Promotion Sorting

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Abstract

Schützenberger's promotion operator is an extensively-studied bijection that permutes the linear extensions of a finite poset. We introduce a natural extension ∂ of this operator that acts on all labelings of a poset. We prove several properties of ∂; in particular, we show that for every labeling L of an n-element poset P, the labeling $\partial^{n-1}(L)$ is a linear extension of P. Thus, we can view the dynamical system defined by ∂ as a sorting procedure that sorts labelings into linear extensions. For all $0 \leq k \leq n-1$, we characterize the n-element posets P that admit labelings that require at least $n-k-1$ iterations of ∂ in order to become linear extensions. The case in which $k = 0$ concerns labelings that require the maximum possible number of iterations in order to be sorted; we call these labelings *tangled*. We explicitly enumerate tangled labelings for a large class of posets that we call inflated rooted forest posets. For an arbitrary finite poset, we show how to enumerate the sortable labelings, which are the labelings L such that $\partial(L)$ is a linear extension.

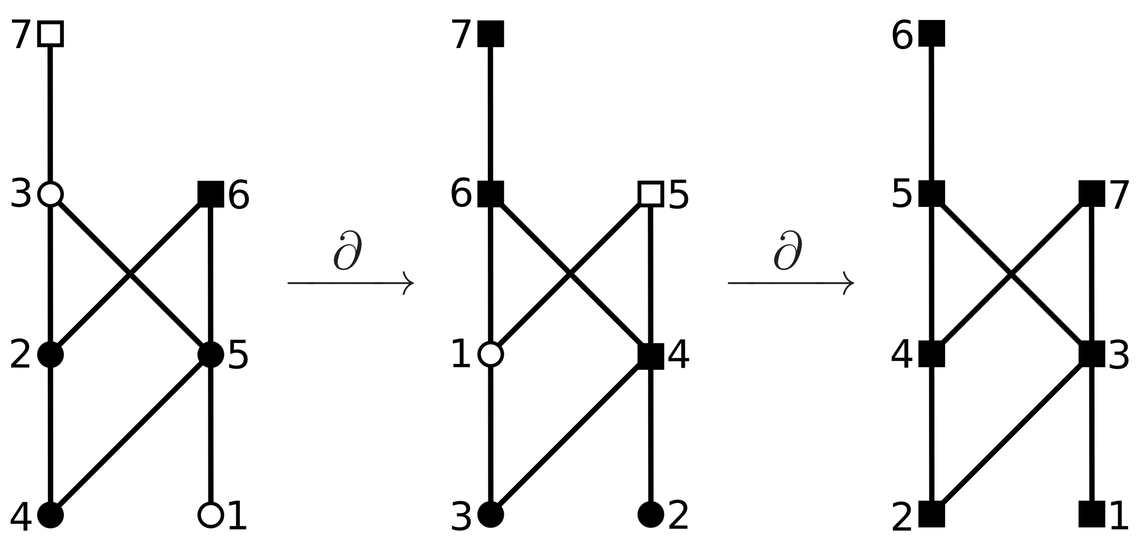
Defining Extended Promotion

A *labeling* of an n-element poset P is a bijection $L : P \to [n]$. A labeling is a *linear extension* if $x <_P y$ implies $L(x) < L(y)$. Let $\mathcal{L}(P)$ be the set of linear extensions of P.

Let $L : P \to [n]$ be a labeling of an n-element poset P. Suppose $x \in P$ is not a maximal element. This means that there are elements of P that are greater than x; among all such elements, let y be the one with the smallest label. We call y the *L-successor* of x. Notice that y does not necessarily cover x in P. Now let $v_1 = L^{-1}(1)$. If v_1 is not a maximal element of P, then let v_2 be its L-successor. If v_2 is not maximal, let v_3 be its L-successor. Continue in this fashion until obtaining an element v_m that is maximal. We define the labeling $\partial(L)$ by

$$\partial(L)(z) = \begin{cases} L(z) - 1, & \text{if } z \notin \{v_1, \ldots, v_m\}; \\ L(v_{i+1}) - 1, & \text{if } z = v_i \text{ for some } i \in \{1, \ldots, m-1\}; \\ n, & \text{if } z = v_m. \end{cases}$$

We call the map ∂ *extended promotion*; when restricted to the set $\mathcal{L}(P)$ of linear extensions of P, the map ∂ is the usual promotion bijection from $\mathcal{L}(P)$ to $\mathcal{L}(P)$.



Sorting with Extended Promotion

Proposition. *If $L : P \to [n]$ is a labeling of an n-element poset P, then $\partial^{n-1}(L) \in \mathcal{L}(P)$.*

Proof Idea. Let $U_j = \{y \in P : L(y) \geq j\}$. We say an element $x \in P$ is *frozen with respect to L* if the sets $U_n, U_{n-1}, \ldots, U_{L(x)}$ are all upper order ideals of P. We show that when we apply ∂ to a labeling that is not a linear extension, the set of frozen elements strictly increases. □

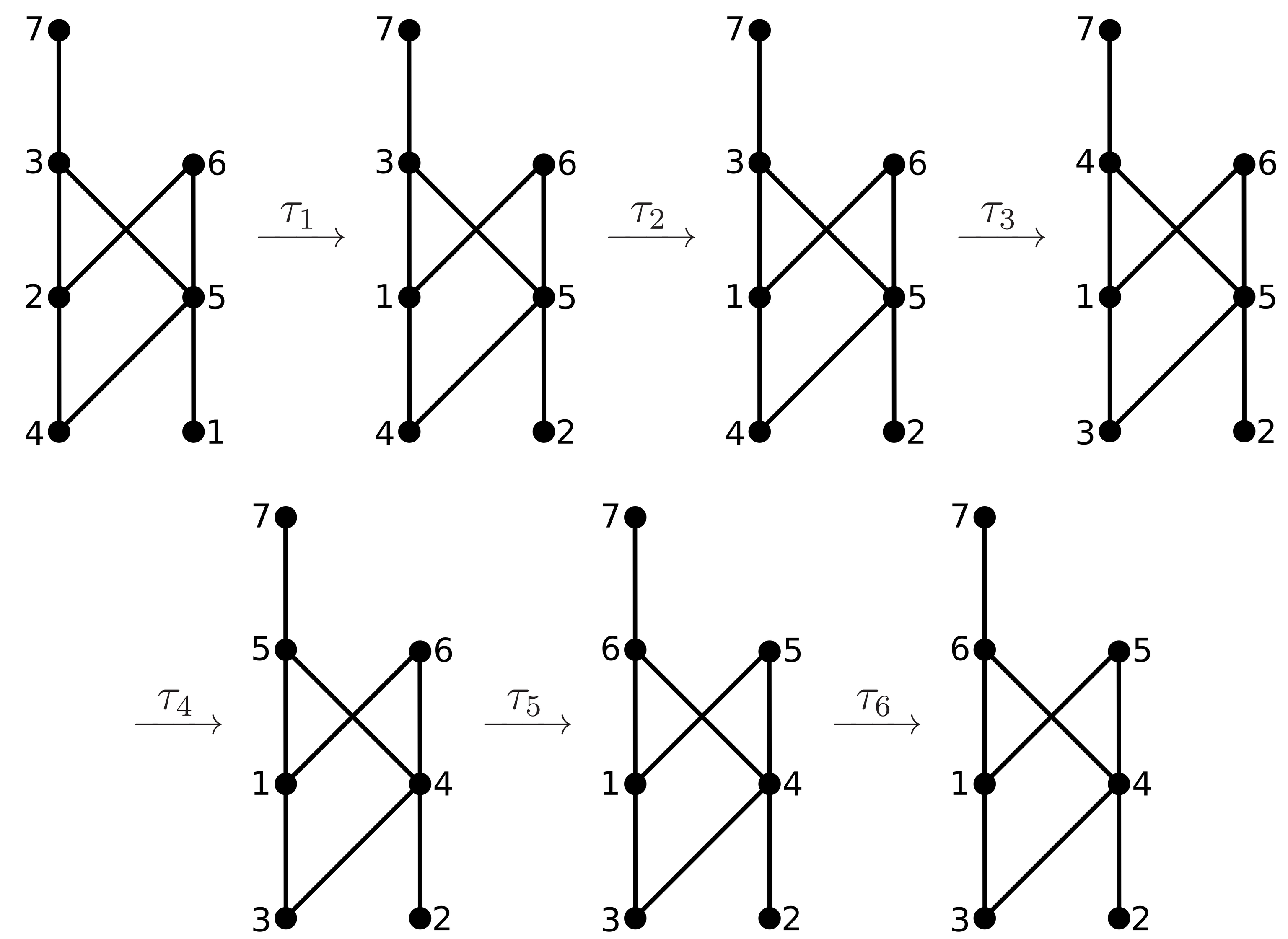
Definition. Let P be an n-element poset. A labeling $L : P \to [n]$ is *k-untangled* if the labeling $\partial^{n-k-2}(L)$ is not a linear extension. We say that L is *tangled* if it is 0-untangled. We make the convention that L is not k-untangled if $n \leq k+1$; in particular, the unique labeling of a 1-element poset is not tangled.

Alternative Description: Toggles

Suppose $L : P \to [n]$ is a labeling of P. If $L^{-1}(i) <_P L^{-1}(i+1)$, let $\tau_i(L) = L$. Otherwise, let $\tau_i(L)$ be the labeling obtained by swapping the labels i and $i+1$.

Proposition. *Let P be an n-element poset, and let $L : P \to [n]$ be a labeling of P. Then*

$$\partial(L) = (\tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1)(L).$$



Posets with k-untangled Labelings

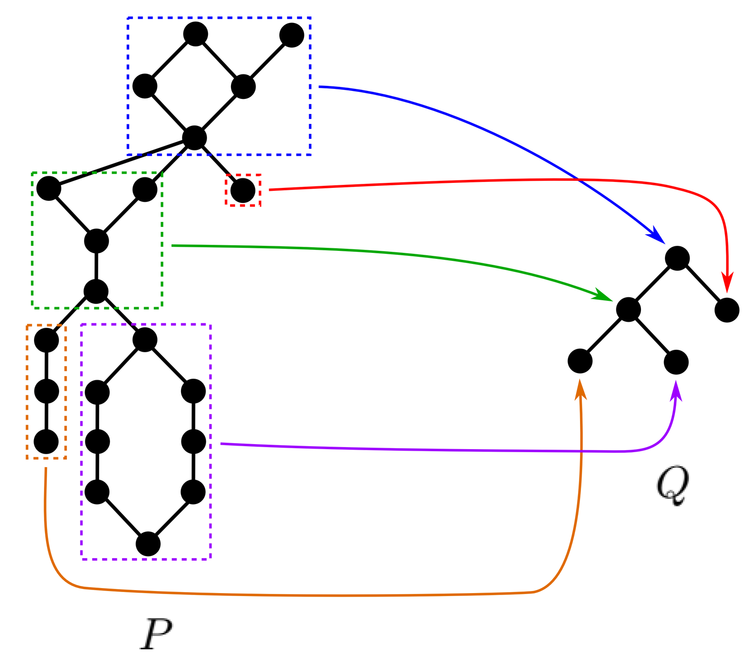
Theorem. *A poset has a k-untangled labeling if and only if it has a lower order ideal of size $k+2$ that is not an antichain.*

Inflated Rooted Forest Posets

Definition. Let Q be a finite poset. An *inflation* of Q is a poset P such that there exists a surjective map $\varphi : P \to Q$ with the following properties:

1. For each $v \in Q$, the subposet $\varphi^{-1}(v)$ of P has a unique minimal element.
2. If $x, y \in P$ are such that $\varphi(x) \neq \varphi(y)$, then $x <_P y$ if and only if $\varphi(x) <_Q \varphi(y)$.

If Q is a rooted forest poset, then P is an *inflated rooted forest poset*.



In our paper, we give a formula for the number of tangled labelings of an inflated rooted forest poset involving several parameters. Here are some simple corollaries:

1. If P is any n-element poset with r connected components, each of which has a unique minimal element, then P has $(n-r)(n-2)!$ tangled labelings.
2. Suppose Q has a single root v^* and $s \geq 1$ leaves that are all children of v^* (so Q has $s+1$ elements). If P is an inflation of Q with map $\varphi : P \to Q$, then the number of tangled labelings of P is $(n-1)!\left(\frac{|\varphi^{-1}(v^*)| - s}{\varphi^{-1}(v^*) - 1}\right)$.

Sortable Labelings

Theorem. *Let P be an n-element poset, and let $\mathcal{M}$ denote the set of maximal elements of P. For each $x \in \mathcal{M}$, let $\mathscr{C}_x$ be the number of chains of P that contain x. The number of labelings of P such that $\partial(L) \in \mathcal{L}(P)$ is*

$$\sum_{x \in \mathcal{M}} \mathscr{C}_x |\mathcal{L}(P \setminus \{x\})|.$$

Suggestions for Future Work

Conjecture. *Every n-element poset has at most $(n-1)!$ tangled labelings.*

It could be interesting to ask some specific questions about the dynamics of ∂ when the poset P is restricted to a narrow class of posets. Some questions include:

- How many labelings L of P satisfy $\partial^2(L) \in \mathcal{L}(P)$?
- What is the average number of iterations of ∂ needed to send a random labeling of P to a linear extension?