On the shifted Littlewood-Richardson coefficients and the Littlewood-Richardson coefficients

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Introduction

Let λ, μ, ν be partitions. Let $l(\lambda)$ be the length of λ, and s_λ be the Schur function associated to the partition λ. The Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ appear in the expansion

$$s_\lambda s_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} s_\nu.$$

If now λ, μ, ν are strict partitions, let Q_λ be the shifted Schur Q-function associated to the strict partition λ. The shifted Littlewood-Richardson coefficients appear in the expansion

$$Q_\lambda Q_\mu = \sum_{\nu} 2^{l(\lambda)+l(\mu)-l(\nu)} f^{\nu}_{\lambda\mu} Q_\nu.$$

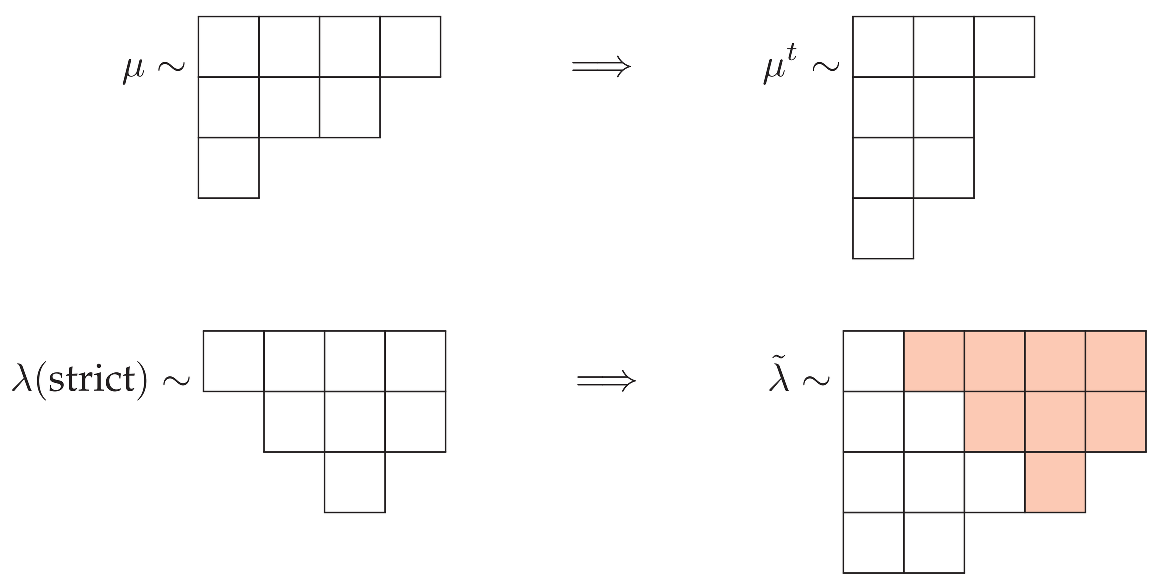
For any strict partition λ, and a partition μ of the same integer, the coefficients $g_{\lambda\mu}$ appear in the decomposition ([4])

$$Q_\lambda = 2^{l(\lambda)} \sum_{\mu} g_{\lambda\mu} s_\mu.$$

The coefficients $g_{\lambda\mu}$ can be considered as shifted Littlewood-Richardson coefficients by the identity ([4])

$$g_{\lambda\mu} = f^{\mu+\delta}_{\lambda\delta},$$

where $\delta = (l, l-1, \dots, 1)$ with $l = l(\mu)$.



We use the shifted Littlewood-Richardson rule given by Stembridge [4] to obtain a new combinatorial models for the coefficients $f^{\nu}_{\lambda\mu}$ and $g_{\lambda\mu}$. The advantage of the new results allows us to compute the coefficients easier and to realize the connections with Littlewood-Richardson coefficients. The motivation of our work comes from the work of P. Belkale, S. Kumar and N. Ressayre [1]. The main results in the article [1] raised up some first clues about relations between shifted Littlewood-Richardson coefficients with Littlewood-Richardson coefficients. N. Ressayre conjectures an inequality between them in [3]

$$g_{\lambda\mu} \le c^{\tilde{\lambda}}_{\mu^t\mu}.$$

We do not use the approach from geometry as in [1], but we try to develop the combinatorial model of Stembridge [4] to discover the bridge between coefficients.

References

[1] Prakash Belkale, Shrawan Kumar, and Nicolas Ressayre. "A generalization of Fulton's conjecture for arbitrary groups". In: *Math. Ann.* 354.2 (2012), pp. 401–425.

[2] Nguyen Duc Khanh. "On the shifted Littlewood-Richardson coefficients and Littlewood-Richardson coefficients". ArXiv e-prints 2004.01121. 2020.

[3] N. Ressayre. *Private communication*. 2019.

[4] John R. Stembridge. "Shifted tableaux and the projective representations of symmetric groups". In: *Adv. Math.* 74.1 (1989), pp. 87–134.

A new combinatorial models for the coefficients $f^{\nu}_{\lambda\mu}$

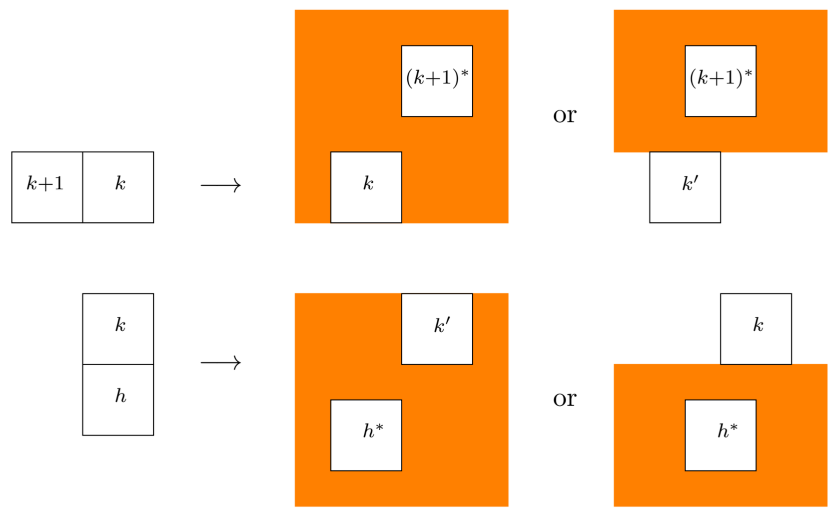
Given a skew shifted shape ν/μ, we number the boxes from top to bottom and right to left in each row by $1, 2, \dots, \nu/\mu$, respectively. The result is denoted by $\widetilde{T}_{\nu/\mu}$.

Let l be the number of boxes in the shifted skew diagram of ν/μ. For each $k = 1, 2, \dots, l$, let k^* to be meant k or k'.

Let $\widetilde{\mathcal{O}}(\nu/\mu)$ be the set of all tableaux T of size l of unshifted shape constructed from $\widetilde{T}_{\nu/\mu}$, satisfying the following conditions:

(C1) If k and $k+1$ appear in the same row of $\widetilde{T}_{\nu/\mu}$, then $(k+1)^*$ appears weakly above k or $(k+1)^*$ appears strictly above k' in T.

(C2) If h appears in the box directly below k in $\widetilde{T}_{\nu/\mu}$, then h^* appears weakly below k' or h^* appears strictly below k in T.



(C3) T is filled by the alphabet $\{1' < 1 < 2' < 2 < \dots < l' < l\}$ such that only one of k or k' appears in T for each $k = 1, 2, \dots, l$. The rightmost letter in each row of T is unmarked.

(C4) For each $j = 1, 2, \dots, n-1$, let $T^{j\downarrow}$ be the result of T by removing the boxes with entries k' or $k > j$ if exists. Suppose that the shape of $T^{j\downarrow}$ is $(\tau_1, \tau_2, \dots)$. Then $\tau_1 \ge \tau_2 \ge \dots$ and if $\tau_{i-1} = \tau_i$ for some i, the entry $(j+1)^*$ does not belong to the i^{th} row of T.

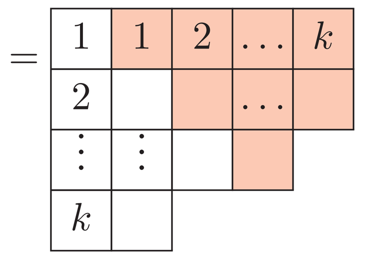
(C5) For each $j = n, n-1, \dots, 2$, let $T^{j\uparrow}$ be the result of T by changing k' to k with $k \ge j$, removing the boxes with entries $k' < j$ if exists. Suppose that the shape of $T^{j\uparrow}$ is $(\tau_1, \tau_2, \dots)$. Then $\tau_1 \ge \tau_2 \ge \dots$ and if $\tau_{i-1} = \tau_i$ for some i, the entry $j-1$ does not belong to the $(i-1)^{th}$ row of T and the entry $(j-1)'$ does not belong to the i^{th} row of T.

Theorem 0.1. *Let λ, μ, ν be strict partitions. Then the coefficient $f^{\nu}_{\lambda\mu}$ is the number of the tableaux T in $\widetilde{\mathcal{O}}(\nu/\mu)$ of shape λ.*

Theorem 0.2. *Let λ be a strict partition and μ be a partition. Then the coefficient $g_{\lambda\mu}$ is the number of the tableaux T in $\widetilde{\mathcal{O}}(\mu+\delta/\delta)$ of shape λ.*

On the coefficients $g_{\lambda\mu}$

$s(\mathcal{T}_\lambda) =$



$\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda)) = \{$pairs of tableaux (T, U) of shape $(\mu^t, \mu) \mid T.U = s(\mathcal{T}_\lambda)\}$

$\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))} = \{$pairs of tableaux (U^t, U) of shape $(\mu^t, \mu) \mid U^t.U = s(\mathcal{T}_\lambda)\}$

Theorem 0.3. *Let λ be a strict partition and μ be a partition. Then $g_{\lambda\mu} = \#\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$.*

Theorem 0.4. *Let λ be a strict partition and μ be a partition. Then $g_{\lambda\mu} \le c^{\tilde{\lambda}}_{\mu^t\mu}$.*

Conjecture 0.5. *Let λ be a strict partition and μ be a partition. Then $g^2_{\lambda\mu} \le c^{\tilde{\lambda}}_{\mu^t\mu}$.*

Conjecture 0.6. *There exists a bijection $\mathcal{S}$ from the set $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$ to the set $\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))$ such that*

1. *The restriction of the map $\mathcal{S}$ on the set $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ is a bijection onto the set $\overline{\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))}$.*
2. *The elements of the set $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$ have the form (U^t_α, U_α), with index α. Let (V_α, V^t_α) be the image of (U^t_α, U_α) through the bijection $\mathcal{S}$. Let (U^t_α, U_α) and (U^t_β, U_β) be elements of the set $\overline{\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))}$. If (U^t_α, U_β) is not in the set $\mathcal{T}(\mu^t, \mu, s(\mathcal{T}_\lambda))$, then (V_α, V^t_β) is in the set $\mathcal{T}(\mu, \mu^t, s(\mathcal{T}_\lambda))$.*

Remark 0.7. *We describe the map $\mathcal{S}$ in [2].*

Proposition 0.8. *Let λ be a strict partition and μ be a partition. Then $g_{\lambda\mu} = g_{\lambda\mu^t}$.*

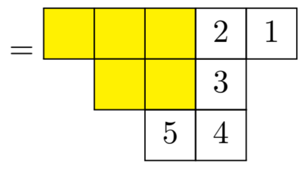
Proposition 0.9. *Suppose that Conjecture 0.6 holds. Then we have $g^2_{\lambda\mu} \le c^{\tilde{\lambda}}_{\mu^t\mu}$.*

Remark 0.10. *Thanks to Theorem 0.3, the validity of Conjecture 0.6 1. implies Proposition 0.8.*

Example

For $\lambda = (3,2), \mu = (3,2), \nu = (5,3,2)$, we have

$\widetilde{T}_{\nu/\mu} =$



The set $\widetilde{\mathcal{O}}_{\nu/\mu}$ contains two tableaux below. Hence $f^{\nu}_{\lambda\mu} = 2$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3' & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2' & 5 \\ \hline 3' & 4 \\ \cline{1-2}\end{array}$$

The computation data below show the inequality conjecture $g^2_{\lambda\mu} \le c^{\tilde{\lambda}}_{\mu^t\mu}$ and the conjecture between combinatorial models for $\lambda = (5,2), \mu = (4,2,1)$

$\lvert\lambda\rvert$	λ	μ	$g_{\lambda\mu}$	$c^{\tilde{\lambda}}_{\mu^t\mu}$
11	(7,3,1)	(3, 3, 2, 1, 1, 1)	2	6
11	(7,3,1)	(4, 2, 2, 1, 1, 1)	2	5
11	(7,3,1)	(4, 3, 1, 1, 1, 1)	2	5
11	(7,3,1)	(4, 3, 2, 1, 1)	3	13
11	(7,3,1)	(5, 2, 2, 1, 1)	2	5
11	(7,3,1)	(5, 3, 1, 1, 1)	2	5
11	(7,3,1)	(5, 3, 2, 1)	3	13
11	(7,3,1)	(6, 2, 2, 1)	2	5
11	(7,3,1)	(6, 3, 1, 1)	2	5
11	(7,3,1)	(6, 3, 2)	2	6
11	(6,4,1)	(3, 3, 2, 2, 1)	2	6
11	(6,4,1)	(4, 2, 2, 2, 1)	2	5
11	(6,4,1)	(4, 3, 2, 1, 1)	3	14
11	(6,4,1)	(4, 3, 2, 2)	2	7
11	(6,4,1)	(4, 3, 3, 1)	2	4