The antiprism triangulation

(Joint work with C. Athanasiadis and J.-M. Brunink)

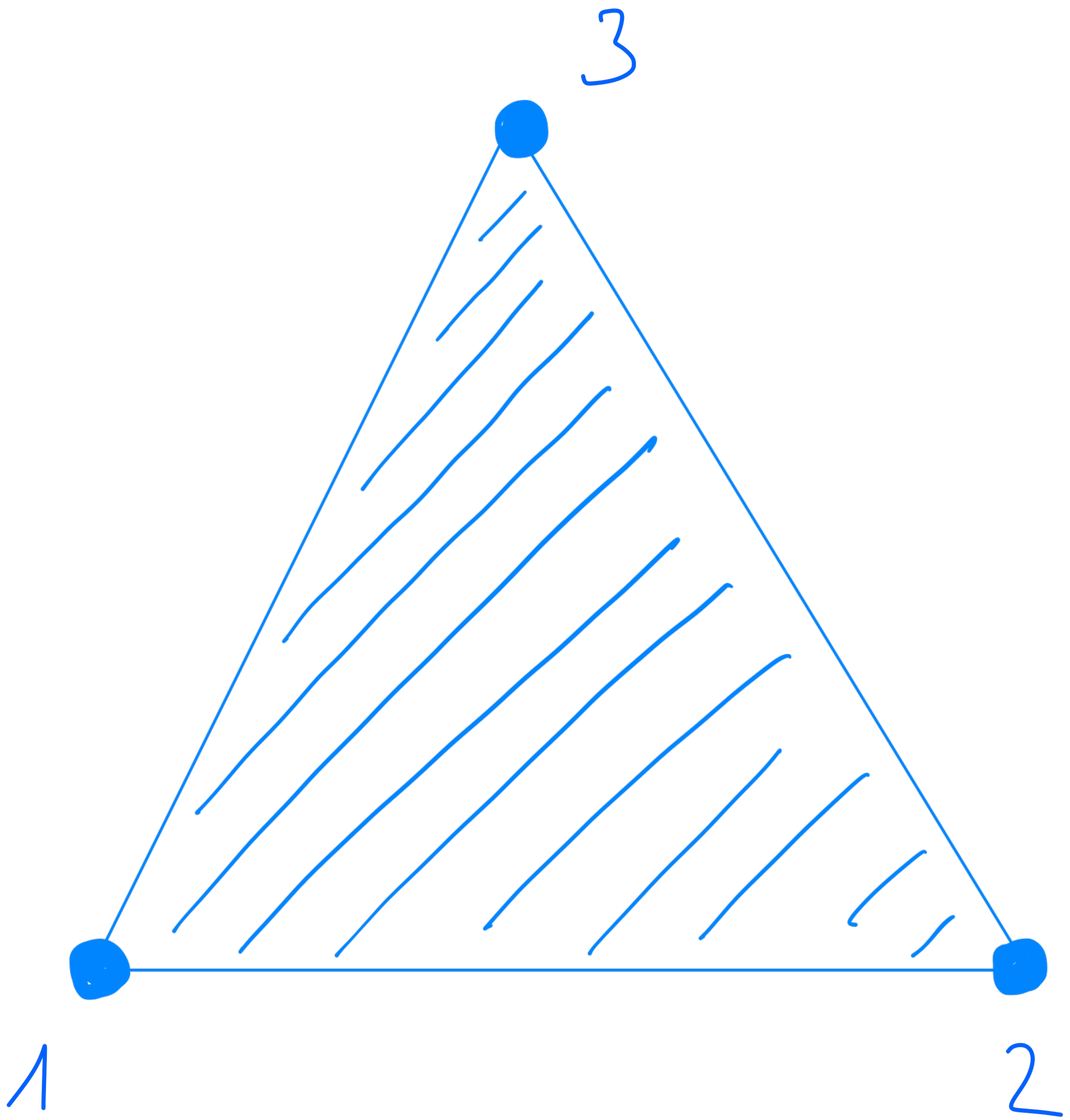
AlCoVE

Martina Juhnke, University of Osnabrück

The barycentric subdivision $\mathrm{sd}(\Delta)$

Δ simplicial complex on vertex set V

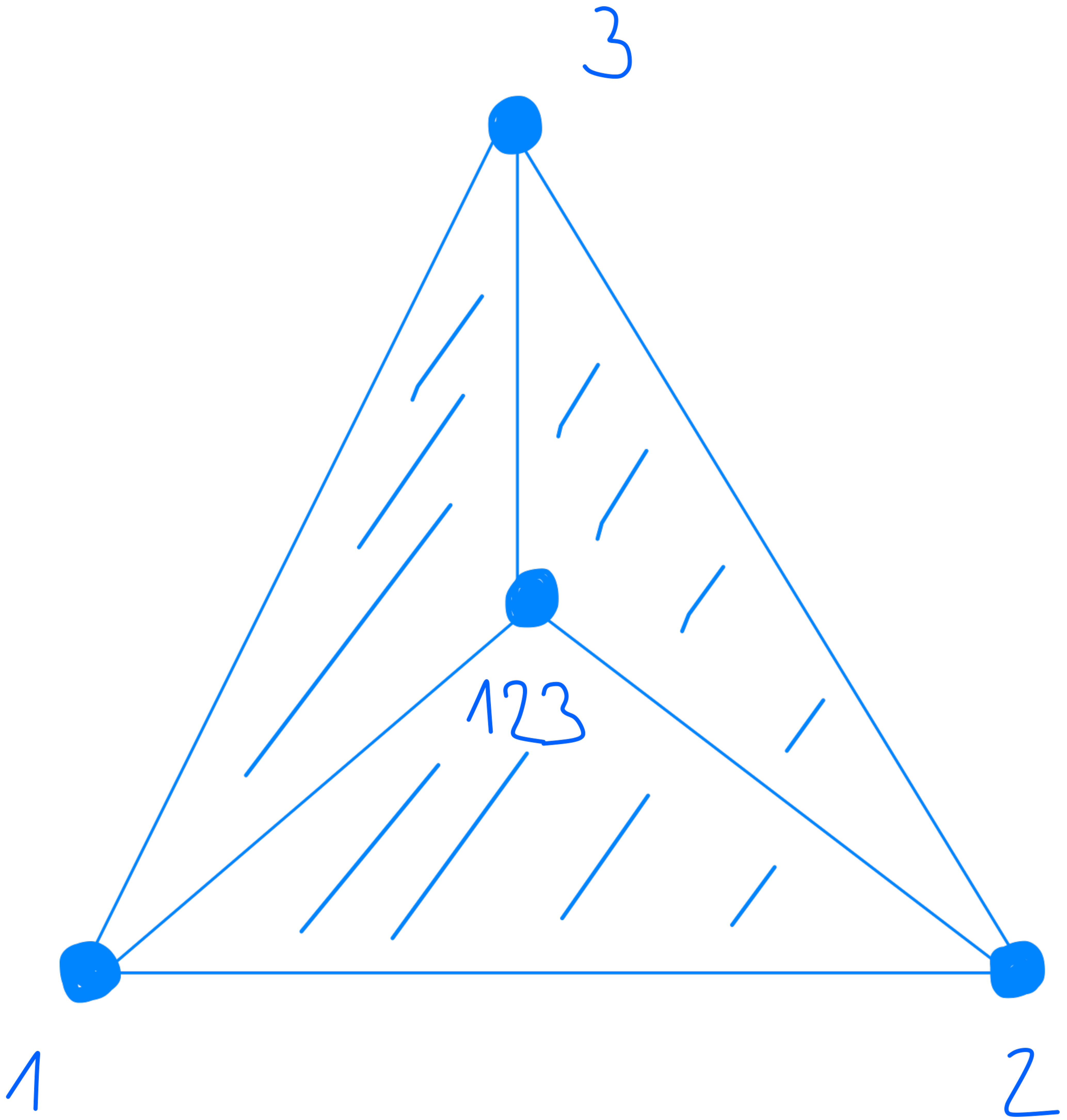
Geometrically:
stellar subdivide faces by decreasing dimension



The barycentric subdivision $\mathrm{sd}(\Delta)$

Δ simplicial complex on vertex set V

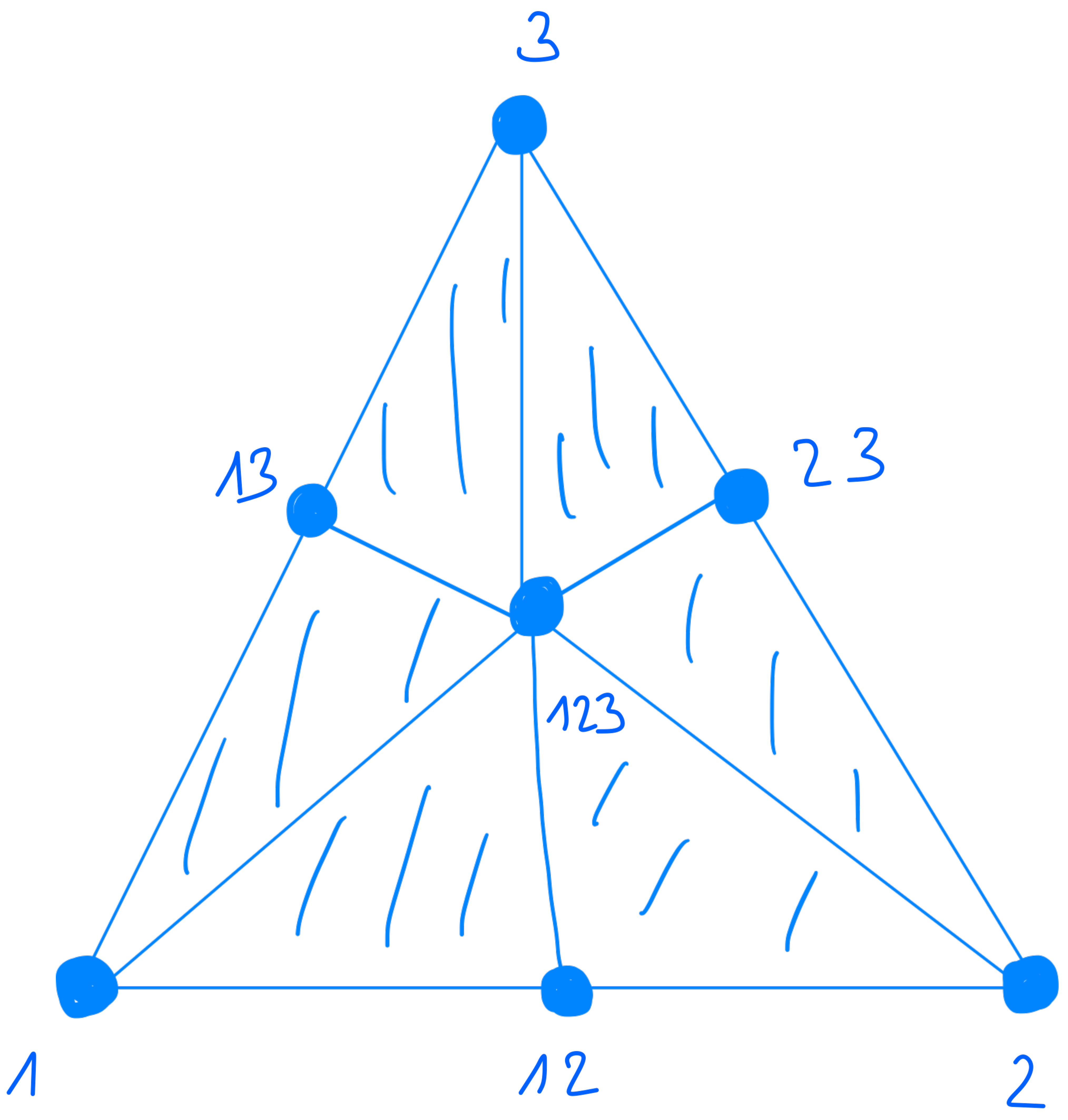
Geometrically:
stellar subdivide faces by decreasing dimension



The barycentric subdivision $\mathrm{sd}(\Delta)$

Δ simplicial complex on vertex set V

Geometrically:
stellar subdivide faces by decreasing dimension



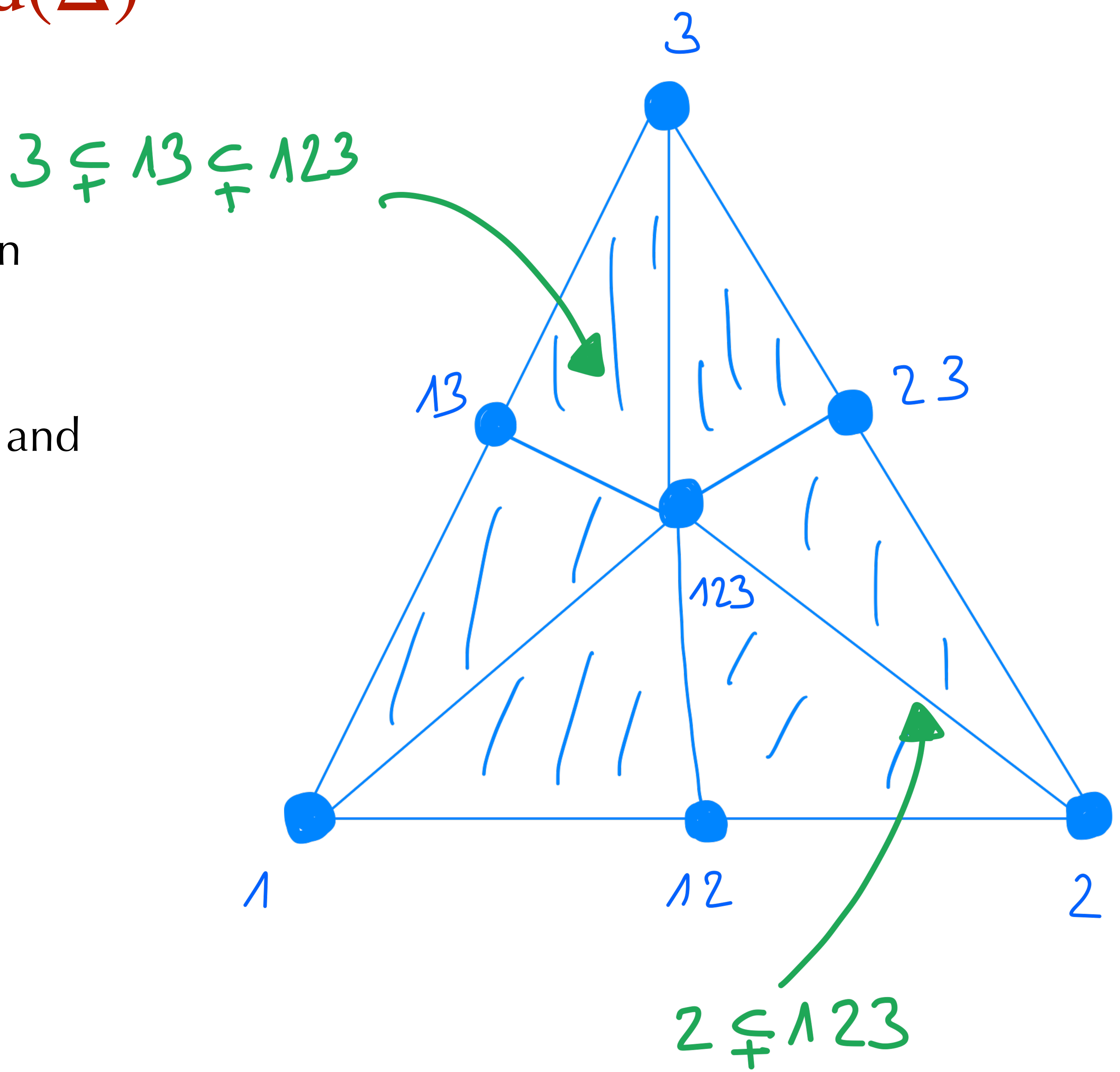
The barycentric subdivision $\mathrm{sd}(\Delta)$

Δ simplicial complex on vertex set V

Geometrically:
stellar subdivide faces by decreasing dimension

Combinatorially:
clique complex of graph on vertex set $\Delta \setminus \{\emptyset\}$ and edges (F, G) if $F \subsetneq G$ or $G \subsetneq F$

k-faces = chains
$\emptyset \neq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$
where $F_i \in \Delta$ for $0 \leq i \leq k$



The barycentric subdivision $\mathrm{sd}(\Delta)$

Δ simplicial complex on vertex set V

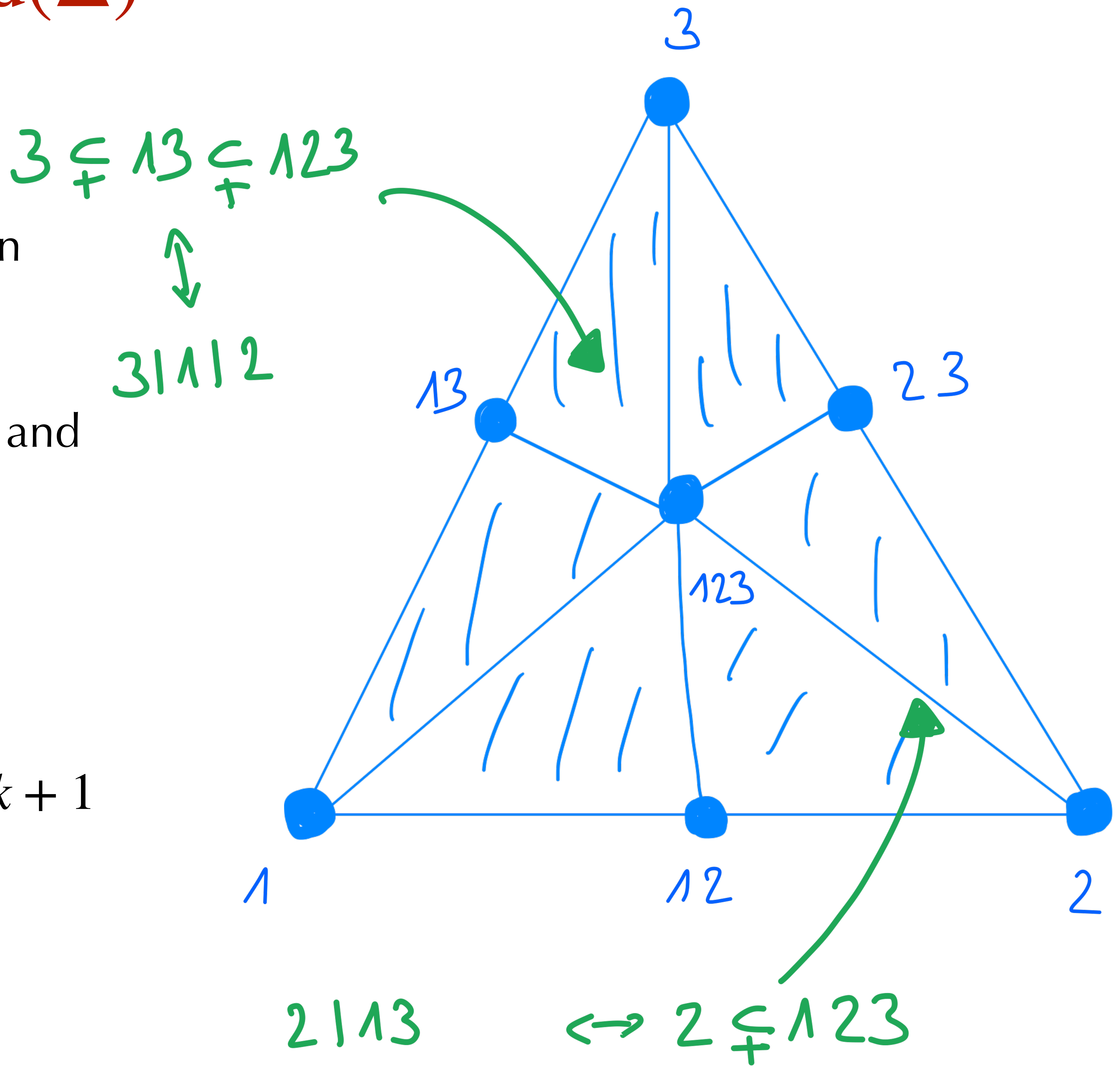
Geometrically:
stellar subdivide faces by decreasing dimension

Combinatorially:
clique complex of graph on vertex set $\Delta \backslash \{\varnothing\}$ and edges (F, G) if $F \subsetneq G$ or $G \subsetneq F$

k-faces = chains
$\varnothing \neq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$
where $F_i \in \Delta$ for $0 \leq i \leq k$

= partial ordered partitions of faces of Δ with $k+1$ blocks

$$B_0 \,|\, B_1 \,|\, \cdots \,|\, B_k$$



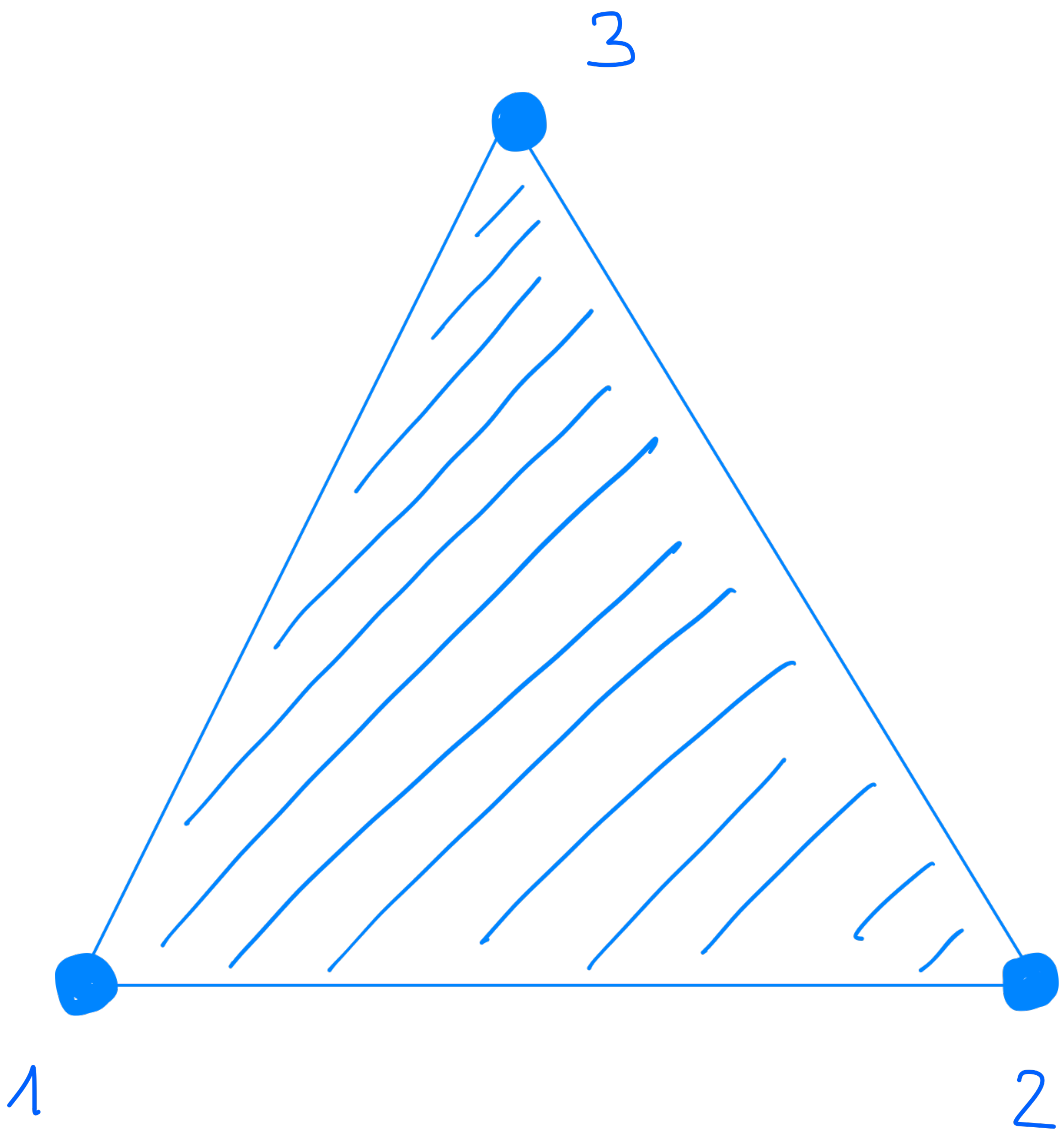
Results for barycentric subdivisions

- f- and h-vector transformations (Brenti, Welker)
- real-rootedness of the h-polynomial if $h(\Delta) \geq 0$ (Brenti, Welker)
- combinatorial interpretations of the γ-vector (Nevo, Petersen, Tenner)
- combinatorial interpretations of the local h-vector (Stanley)
- non-negativity of the local γ-vector for CW-regular subdivisions (Athanasiadis, Savvidou; J., Murai, Sieg)
- almost strong Lefschetz property if Δ is shellable (J., Nevo)

The antiprism triangulation $\mathrm{sd}_{\mathscr{A}}(\Delta)$

Δ simplicial complex on vertex set V

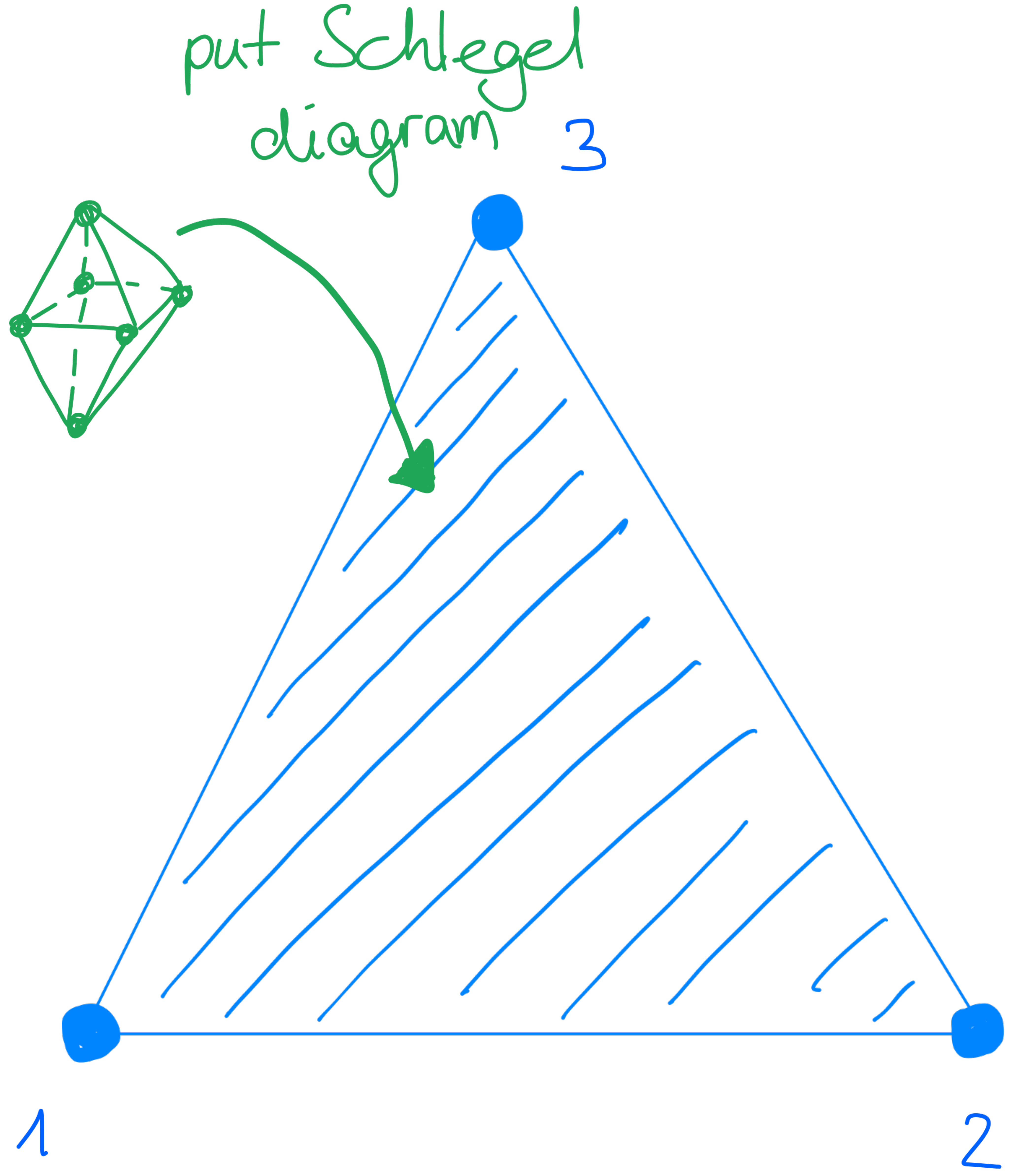
Geometrically:
perform crossings by decreasing dimension



The antiprism triangulation $\mathrm{sd}_{\mathscr{A}}(\Delta)$

Δ simplicial complex on vertex set V

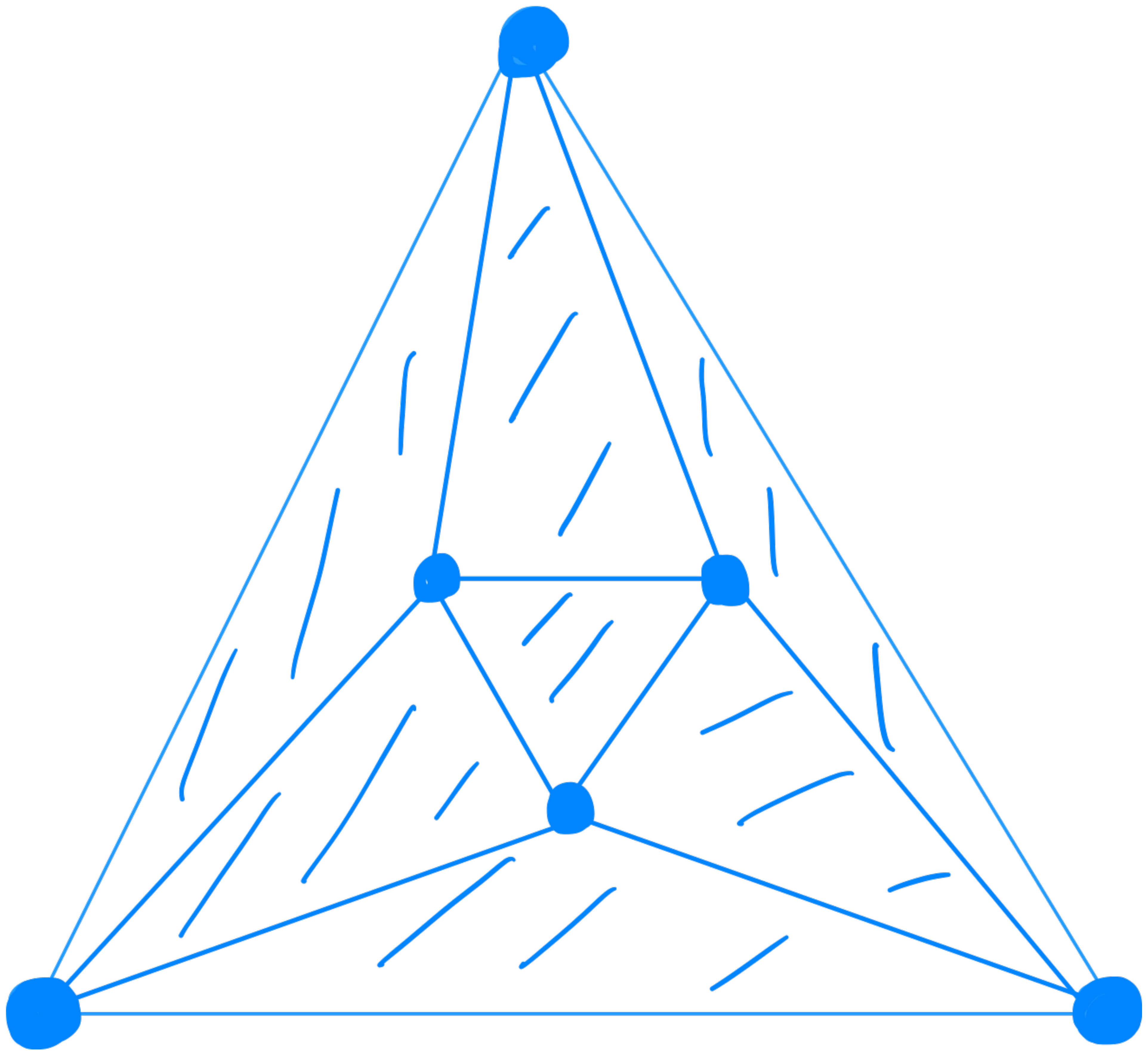
Geometrically:
perform crossings by decreasing dimension



The antiprism triangulation $\mathrm{sd}_{\mathscr{A}}(\Delta)$

Δ simplicial complex on vertex set V

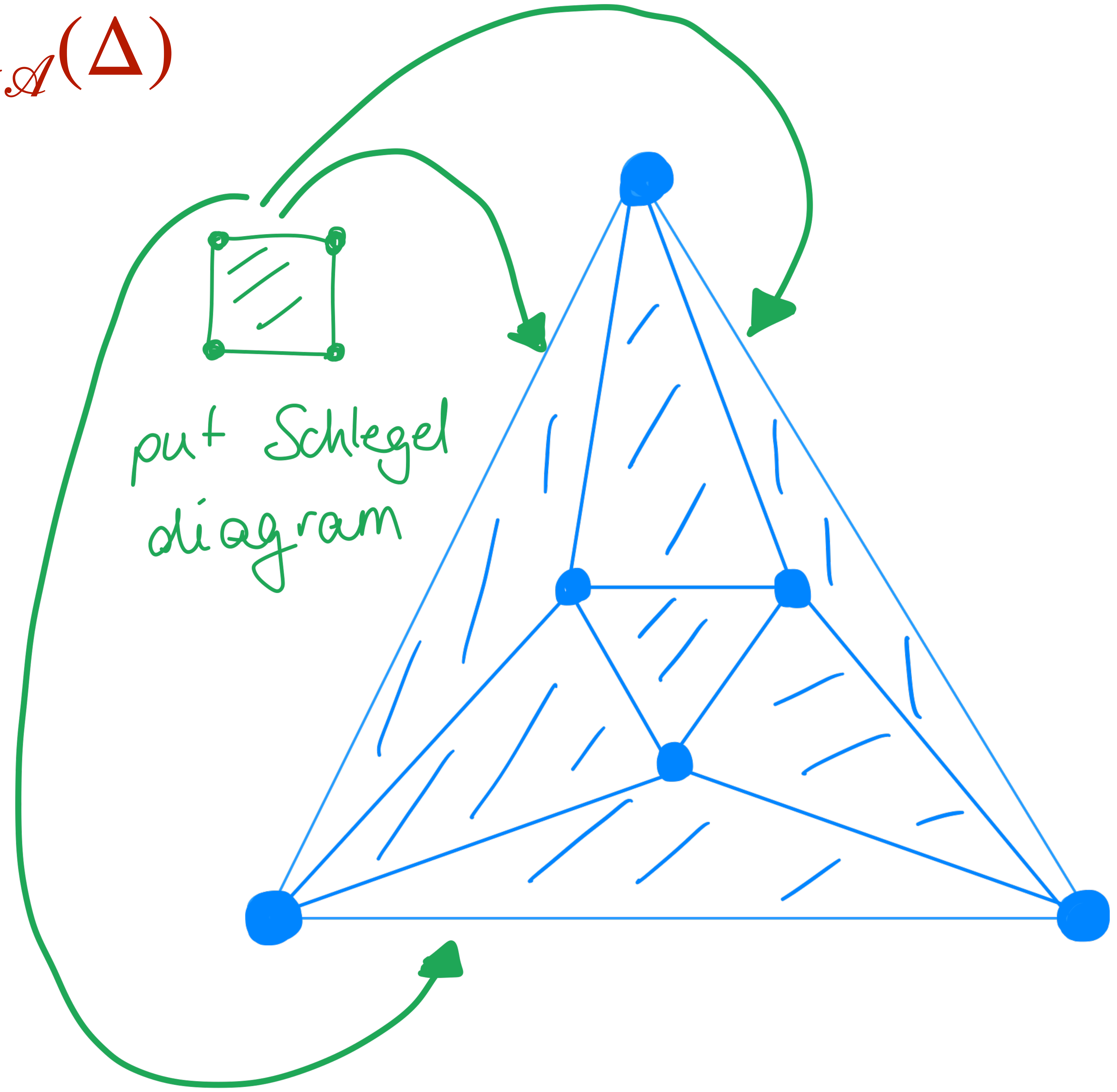
Geometrically:
perform crossings by decreasing dimension



The antiprism triangulation $\mathrm{sd}_{\mathscr{A}}(\Delta)$

Δ simplicial complex on vertex set V

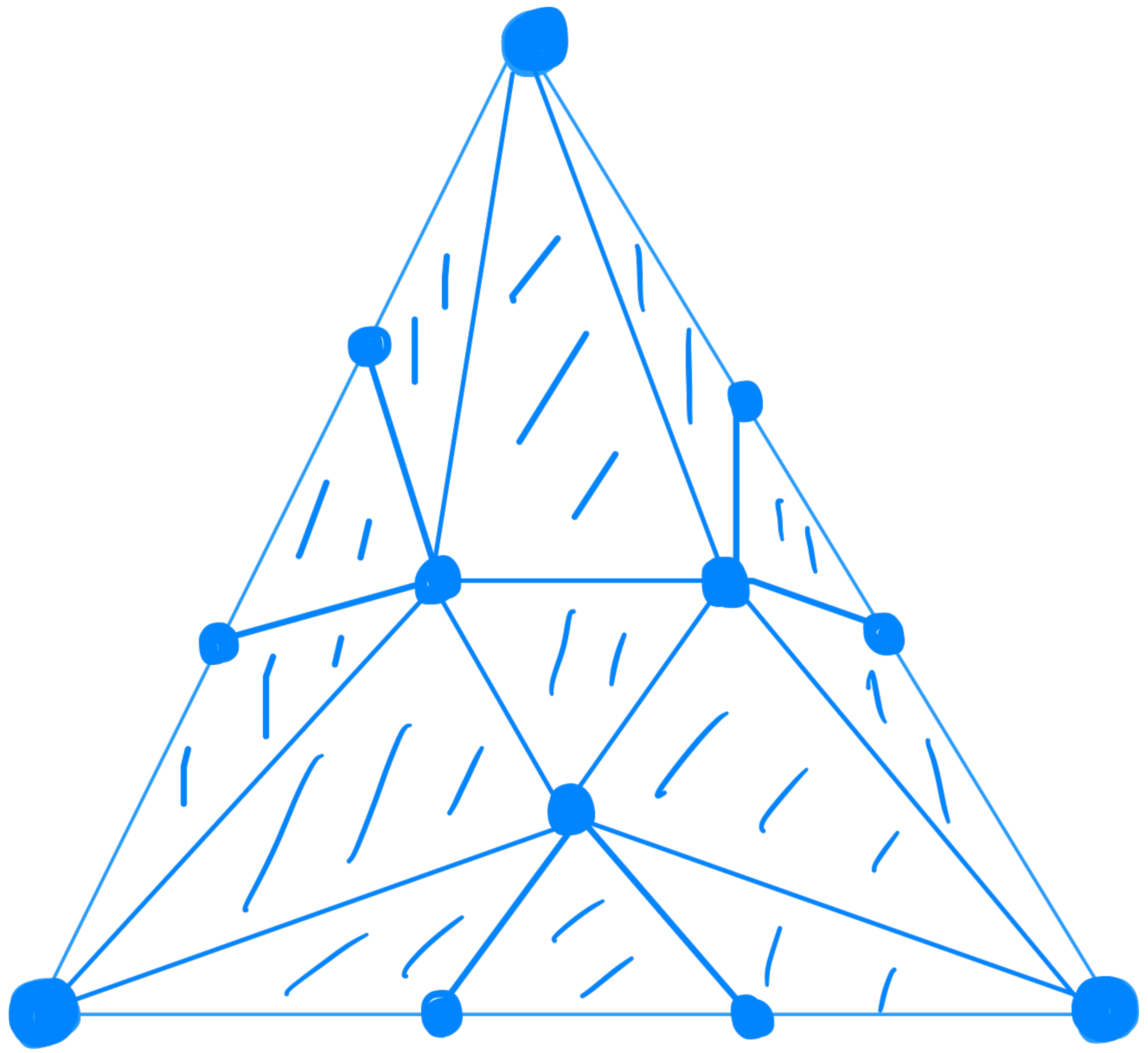
Geometrically:
perform crossings by decreasing dimension



The antiprism triangulation $\mathrm{sd}_{\mathscr{A}}(\Delta)$

Δ simplicial complex on vertex set V

Geometrically:
perform crossings by decreasing dimension



The antiprism triangulation $\mathrm{sd}_{\mathscr{A}}(\Delta)$

Δ simplicial complex on vertex set V

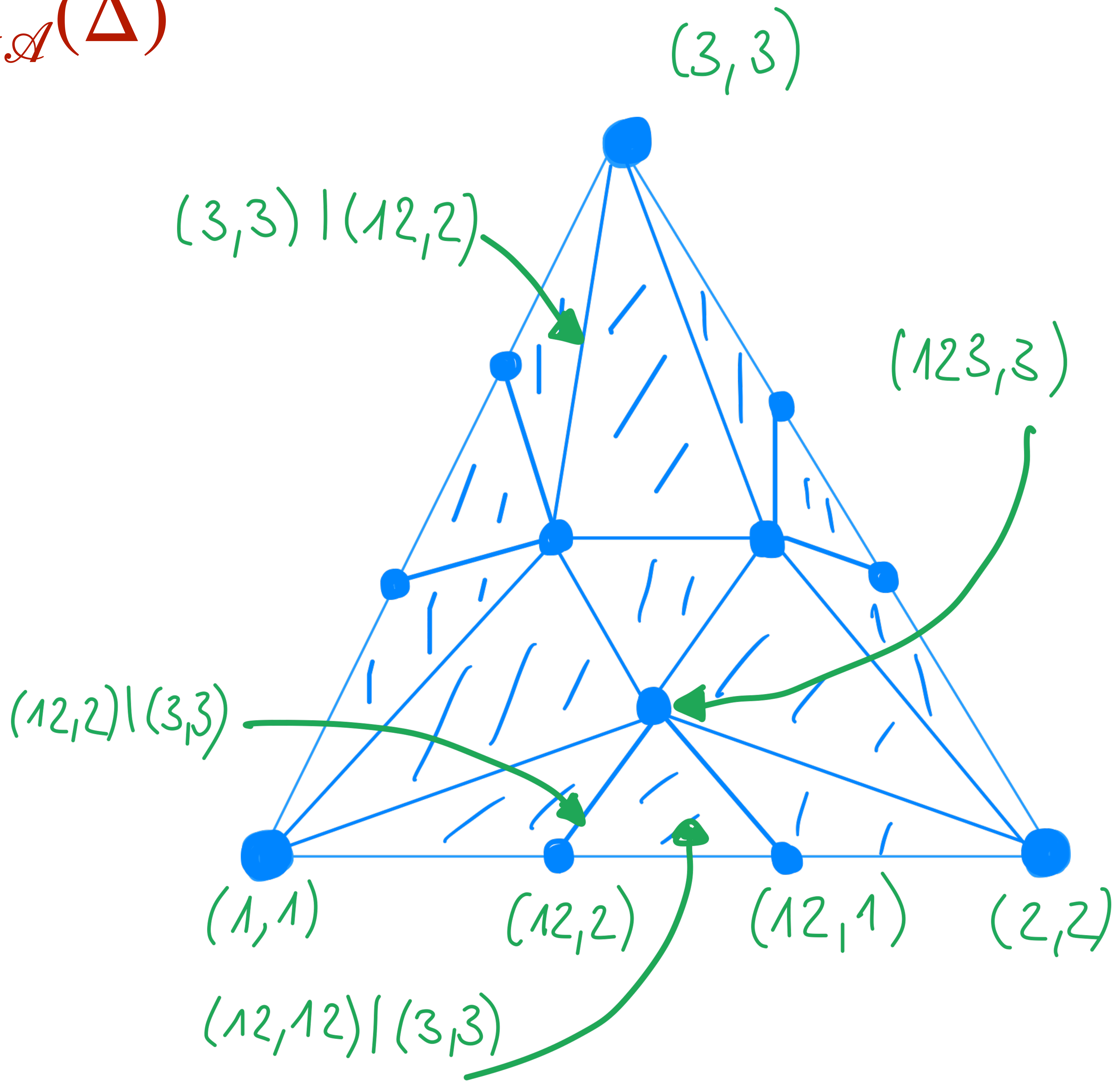
Geometrically:
perform crossings by decreasing dimension

Combinatorially:
clique complex of graph on vertex set:
$\{(F, v) \mid v \in F \in \Delta\}$ and edges $(F, v), (G, w)$ if $F = G$, or $F \subsetneq G$ and $w \in G \backslash F$ or vice versa

k-faces = multipointed ordered partitions of faces of Δ of weight $k + 1$

$$(B_1, C_1) \,|\, (B_2, C_2) \,|\, \cdots \,|\, (B_m, C_m)$$

with $B_i \supsetneq C_i$



Face enumeration

Let Δ be an $(n-1)$-dimensional simplicial complex.

The vector $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{n-1}(\Delta))$, where

$$f_i(\Delta) = \#\{F \in \Delta \mid \dim F = i\}$$

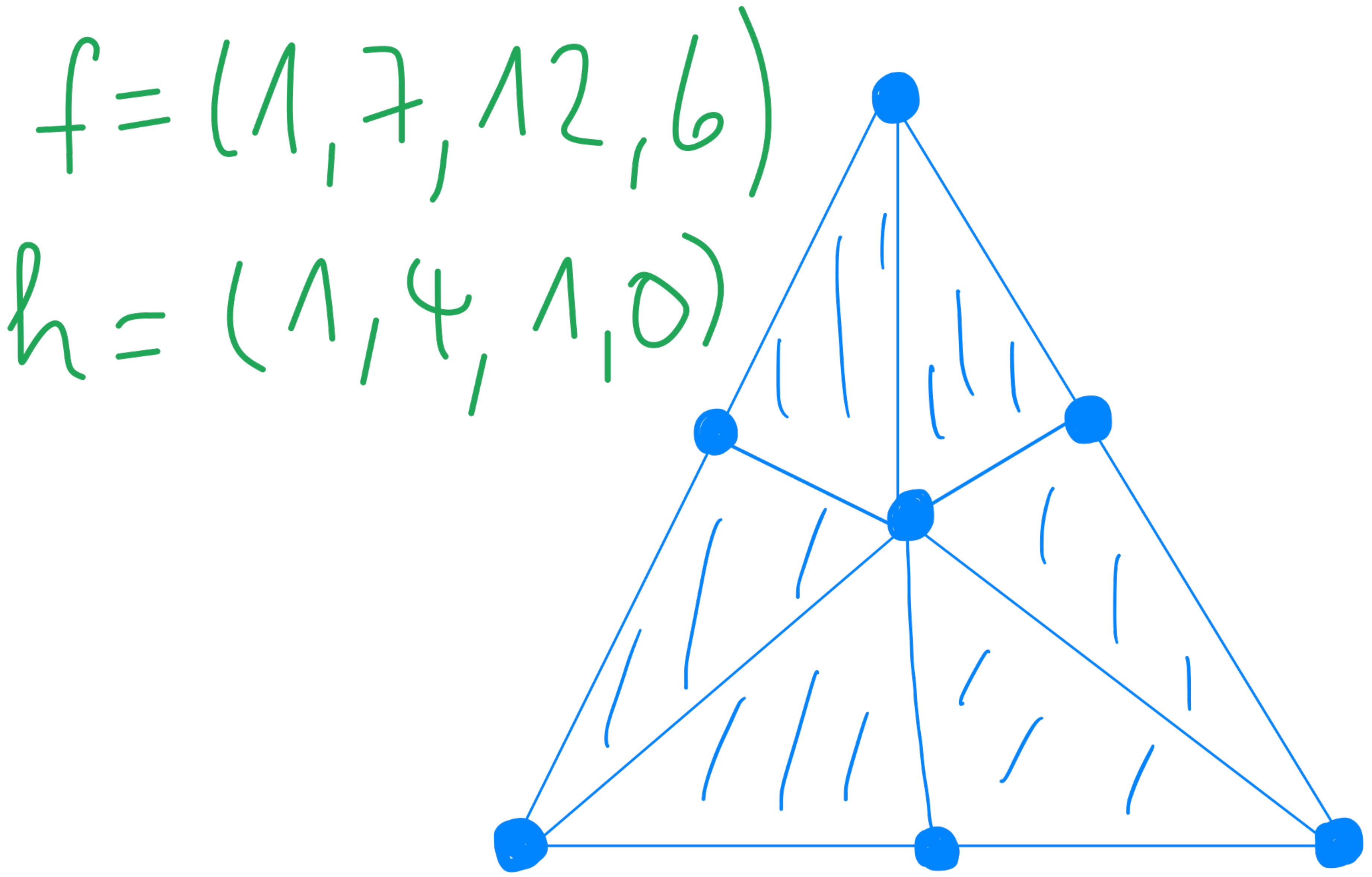
is called *f*-vector of Δ.

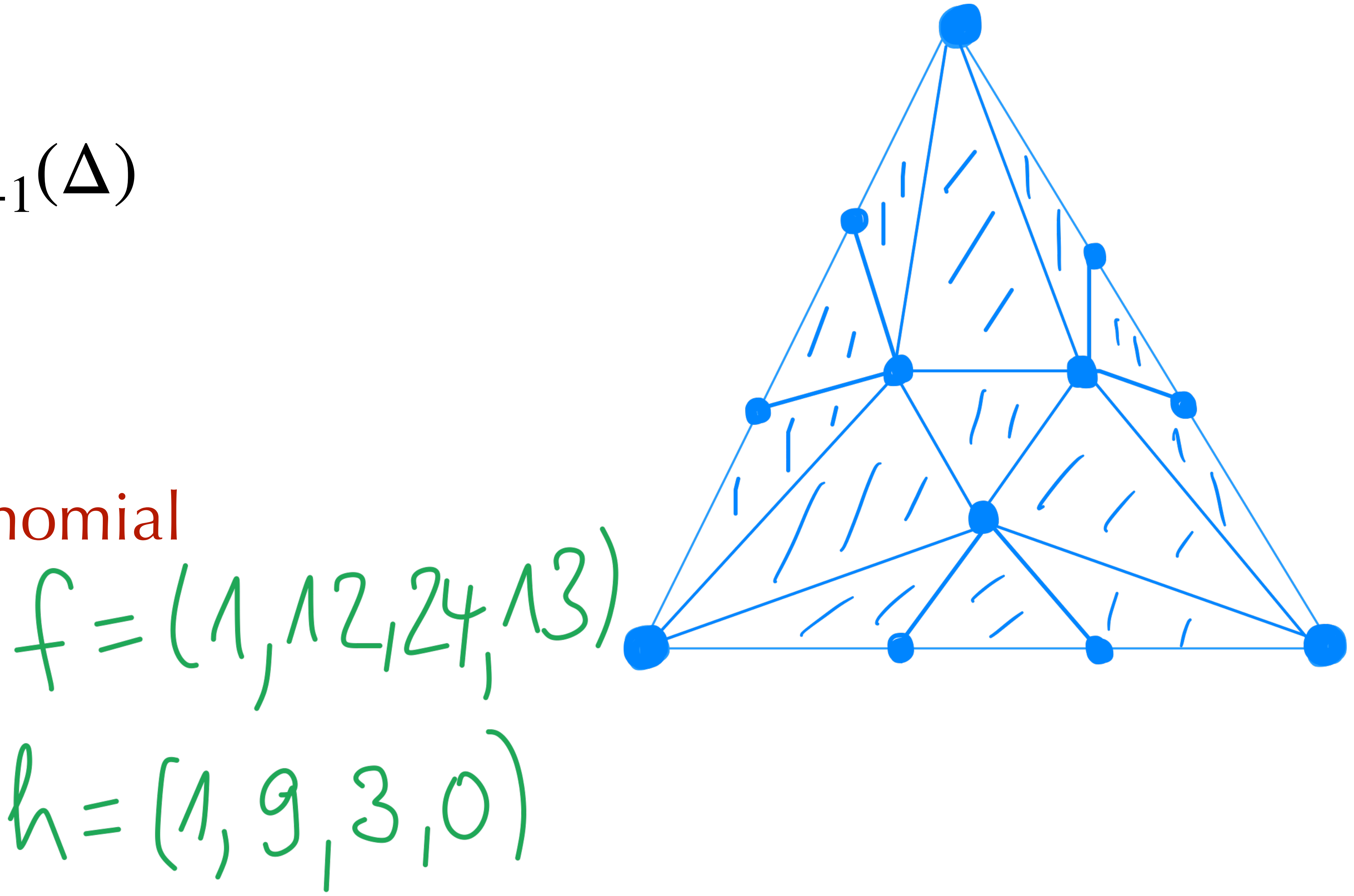
The vector $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta))$, where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{i-j} f_{j-1}(\Delta)$$

is called *h*-vector of Δ.

The polynomial $h(\Delta, x) = \sum_{i=0}^{n} h_i(\Delta) x^i$ is called *h*-polynomial of Δ.





The antiprism triangulation of the simplex

Theorem (Athanasiadis, Brunink, J.; 2020)

Let σ_n be an $(n-1)$-simplex and $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n)) = (h_0, \ldots, h_n)$ be the h-vector of $\mathrm{sd}_{\mathscr{A}}(\sigma_n)$. Then h_i is equal to:

- the number of proper multipointed partial ordered partitions of $[n]$ of weight i,
- the number of ordered partitions $\pi = (B_1 \,|\, \cdots \,|\, B_m)$ of $[n]$ such that $\# \bigcup_{j=1}^{\lfloor m/2 \rfloor} B_j = i$,
- $\binom{n}{i}$ times the number of permutations in $\mathfrak{S}_n$ with excedance set $[i]$.

The antiprism triangulation of the simplex

$n=3$, $i=2$

$h_2 = 3$

Theorem (Athanasiadis, Brunink, J.; 2020)

Let σ_n be an $(n-1)$-simplex and $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n)) = (h_0, \dots, h_n)$ be the h-vector of $\mathrm{sd}_{\mathscr{A}}(\sigma_n)$. Then h_i is equal to:

- the number of proper multipointed partial ordered partitions of $[n]$ of weight i,
- the number of ordered partitions $\pi = (B_1 \mid \cdots \mid B_m)$ of $[n]$ such that $\# \bigcup_{j=1}^{\lfloor m/2 \rfloor} B_j = i$,
- $\binom{n}{i}$ times the number of permutations in $\mathfrak{S}_n$ with excedance set $[i]$. 231

$\binom{n}{i} = 3$

(123|23)

(123|13)

(123|23)

1|2|3

1|3|2

23|1

The antiprism triangulation of the simplex

Theorem (Athanasiadis, Brunink, J.; 2020)

The polynomial $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x)$ is real-rooted and interlaces $h(\mathrm{sd}_{\mathscr{A}}(\sigma_{n+1}), x)$ for every $n \in \mathbb{N}$.

Moreover, $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x)$ has a real-rooted and interlacing symmetric decomposition w.r.t. $n-1$:

$$h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) = h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x) + (h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) - h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x)).$$

The antiprism triangulation of the simplex

Theorem (Athanasiadis, Brunink, J.; 2020)

The polynomial $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x)$ is real-rooted and interlaces $h(\mathrm{sd}_{\mathscr{A}}(\sigma_{n+1}), x)$ for every $n \in \mathbb{N}$.

Moreover, $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x)$ has a real-rooted and interlacing symmetric decomposition w.r.t. $n-1$:

$$h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) = h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x) + (h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) - h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x)).$$

$n=3$: $1 + 9x + 3x^2 = (1+7x+x^2) + \underbrace{(2x+2x^2)}_{= x(2+2x)}$

The antiprism triangulation of the simplex

Theorem (Athanasiadis, Brunink, J.; 2020)

The polynomial $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x)$ is real-rooted and interlaces $h(\mathrm{sd}_{\mathscr{A}}(\sigma_{n+1}), x)$ for every $n \in \mathbb{N}$.

Moreover, $h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x)$ has a real-rooted and interlacing symmetric decomposition w.r.t. $n-1$:

$$h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) = h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x) + (h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) - h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x)).$$

Conjecture: $h(\mathrm{sd}_{\mathscr{A}}(\sigma_{n-1}), x)$ interlaces $(h(\mathrm{sd}_{\mathscr{A}}(\sigma_n), x) - h(\mathrm{sd}_{\mathscr{A}}(\partial\sigma_n), x))$ for every $n \in \mathbb{N}$.

This would imply that $h(\mathrm{sd}_{\mathscr{A}}(\Delta), x)$ is real-rooted for every simplicial complex Δ with $h(\Delta) \geq 0$.

Face vector transformation — the f-vector

Let Δ be an $(n-1)$-dimensional simplicial complex. Then

$$f_{j-1}(\mathrm{sd}_{\mathscr{A}}(\Delta)) = \sum_{k=j}^{n} q_{\mathscr{A}}(k,j) f_{k-1}(\Delta),$$

where $q_{\mathscr{A}}(k,j)$ equals the number of multipointed ordered partitions of $[k]$ of weight j:

$$q_{\mathscr{A}}(k,j) = \binom{k}{j} \sum_{i=0}^{j} i!S(j,i)i^{k-j}.$$

Remark: The existence and non-negativity of the numbers $q_{\mathscr{A}}(k,j)$ follows from a more general result by Athanasiadis for uniform triangulations.

Face vector transformation — the h-vector

Let Δ be an $(n-1)$-dimensional simplicial complex. Then

$$h_j(\mathrm{sd}_{\mathscr{A}}(\Delta)) = \sum_{k=0}^{n} p_{\mathscr{A}}(n,k,j) h_k(\Delta),$$

where $p_{\mathscr{A}}(n,k,j)$ equals the number of ordered partitions π of sets $[k] \subseteq S \subseteq [n]$ with:

- j elements colored black and the remaining ones white, and:
- If π has a monochromatic block B, then:

    B is the first block,

    all elements of B are black, and

    $B \subseteq [k]$.

Face vector transformation — the h-vector

Let Δ be an $(n-1)$-dimensional simplicial complex. Then

$$h_j(\mathrm{sd}_{\mathscr{A}}(\Delta)) = \sum_{k=0}^{n} p_{\mathscr{A}}(n,k,j) h_k(\Delta),$$

where $p_{\mathscr{A}}(n,k,j)$ equals the number of ordered partitions π of sets $[k] \subseteq S \subseteq [n]$ with:

- j elements colored black and the remaining ones white, and:
- If π has a monochromatic block B, then:

  B is the first block,

  all elements of B are black, and

  $B \subseteq [k]$.

$n=3,\ k=0,\ j=1$

$[k] = \emptyset$

12 12

1 23 123 123

23 32

13 31

$p_A(3,0,1) = 9$

$= h_1(\mathrm{sd}_A(B_3))$

Lefschetz properties

Let Δ be an $(n-1)$-dimensional Cohen-Macaulay omplex with Stanley-Reisner ring $\mathbb{F}[\Delta]$.

Δ is called almost strong Lefschetz if there exists an linear system of parameters Θ and a linear form ω such that

$$\times\, \omega^{n-2i-1} : (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_i \to (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{n-1-i}$$

$$f \mapsto \omega^{n-2i-1} \cdot f$$

is injective for $0 \le i \le \lfloor (n-1)/2 \rfloor$.

Lefschetz properties

Let Δ be an $(n-1)$-dimensional Cohen-Macaulay omplex with Stanley-Reisner ring $\mathbb{F}[\Delta]$.

Δ is called almost strong Lefschetz if there exists an linear system of parameters Θ and a linear form ω such that

$$\times\,\omega^{n-2i-1} : (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_i \to (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{n-1-i}$$
$$f \mapsto \omega^{n-2i-1}\cdot f$$

is injective for $0 \leq i \leq \lfloor (n-1)/2 \rfloor$.

Consequences:

- $g(\Delta) = (1, h_1(\Delta) - h_0(\Delta), \ldots, h_{\lfloor n/2 \rfloor}(\Delta) - h_{\lfloor n/2 \rfloor - 1}(\Delta))$ is an M-sequence.
- $h_i(\Delta) \leq h_{n-1-i}(\Delta)$ for $0 \leq i \leq \lfloor (n-1)/2 \rfloor$
- $h_0(\Delta) \leq h_1(\Delta) \leq \cdots \leq h_{\lfloor n/2 \rfloor}(\Delta)$

Lefschetz property for antiprism triangulations

Theorem (Athanasiadis, Brunink, J.; 2020)

Let Δ be a shellable simplicial complex. Then $\mathrm{sd}_{\mathscr{A}}(\Delta)$ is almost strong Lefschetz.

In particular, $h(\mathrm{sd}_{\mathscr{A}}(\Delta))$ is unimodal.

Lefschetz property for antiprism triangulations

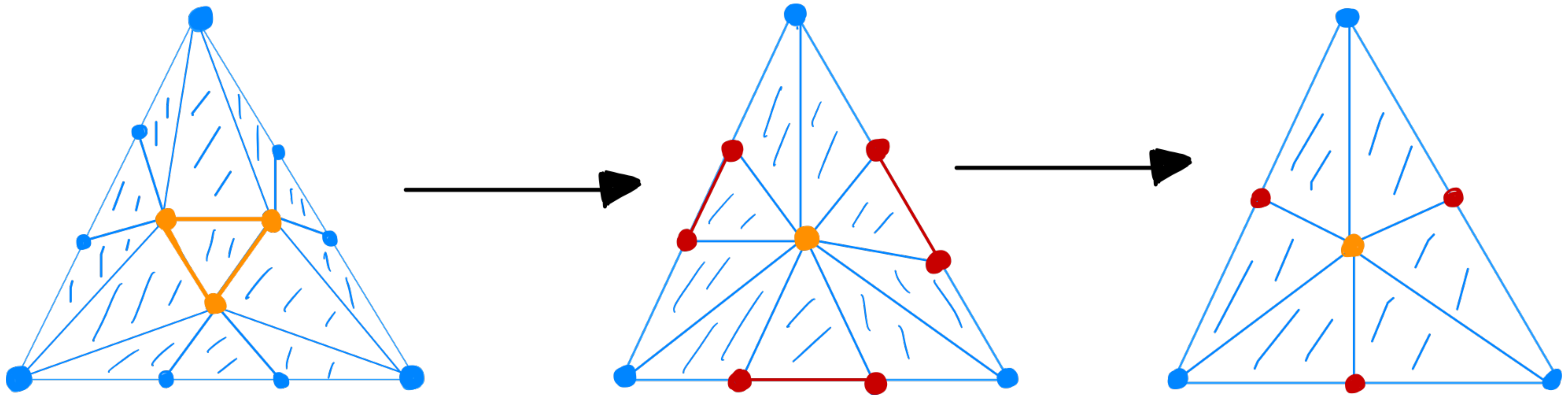
Theorem (Athanasiadis, Brunink, J.; 2020)

Let Δ be a shellable simplicial complex. Then $\mathrm{sd}_{\mathscr{A}}(\Delta)$ is almost strong Lefschetz.

In particular, $h(\mathrm{sd}_{\mathscr{A}}(\Delta))$ is unimodal.

Sketch of the proof:

- Induction on the number of facets of Δ and $\dim \Delta$
- Base case: Using nice edge contractions $\mathrm{sd}_{\mathscr{A}}(\sigma_n)$ can be transformed into $\mathrm{sd}(\sigma_n)$, which is known to be almost strong Lefschetz.



Lefschetz property for antiprism triangulations

Theorem (Athanasiadis, Brunink, J.; 2020)

Let Δ be a shellable simplicial complex. Then $\mathrm{sd}_{\mathscr{A}}(\Delta)$ is almost strong Lefschetz.

In particular, $h(\mathrm{sd}_{\mathscr{A}}(\Delta))$ is unimodal.

Sketch of the proof:

- Induction on the number of facets of Δ and $\dim \Delta$
- Base case: Using nice edge contractions $\mathrm{sd}_{\mathscr{A}}(\sigma_n)$ can be transformed into $\mathrm{sd}(\sigma_n)$, which is known to be almost strong Lefschetz.
- The induction step works as in the case for barycentric subdivisions

Conclusion

Barycentric subdivision vs. antiprism triangulation

	Barycentric subdivision	Antiprism triangulation
f- and h-vector transformations	✓	✓
Real-rootedness	✓	conjectured
Local h-vector	✓	✓
Lefschetz property	✓	✓

Conclusion

Barycentric subdivision vs. antiprism triangulation

	Barycentric subdivision	Antiprism triangulation
f- and h-vector transformations	✓	✓
Real-rootedness	✓	conjectured
Local h-vector	✓	✓
Lefschetz property	✓	✓

Thank you!