A Combinatorial Formula for the Kazhdan-Lusztig Polynomials of Sparse Paving Matroids

Kyungyong Lee[†], George Nasr, and Jamie Radcliffe

University of Alabama[†] and University of Nebraska-Lincoln

arXiv:2006.10209

Abstract

We prove the positivity of Kazhdan-Lusztig polynomials for sparse paving matroids, which are known to be logarithmically almost all matroids, but are conjectured to be almost all matroids. The positivity follows from a remarkably simple combinatorial formula we discovered for these polynomials using skew young tableaux. This supports the conjecture that Kazhdan-Lusztig polynomials for all matroids have non-negative coeffiecients.

Background

Sparse Paving Matroids

Let M be a rank d matroid on $m+d$ elements. Let $\mathcal{B}$ be its set of bases and let

$$\mathcal{CH} = \binom{[m+d]}{d} \setminus \mathcal{B}.$$

Then M is *sparse paving* if any (and hence all) of the following hold.

- $\mathcal{CH}$ is the set of circuit-hyperplanes for M.
- For distinct $C, C' \in \mathcal{CH}$, we have $|C \triangle C'| \geq 4$.

Sparse paving matroids draw research interest due to a conjecture in [3], based on a prediction in [1]. The conjecture is that

$$\lim_{n\to\infty} \frac{s_n}{m_n} = 1,$$

where s_n (respectively, m_n) is the number of sparse paving matroids (respectively, matroids) on n elements. So far, it is known thanks to Pendavingh and van der Pol [5] that

$$\lim_{n\to\infty} \frac{\log s_n}{\log m_n} = 1.$$

Background Continued

Kazhdan-Lusztig Polynomials

Let M be an arbitrary matroid. Let F be a flat of the matroid M, rk be the rank function on M, and χ_M be the characteristic function for M. We denote M^F (respectively M_F) for the localization (respectively contraction) for M at F.

The *Kazhdan-Lusztig polynomial for* M, denoted $P_M(t)$, was introduced by Elias, Proudfoot, and Wakefield [2], so that

- If $\mathrm{rk}\, M = 0$, then $P_M(t) = 1$;
- If $\mathrm{rk}\, M > 0$, then $\deg P_M(t) < \frac{1}{2}\mathrm{rk}\, M$;
- $t^{\mathrm{rk} M} P_M(t^{-1}) = \sum_{F \text{ a flat}} \chi_{M^F}(t) P_{M_F}(t)$.

These polynomials are conjectured to have non-negative coefficients and real roots. The non-negative conjecture has been proven when M is representable [2], and many have found formulas for specific examples of such matroids.

Skew Young Tableaux

Consider the fillings of the following shapes.

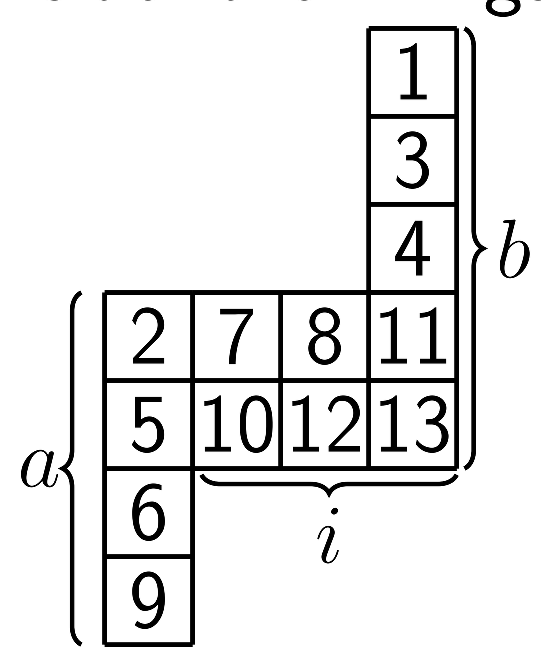

Figure 1

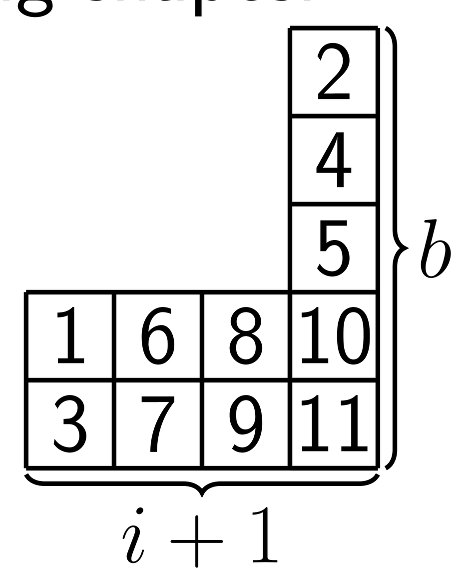

Figure 2

We refer to a legal filling as in Figure 1 as a *skew young tableau*, and denote $\mathrm{Skyt}(a,i,b)$ as the set of such fillings. We let $\mathrm{skyt}(a,i,b) := \#\,\mathrm{Skyt}(a,i,b)$.

Background Continued

We also define $\overline{\mathrm{Skyt}}(i,b)$, the subset of $\mathrm{Skyt}(2,i,b)$ so that 1 is always the entry at the top of the left-most column. See Figure 2. Similarly to before, we let $\overline{\mathrm{skyt}}(i,b) := \#\,\overline{\mathrm{Skyt}}(i,b)$.

Main Result and Special Cases

Theorem. *Let c_i be the i-th coefficient for the Kazhdan-Lusztig polynomial for the sparse paving matroid M with rank d on $m+d$ elements with circuit-hyperplanes $\mathcal{CH}$. Then*

$c_i = \mathrm{skyt}(m+1, i, d-2i+1) - |\mathcal{CH}| \cdot \overline{\mathrm{skyt}}(i, d-2i+1)$.

Moreover, this formula is always non-negative.

That is, fixing m, d, and i, the value of c_i is completely determined by the number of circuit-hyperplanes for M.

When $\mathcal{CH} = \emptyset$, observe that M is a uniform matroid.

If $m+1 = 2$ or $d-2i+1 = 2$, then $\mathrm{skyt}(m+1, i, d-2i+1)$ becomes equal to the number of polygon dissections [6]. Hence, when $m+1 = d-2i+1 = 2$, it becomes a Catalan number.

Finally, note that asymptotically, most sparse paving matroids will not be representable [4]. Hence, the above result establishes non-negativity for many more matroids than what was known before.

References

[1] H. H. Crapo and G-C. Rota. (1970)

[2] B. Elias, N. Proudfoot, M. Wakefield. (2016)

[3] D. Mayhew, M. Newman, D. Welsh, G. Whittle. (2011)

[4] P. Nelson. (2018)

[5] R. Pendavingh, J. van der Pol. (2015)

[6] R. Stanley. (1996)