Dimers, double-dimers, and the PT/DT correspondence

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joint work with Gautam Webb and Ben Young

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AICoVE

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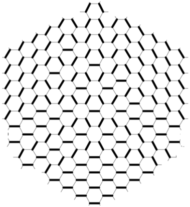
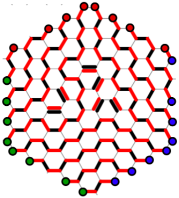
Outline

(1) The dimer model



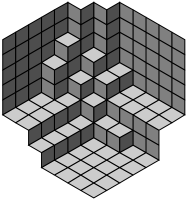
(2) The double-dimer model



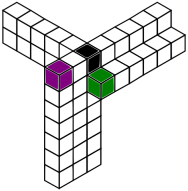
(3) Main result

(4) Proof sketch

(4a) Box counting in DT theory



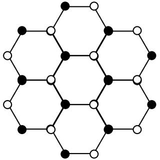
(4b) Box counting in PT theory



Dimer configurations

Assumptions:

- $G = (V_1, V_2, E)$ is finite, bipartite, planar
- G has a fixed embedding in the plane (dividing the plane into *faces*, one of which is unbounded)



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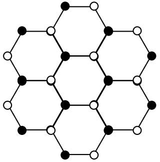
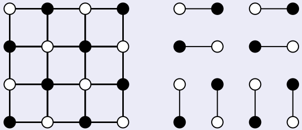


Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

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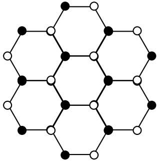
- $|V_1| = |V_2|$



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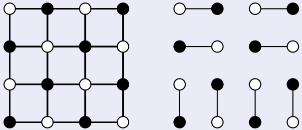

A collection of edges that covers each vertex exactly once

Kuo condensation

Let $Z^D(G) = \sum_M w(M)$, where $w(M) = \prod_{e \in M} w(e)$

(If all edge weights $= 1$, $Z^D(G) = \#$ of dimer configs of G)

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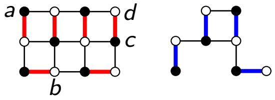

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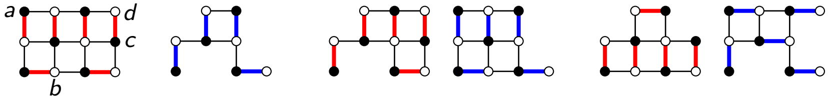

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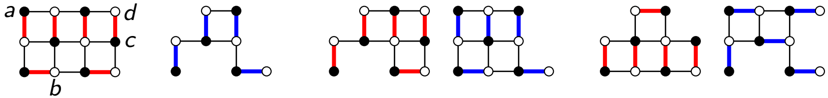

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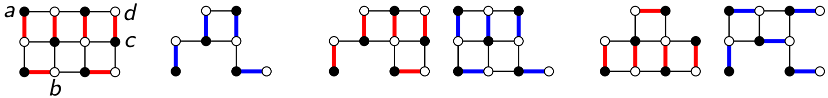

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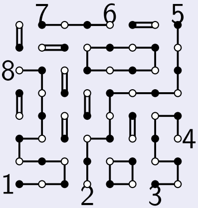
Non-bijective proofs:

- Desnanot-Jacobi identity/Dodgson condensation (see Zeilberger, 1997)
- Pfaffian identities, e.g. Fulmek, 2010
- Plücker relations, e.g. Speyer, 2016

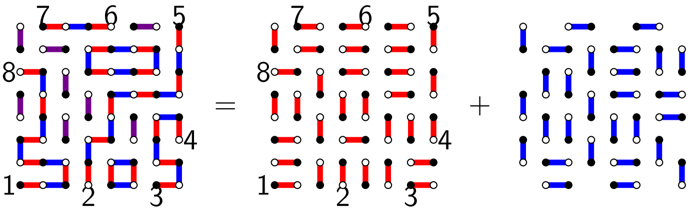
Double-dimer configurations

N is a set of special vertices called *nodes* on the outer face of G (with $|\mathbf{N}|$ even)

Definition (Double-dimer configuration on $(G, \mathbf{N})$)

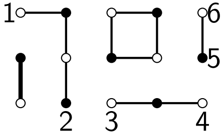


- A "subgraph" (doubled edges allowed) of degree 1 at $v \in \mathbf{N}$ and degree 2 at all other vertices.
- Equivalently, a configuration of
  - ℓ disjoint loops
  - Doubled edges
  - Paths connecting nodes in pairs

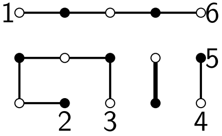


Double-dimer configurations

Each double-dimer configuration is associated with a planar pairing of **N**



$$\sigma = \begin{array}{c|c|c} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}$$



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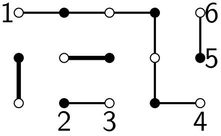

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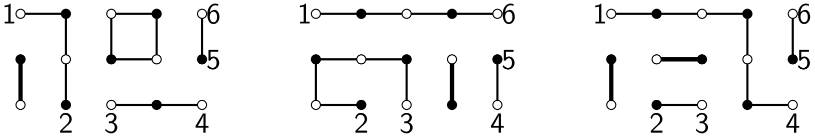

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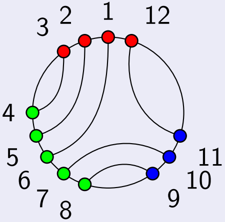
$Z_\sigma^{DD}(G,\mathbf{N})$ = weighted sum of all DD configs with pairing σ

A condensation-type recurrence holds for $Z_\sigma^{DD}(G,\mathbf{N})$ when σ is *tripartite*

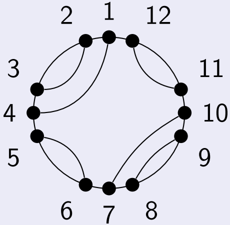
Tripartite pairings

Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



Tripartite



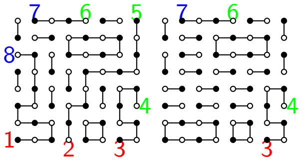
Not tripartite

We often color the nodes in the sets red, green, and blue, in which case σ has no monochromatic pairs.

Double-dimer condensation

$Z_\sigma^{DD}(G, \mathrm{N}) =$ weighted sum of all DD config with pairing σ

$$Z_\sigma^{DD}(\mathrm{N}) Z_{\sigma_{1258}}^{DD}(\mathrm{N}-1,2,5,8) = Z_{\sigma_{12}}^{DD}(\mathrm{N}-1,2) Z_{\sigma_{58}}^{DD}(\mathrm{N}-5,8) + Z_{\sigma_{18}}^{DD}(\mathrm{N}-1,8) Z_{\sigma_{25}}^{DD}(\mathrm{N}-2,5)$$



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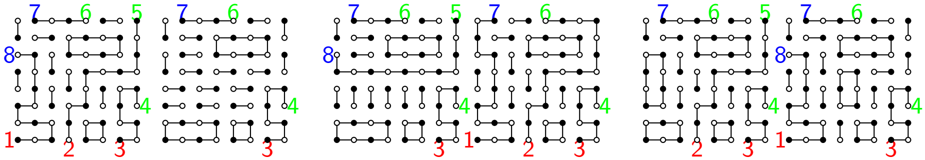

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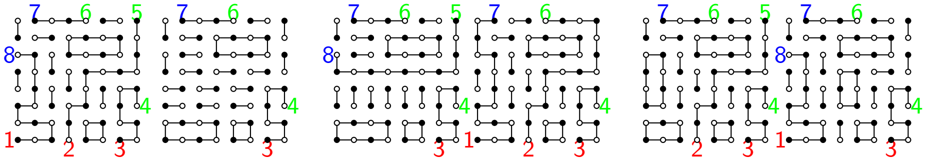


Theorem (J., 2019)

Divide N *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathrm{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and* x, y, w, v *appear in cyclic order then*

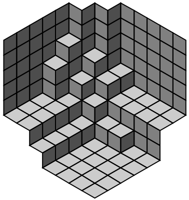
$$Z_{\sigma}^{DD}(G, \mathrm{N})Z_{\sigma_{xywv}}^{DD}(G, \mathrm{N}-\{x, y, w, v\}) =$$

$$Z_{\sigma_{xy}}^{DD}(G, \mathrm{N}-\{x, y\})Z_{\sigma_{wv}}^{DD}(G, \mathrm{N}-\{w, v\}) + Z_{\sigma_{xv}}^{DD}(G, \mathrm{N}-\{x, v\})Z_{\sigma_{wy}}^{DD}(G, \mathrm{N}-\{w, y\})$$

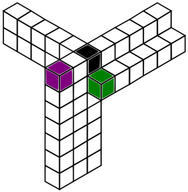
Proof uses Dodgson condensation, technical extension of Kenyon-Wilson

Main result

Donaldson-Thomas theory and Pandharipande-Thomas theory have generating fns known as the *combinatorial Calabi-Yau topological vertices* which count *plane-partition-like* objects.



DT theory gen fn: $V(\mu_1, \mu_2, \mu_3)$



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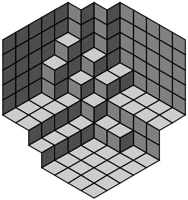

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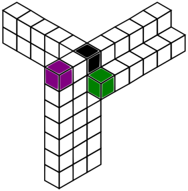

PT theory gen fn: $W(\mu_1, \mu_2, \mu_3)$

Theorem (J.-Webb-Young, Calabi-Yau case of Conj 4 from PT2009)

$$V(\mu_1, \mu_2, \mu_3) = M(q)W(\mu_1, \mu_2, \mu_3),$$

where $M(q)$ is the generating function for plane partitions.

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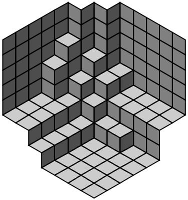

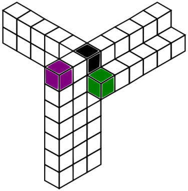

DT theory gen fn: $V(\mu_1, \mu_2, \mu_3)$
dimer interpretation ⟶

PT theory gen fn: $W(\mu_1, \mu_2, \mu_3)$
double-dimer interpretation ⟶

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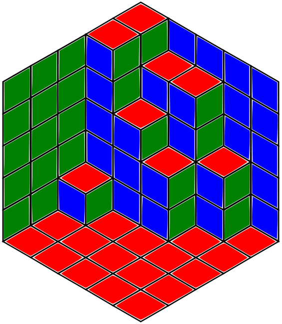
Plane partitions



5	4	4	2
5	3	2	
1			

Definition (Plane partition)

A plane partition is a finite array π of positive integers that is nonincreasing in rows and columns.

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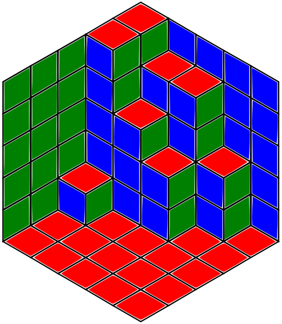

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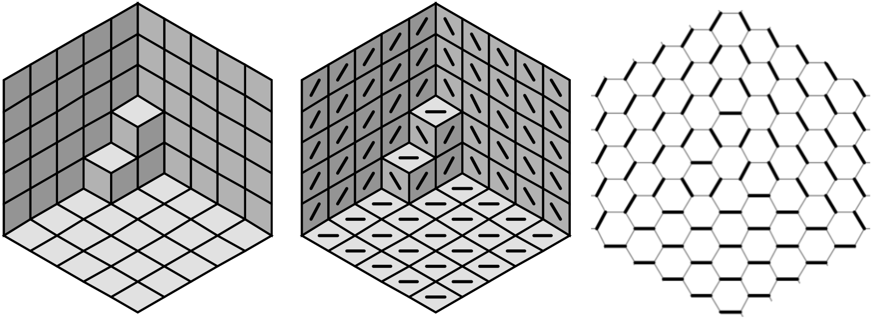
Theorem (MacMahon, 1916)

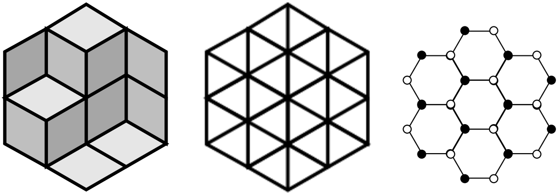
$$M(q) = \sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}, \quad \textit{where } |\pi| = \sum \pi_{i,j}.$$

Connection to the dimer model

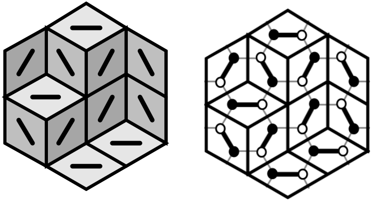


Connection to the dimer model

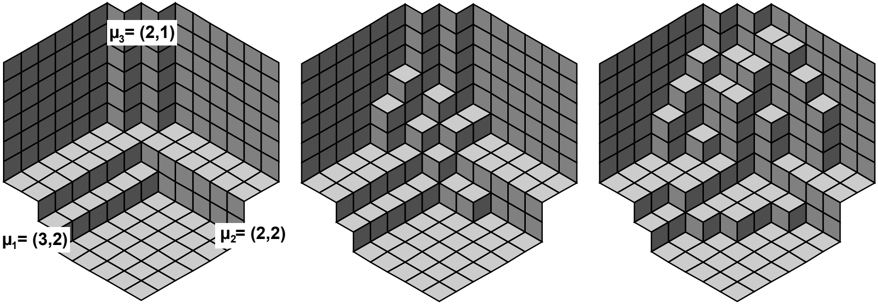
Let $\mathcal{B}(r, s, t)$ denote a "room" with dimensions $r \times s \times t$.
$\pi \in \mathcal{B}(r, s, t)$ is a rhombus tiling of a hexagonal region of triangles



This is equivalent to a dimer configuration of the dual graph.

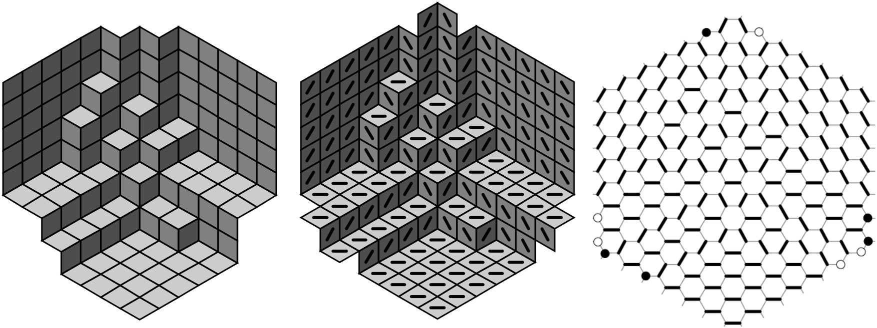


In DT theory, we count *plane partitions asymptotic to* (μ_1, μ_2, μ_3)



The DT topological vertex is $V(\mu_1, \mu_2, \mu_3) = \sum_{\pi \in P(\mu_1,\mu_2,\mu_3)} q^{w(\pi)}$

Dimer model interpretation



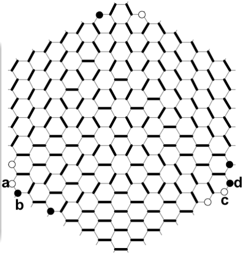
$\pi \in P(\mu_1, \mu_2, \mu_3)$ is equivalent to a dimer configuration on the honeycomb graph with some outer vertices removed.

The condensation recurrence in DT theory

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G - \{a,b,c,d\}) = Z^D(G - \{a,b\})Z^D(G - \{c,d\}) + Z^D(G - \{a,d\})Z^D(G - \{b,c\})$$



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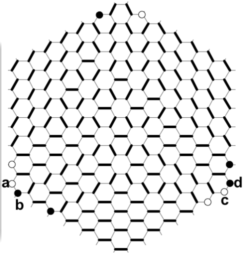

$$qV((3,1),(2,1),\mu_3)V((3,2),(2,2),\mu_3) = qV((3,2),(2,1),\mu_3)V((3,1),(2,2),\mu_3) + V((2,1,1),(3),\mu_3)V((4),(1,1,1),\mu_3)$$

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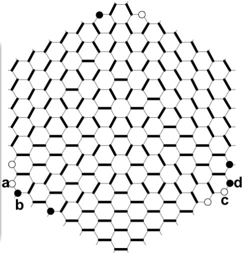
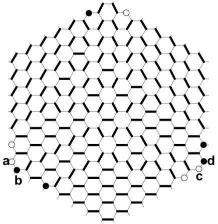
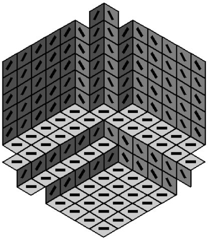
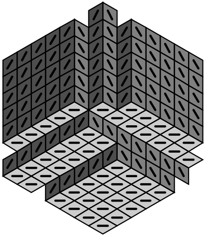
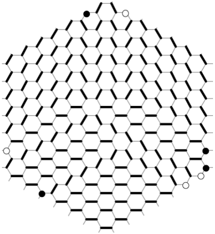


$$qV((3,1),(2,1),\mu_3)V((3,2),(2,2),\mu_3) = qV((3,2),(2,1),\mu_3)V((3,1),(2,2),\mu_3) + V((2,1,1),(3),\mu_3)V((4),(1,1,1),\mu_3)$$



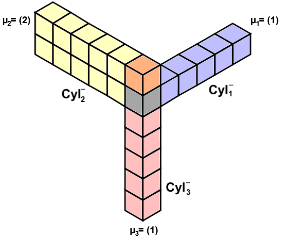






PT box counting

Following PT2009, we first define containers that we will fill with boxes:



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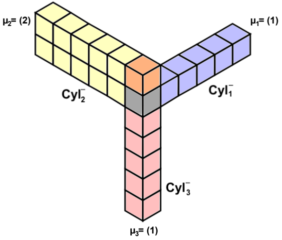
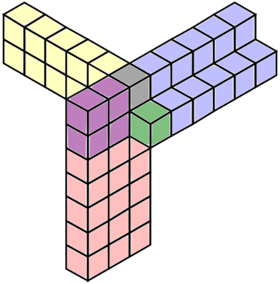
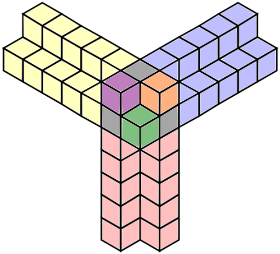


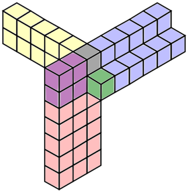
Containers consist of regions:

- $\mathrm{I}^- = \mathrm{Cyl}_1^- \cup \mathrm{Cyl}_2^- \cup \mathrm{Cyl}_3^-$
- $\mathrm{III} = \mathrm{Cyl}_1 \cap \mathrm{Cyl}_2 \cap \mathrm{Cyl}_3$
- $\mathrm{II} = \bigcup_{i \neq j \neq k} (\mathrm{Cyl}_i \cap \mathrm{Cyl}_j \setminus \mathrm{Cyl}_k)$



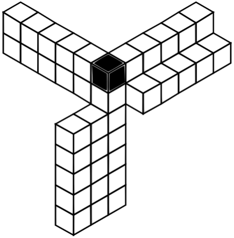


AB configurations

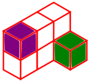


Definition

If $A \subseteq \mathrm{I}^- \cup \mathrm{III}$ and $B \subseteq \mathrm{II} \cup \mathrm{III}$ are finite sets of boxes, then (A, B) is an AB *configuration* if each of A and B satisfies the condition for plane partitions (with gravity pulling in the opposite direction).

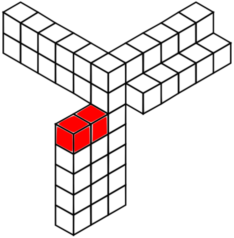


A

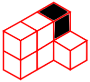


B

Valid AB configuration



A



B

Invalid AB configuration
(A is valid, B is not)

Labelled AB configurations

- Some AB configs can be labelled (following an intricate set of rules).
- Define

$$W(\mu_1, \mu_2, \mu_3) = \sum_{\text{labelled } AB \text{ configs}} q^{|A|+|B|}$$

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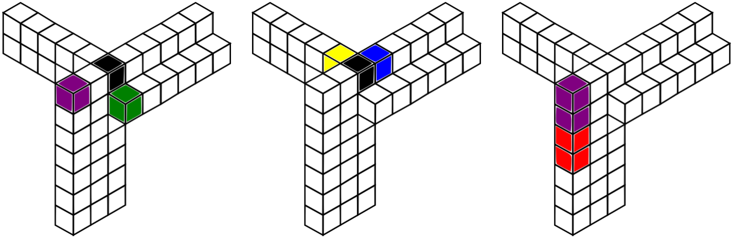
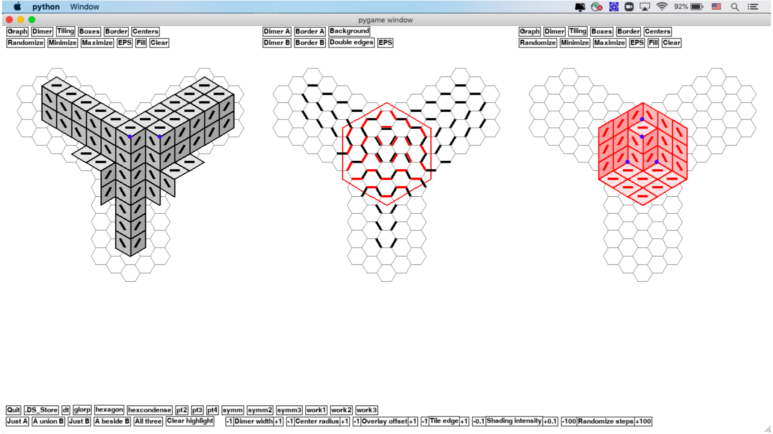
- We prove that labelled AB configurations are a discrete version of *labelled box configurations* used to define the PT topological vertex.



Connection to the tripartite double-dimer model

The observation that a PT labelled box config can be split into an A config and a B config gives a tripartite double-dimer interpretation for the PT topological vertex $W(\mu_1, \mu_2, \mu_3)$



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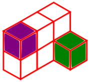
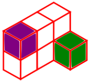
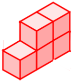


B configuration

Connection to the tripartite double-dimer model

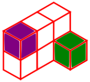


B configuration

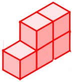


Draw boxes NOT in B

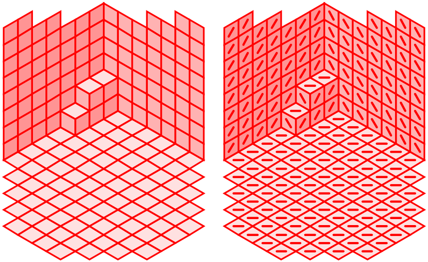
Connection to the tripartite double-dimer model



B configuration

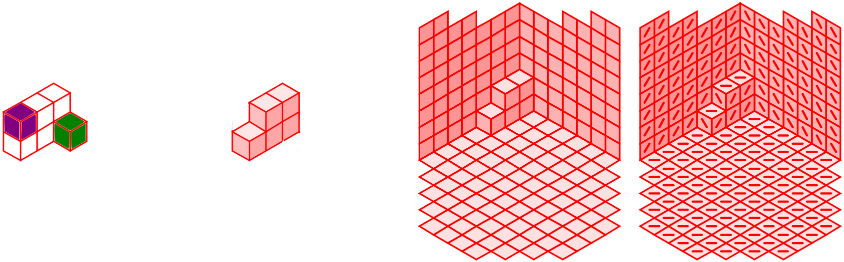


Draw boxes NOT in B

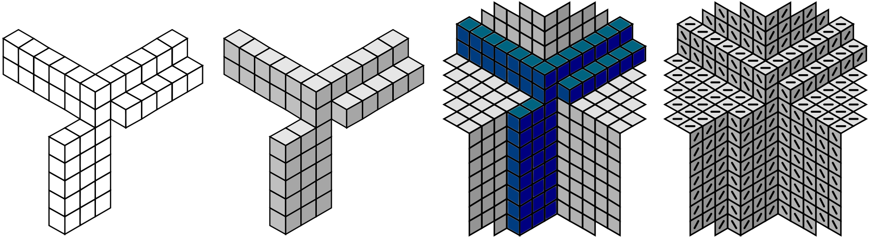


Extend to tiling of the plane

Connection to the tripartite double-dimer model



B configuration — Draw boxes NOT in B — Extend to tiling of the plane



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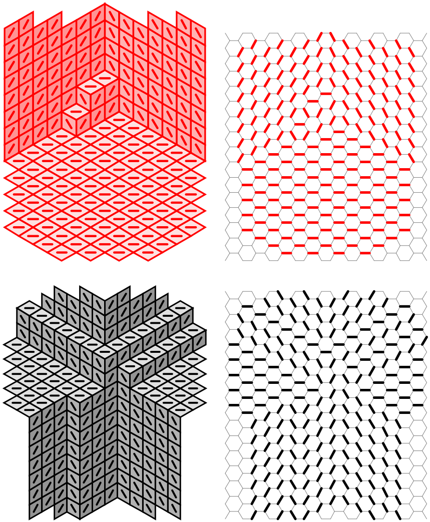

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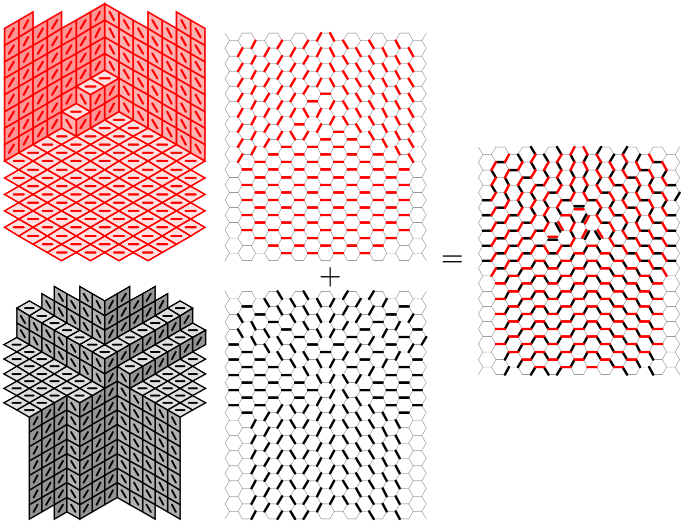

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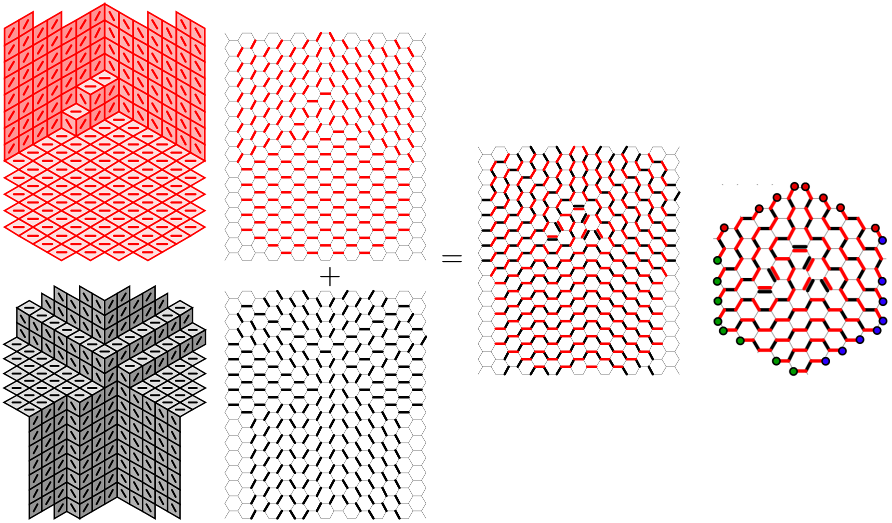
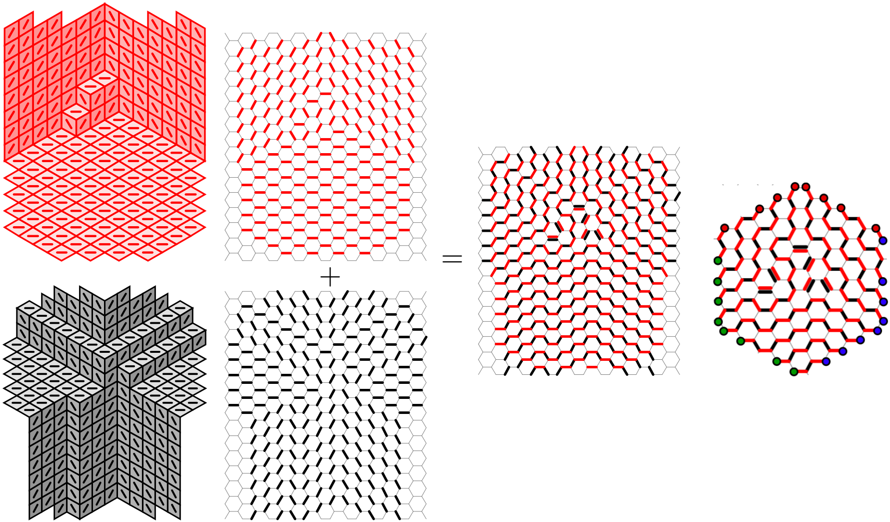


Connection to the tripartite double-dimer model



Apply double-dimer condensation by adding nodes.

Summary

Theorem (J.-Webb-Young, Calabi-Yau case of Conj 4 from PT2009)

$$V(\mu_1, \mu_2, \mu_3) = M(q)W(\mu_1, \mu_2, \mu_3),$$

where $M(q)$ is the generating function for plane partitions.

- $V(\mu_1, \mu_2, \mu_3)$ counts dimer configurations
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- Both are unique solutions of the condensation recurrence (up to a constant)

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Thank you for listening!

References

- M. Fulmek, Graphical condensation, overlapping Pfaffians and superpositions of matchings. *Electron. J. Comb.*, 17, 2010.
- H. Jenne. Combinatorics of the double-dimer model. *arXiv preprint arXiv:1911.04079*, 2019.
- H. Jenne, G. Webb, and B. Young. The Combinatorial PT-DT Correspondence *arXiv preprint arXiv:2012.08484*, 2020.
- R. Kenyon and D. Wilson. Combinatorics of tripartite boundary connections for trees and dimers. *Electron. J Comb.*, 16(1), 2009.
- R. Kenyon and D. Wilson. Boundary partitions in trees and dimers. *Trans. Amer. Math. Soc.*, 363(3):1325-1364, 2011.
- E. H. Kuo. Applications of graphical condensation for enumerating matchings and tilings. *Theoret. Comput. Sci.*, 319(1-3):29-57, 2004.
- R. Pandharipande and R. P. Thomas. The 3-fold vertex via stable pairs. Geom. Topol., 13(4):1835–1876, 2009.
- D. E. Speyer. Variations on a theme of Kasteleyn, with Application to the Totally Nonnegative Grassmannian. *Electron. J. Comb.*, 23(2), 2016.
- D. Zeilberger. Dodgon's Determinant-Evaluation Rule proved by Two-Timing Men and Women, *Electron. J. Comb.*, 4(2), 1997.