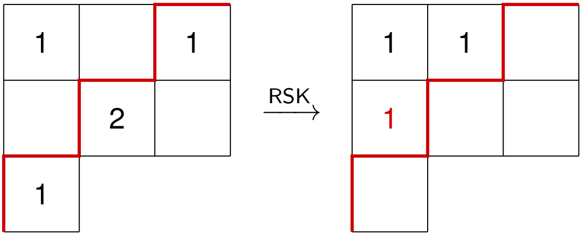

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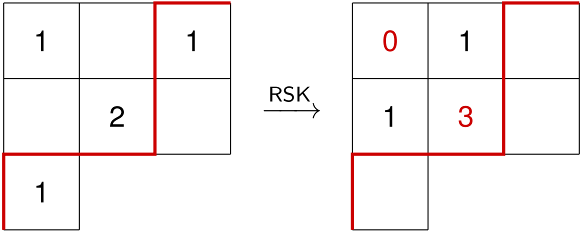

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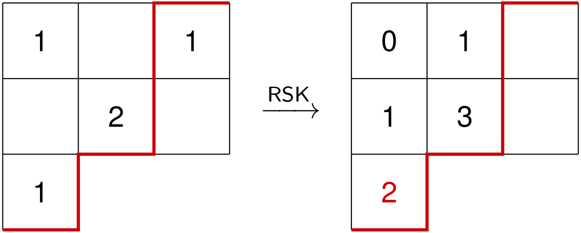

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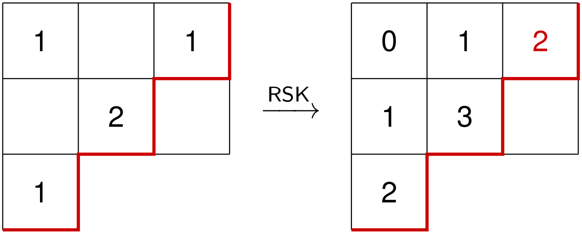

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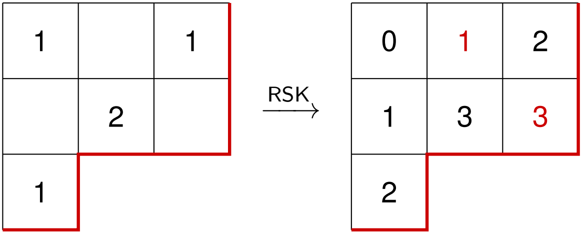

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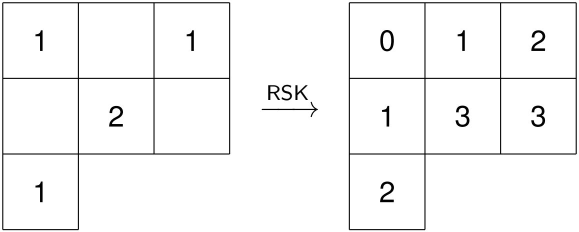


Comparing the definitions

The inductive definition is clearly well-defined and is bijective by induction; it has a “dynamical algebraic combinatorics” flavour.

The GK-invariant definition seems intrinsically interesting, but it isn’t obvious that is bijective.

The agreement goes back to Kirillov-Berenstein.

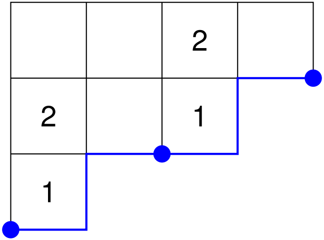
Sam Hopkins has nice notes on his webpage which were our introduction to this story.

Fill(P)

To define affine RSK, we must first define the relevant poset.

Choose a lattice path P of a right steps and b up steps, with $a, b > 0$.

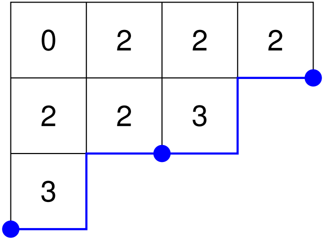
Fill(P) denotes an (a, b)-periodic function from the region above P and its (a, b)-translates, with a finite number of orbits of non-zero entries.



RPP(P)

Our RPPs will be analogous: they are located above the lattice path P and its translates. They are (a, b)-periodic, and they weakly increase down/right. They have only finitely many orbits with non-zero values.

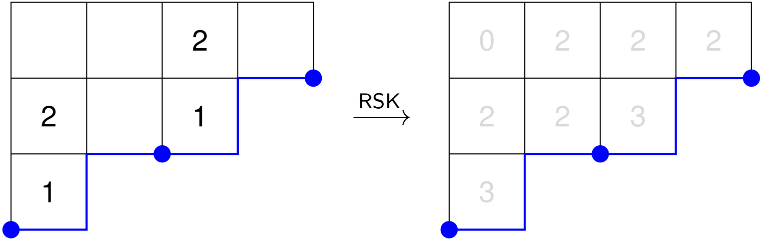
Here is an example of a RPP(P).



GK-invariants definition of affine RSK

For each box on the border of the filling, calculate the GK-invariants of the part of the filling above the box.

Enter them into the corresponding diagonal of the reverse plane partition.



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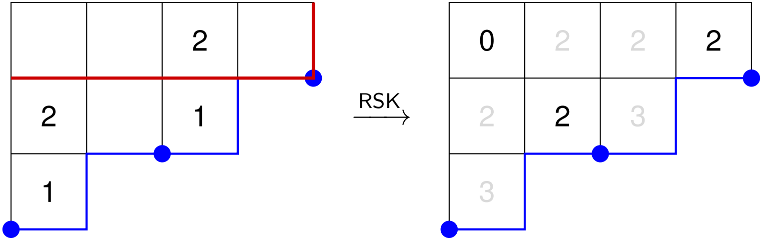

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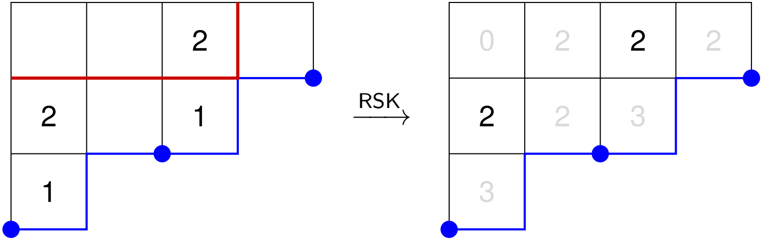

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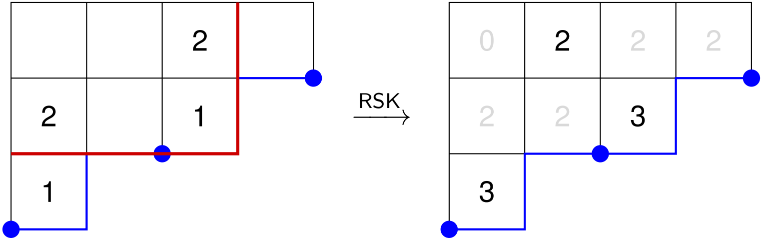

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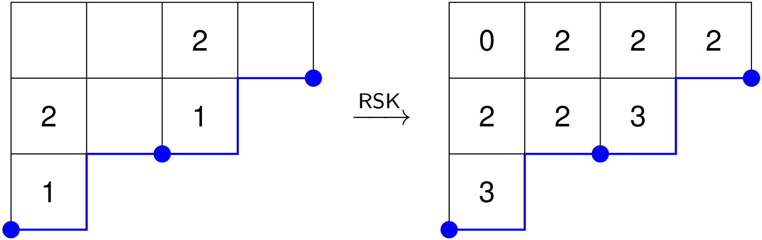


Inductive step for toggle definition of RSK

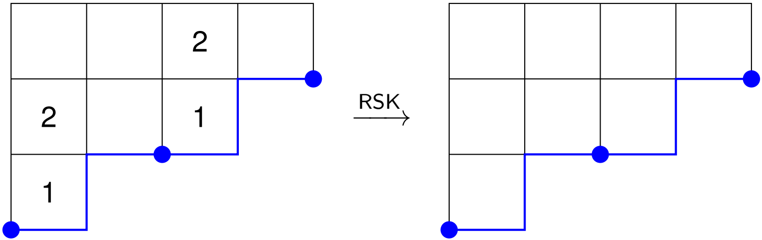
Suppose that P and P' are two lattice paths with a right steps and b up steps, with P' and P agreeing except that there is one box below P' and above P.
Suppose that $\mathrm{RSK}_{P'}$ is already defined.
Then define RSK_P by:

- applying $\mathrm{RSK}_{P'}$ to the part above P,
- toggling along the diagonal of the box between P' and P,
- adding the entry of that box to that position in the RPP.

Inductive toggle definition of RSK

Let $f \in \mathrm{Fill}(P)$.
Choose P' with a right steps and b up steps, such that all the non-zero entries of f are below P'.
Define $\mathrm{RSK}_{P'}$ of the zero filling to be the zero RPP.
Then repeatedly apply the induction step.

Example of toggle definition of RSK



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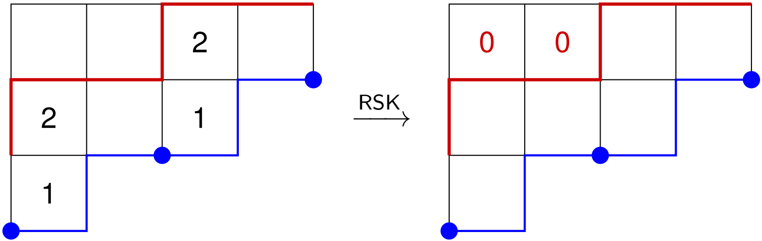

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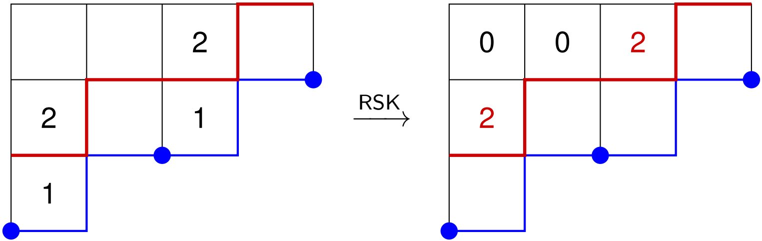

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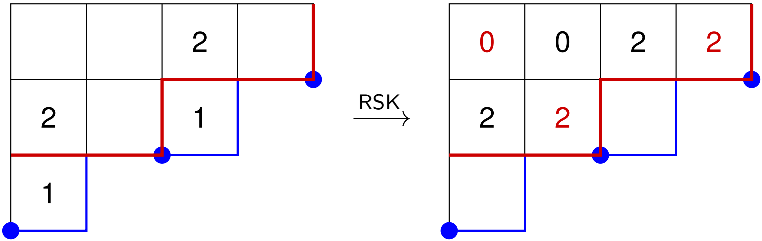

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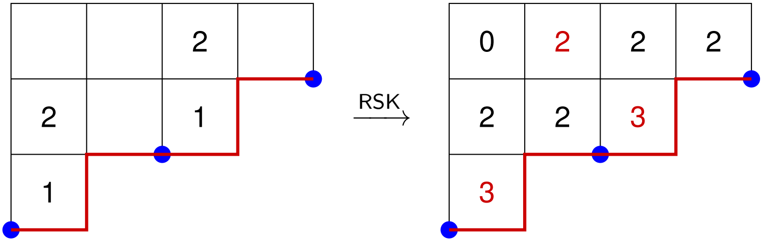

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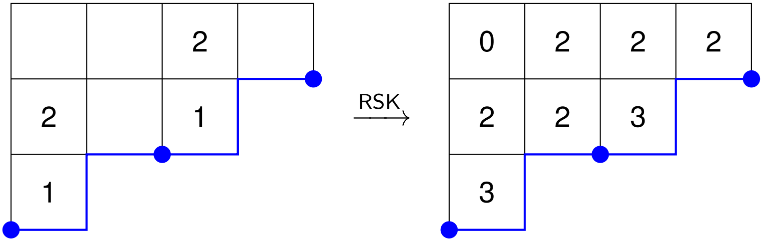


Comparing the definitions

There is a result which will not come as a surprise.

Theorem (Garver–Patrias–T)

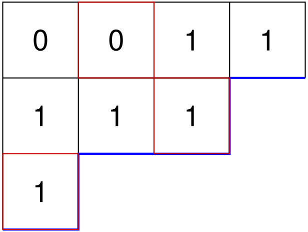
The GK definition and the toggle definition agree.

The toggle definition is injective, because we can reverse the RSK process, at each step identifying the entry from the filling.

However, this RSK is not a bijection!

Example of an RPP not in the image of RSK

Consider what happens with this RPP at the diagonals marked in red:



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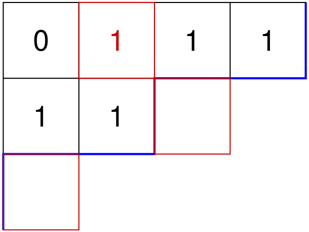


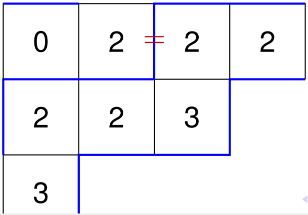
Making RSK a bijection I

One option is to make RSK a bijection by describing the image of RSK inside $\mathrm{RPP}(P)$.

We divide an RPP into *layers* by lattice paths which are translations of P by $(-n, n)$ for $n \in \mathbb{N}$.

Theorem (Garver–Patrias–T)

$f \in RPP(P)$ is in the image of RSK *if and only if, for each pair of successive layers L and L′, there is a pair of adjacent boxes $x \in L$ and $x' \in L'$, such that $f(x) = f(x')$.*



Making RSK a bijection II

We can make RSK a bijection by starting with an arbitrary RPP, and enlarging the set $\mathrm{Fill}(P)$ so that all of $\mathrm{RPP}(P)$ is in the image.

Run the inverse procedure until we obtain the filling doesn't change any further.

At that point, the remaining filling is constant on layers, so it can be described by a weakly decreasing sequence of non-negative integers.

Theorem (Garver–Patrias–T)

RSK defines a bijection:

$$\mathrm{Fill}(P) \times \mathit{Partitions} \to \mathrm{RPP}(P).$$

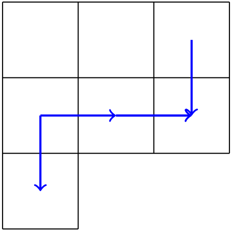
Thank you!

Quivers

In the background of today's talk, there are quiver representations.

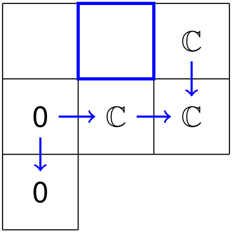
I am briefly going to explain what the quiver story looks like here.

The vertices of the quiver are the boxes of the border strip of ν, with arrows between adjacent vertices, oriented down/right.



Quivers: interpreting Fill(ν)

Each box of ν determines an indecomposable representation, where we put a copy of $\mathbb{C}$ at each border box in its hook, and 0 elsewhere, and identity maps between the copies of $\mathbb{C}$.



In this way, a filling of ν determines a quiver representation as a direct sum of indecomposable representations; the entries in the filling denote the multiplicities of the corresponding representations.

Quivers: interpreting RPP(ν)

The multiplicities of the indecomposable summands in a representation of Q is one way to specify the representation up to isomorphism. The quiver perspective on RSK is that we should think of RPP(ν) as a different invariant of a quiver representation. Then RSK just amounts to passing from one invariant of a representation to a different one, and is therefore bijective.

What is this other invariant? That is something which Al, Becky, and I had to figure out, and it is slightly technical, but here goes:

Quivers: interpreting RPP(ν) II

Given X a representation of Q coming from a filling of ν, we take a generic nilpotent endomorphism of X, say ϕ.

This induces a linear map ϕ_i on the vector space at each vertex of the representation.

We take the Jordan block sizes of each ϕ_i, and put them into the corresponding diagonal of ν (increasing down/right, and padded with zeros if necessary).

It turns out that this defines a reverse plane partition (not obvious), and turns out to be enough information to reconstruct X.