Formulas for Macdonald polynomials arising from the ASEP

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with Sylvie Corteel, Jim Haglund, Sarah Mason, and Lauren Williams

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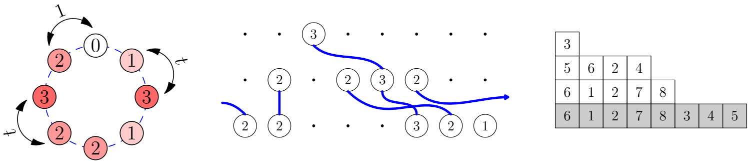
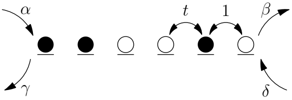


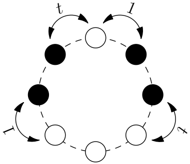
overview

1. stat mech: asymmetric simple exclusion process (ASEP)
2. orthogonal polynomials: Macdonald polynomials
3. combinatorics: multiline queues

I. asymmetric simple exclusion process (ASEP)

- the ASEP is a particle process describing particles hopping on a finite 1D lattice: 1 particle per site, at each time step any two adjacent particles may swap with some probability, with possible interactions at the boundary





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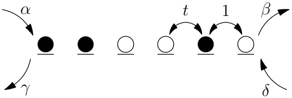

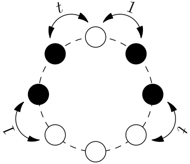

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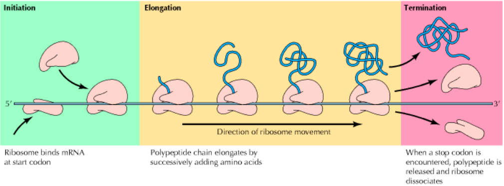

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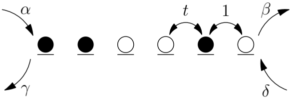

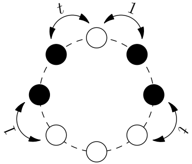

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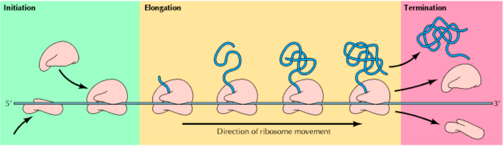


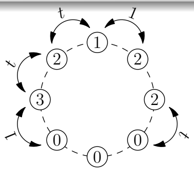
Ribosome binds mRNA at start codon

Polypeptide chain elongates by successively adding amino acids

When a stop codon is encountered, polypeptide is released and ribosome dissociates

- ASEP has beautiful combinatorial structure, deep connections to orthogonal polynomials (Askey-Wilson, Macdonald, Koornwinder). Also connected to random matrix theory, total positivity on the Grassmanian, other statistical mechanics models such as the six-vertex model and the XXZ model..

today's setting: multispecies ASEP on a circle



- now we have particles of types $0, 1, \ldots, L$ with J_i particles of type i, represent the type by $\lambda = (L^{J_L}, \ldots, 1^{J_1}, 0^{J_0})$.

(Here $\lambda = (3, 2, 2, 2, 1, 0, 0, 0)$)

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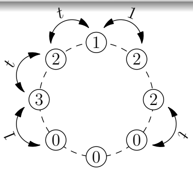

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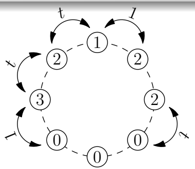

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- possible transitions between states are swaps of adjacent particles:

$$X \textcircled{\scriptsize A} \textcircled{\scriptsize B} Y \underset{t}{\overset{1}{\rightleftharpoons}} X \textcircled{\scriptsize B} \textcircled{\scriptsize A} Y$$

  when $A < B$. $(0 \leq t \leq 1)$

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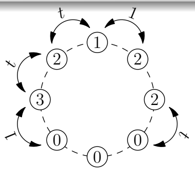

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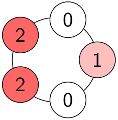
- main question: find an explicit formula for the **stationary probabilities**, i.e. the left eigenvector corresponding to eigenvalue 1 of the transition matrix.

stationary probabilities: an example for $\lambda = (2, 2, 1, 0, 0)$



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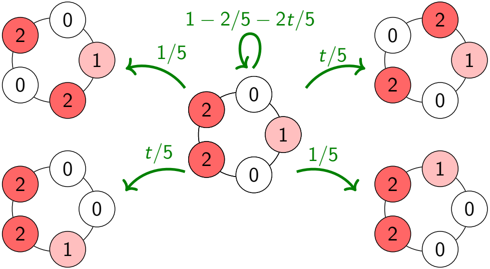

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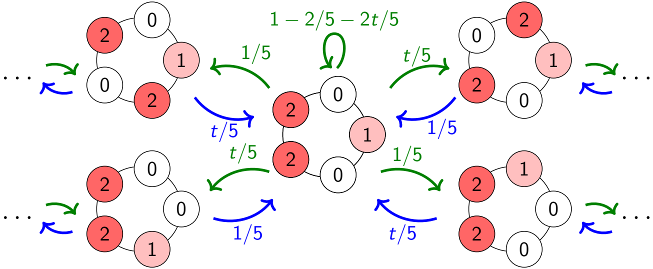

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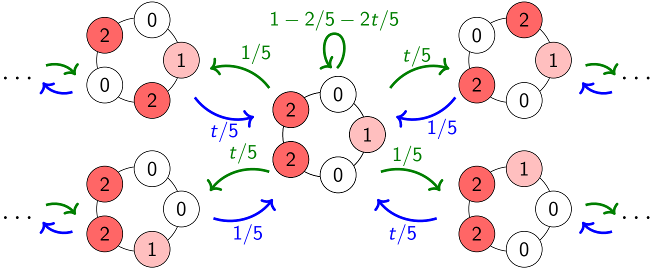

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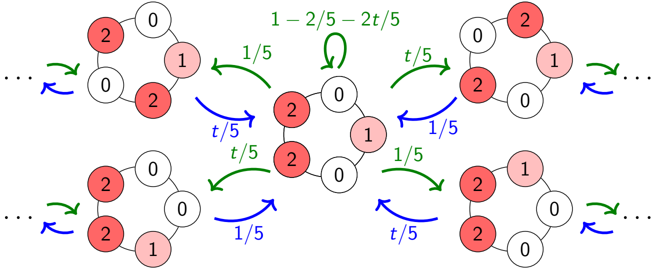

we compute the stationary probabilities:

$$\Pr(2,0,1,0,2) = \frac{1}{Z}(3 + 7t + 7t^2 + 3t^3) \qquad \Pr(0,2,1,0,2) = \frac{1}{Z}(5 + 6t + 7t^2 + 2t^3)$$

$$\Pr(2,1,0,0,2) = \frac{1}{Z}(6 + 7t + 6t^2 + t^3) \qquad \Pr(2,0,0,1,2) = \frac{1}{Z}(1 + 6t + 7t^2 + 6t^3)$$

$$\Pr(2,1,2,0,0) = \frac{1}{Z}(3 + 7t + 7t^2 + 3t^3) \qquad \Pr(2,0,1,2,0) = \frac{1}{Z}(2 + 7t + 6t^2 + 5t^3)$$

$Z = \sum_{\mu} \tilde{Pr}(\mu) = 100 + 200t + 200t^2 + 100t^3$ (partition function)

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- $s_\lambda = \sum_\sigma x^\sigma$ where σ is a semi-standard filling of the Young diagram of shape λ

  E.g. the following are the fillings of shape $(2, 1)$ on 3 letters:

$$\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & \\ \hline 2 & 3 \\ \hline\end{array}$$

$$s_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Macdonald polynomials

- In 1988, Macdonald introduced a new family of homogeneous symmetric polynomials $\{P_\lambda(X; q, t)\}_{\lambda \vdash n}$ in $\Lambda(q, t)$, uniquely determined by the following properties:
    - orthogonal basis for $\Lambda(q, t)$ with respect to the Macdonald inner product $\langle , \rangle_{q,t}$
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- Example:

$$P_{(2,1)}(x_1,x_2,x_3;q,t) = m_{(2,1)} + \frac{(1-t)(2+q+t+2qt)}{1-qt^2} m_{(1,1,1)}.$$

where $m_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots$ and $m_{(1,1,1)} = x_1x_2x_3 + x_2x_2x_4 + \cdots$

some properties of Macdonald polynomials

- $P_\lambda(X; q, t)$ specializes to..
  - Schur functions s_λ at $q = t$
  - Hall-Littlewood polynomials at $q = 0$
  - Jack polynomials at $t = q^\alpha$ and $q \to 1$

$$\begin{tikzcd}
& P_\lambda \arrow[dl, "q=0"'] \arrow[dd, "q=t"] \arrow[dr, "t=0"] & \\
\text{Hall–Littlewood} \arrow[dr, "t=0"] & & q\text{-Whittacker} \arrow[dl, "q=0"] \\
& s_\lambda &
\end{tikzcd}$$

- Haglund–Haiman–Loehr gave the first combinatorial formula in 2005 (HHL tableaux), other formulas by Ram–Yip '11, Lenart '14
- there is a nonsymmetric version E_μ, indexed by compositions
  - introduced by Macdonald in 1995 to study the P_λ's, further studied by Opdam–Heckman, Cherednik

Macdonald polynomials and statistical mechanics

Macdonald polynomials and statistical mechanics

Theorem (Cantini-de Gier-Wheeler '15)

At $x_1 = \cdots = x_n = q = 1$:

- *Let λ be a partition. P_λ specializes to the partition function of the ASEP on a circle with particle types given by λ:*

  $$P_\lambda(1,\ldots,1;1,t) = \sum_\mu \tilde{\mathrm{Pr}}(\mu)$$

  where the sum is over **compositions** *μ that are rearrangements of the parts of λ*

- *When μ is a partition,*

  $$E_\mu(1,\ldots,1;1,t) = \tilde{\mathrm{Pr}}(\mu).$$

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Angel '08, Ferrari-Martin '07 ($t = 0$ case), Martin '18 (current setting)

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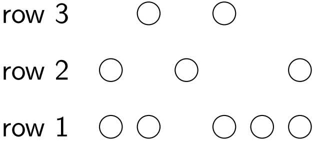

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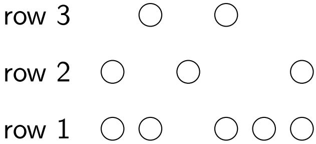

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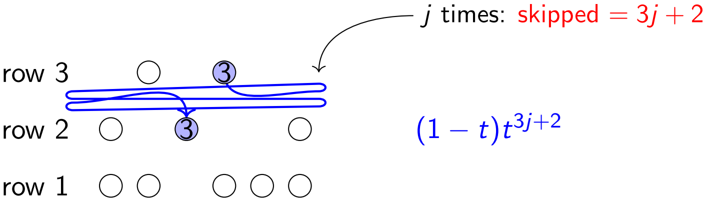

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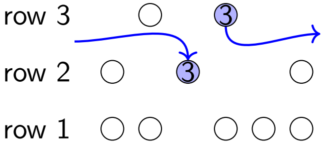

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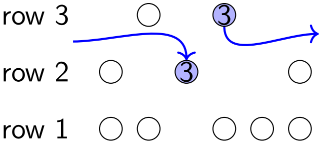

skipped

$$(1-t)\sum_j t^{3j+2} = \frac{t^2(1-t)}{1-t^3}$$

available

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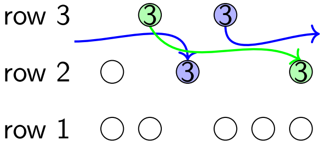

$\frac{t^2(1-t)}{1-t^3} \cdot \frac{(1-t)}{1-t^2}$

III. combinatorics: multiline queues

Angel '08, Ferrari-Martin '07 ($t = 0$ case), Martin '18 (current setting)

- A **multiline queue (MLQ)** for particles of types $0, 1, \ldots, L$ on an ASEP of n locations is:
  - a ball system on a cylinder of L rows and n columns,
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- Each ball chooses an available ball **weakly to the right** to pair with in the row below. t counts the number of available balls skipped: assign weight $t^{\text{total skipped}}(1-t)$.
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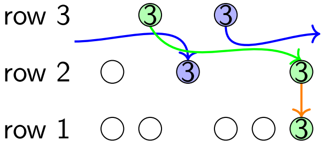

$$\frac{t^2(1-t)}{1-t^3} \cdot \frac{(1-t)}{1-t^2} \cdot 1$$

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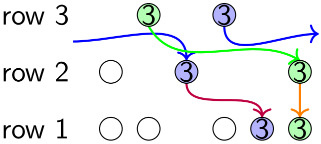

$\frac{t^2(1-t)}{1-t^3} \cdot \frac{(1-t)}{1-t^2} \cdot 1 \cdot \frac{t(1-t)}{1-t^4}$

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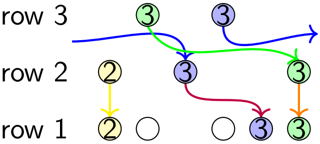

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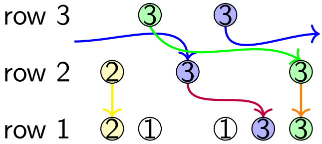

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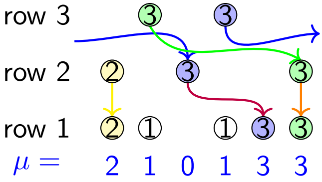
- The **state of the multiline queue** is read off Row 1.



$$\frac{t^2(1-t)}{1-t^3} \cdot \frac{(1-t)}{1-t^2} \cdot 1 \cdot \frac{t(1-t)}{1-t^4} \cdot 1$$

$$\mathrm{wt}(Q) = \prod_{\text{pairing}} t^{\text{skipped}} \frac{(1-t)}{1-t^{\text{available}}}$$

$$= \frac{t^3(1-t)^4}{(1-t^4)(1-t^3)(1-t^2)}$$

multiline queues and the ASEP on a circle

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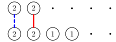
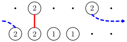
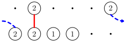


1



$\frac{(1-t)}{1-t^3}$



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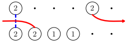

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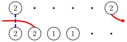

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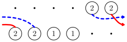
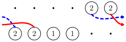


$\frac{(1-t)^2}{(1-t^4)(1-t^3)}$



$\frac{t(1-t)^2}{(1-t^4)(1-t^3)}$

$$\Pr(2,2,1,1,0,0) = \frac{(1-t)(1-t^2)(t^4+t^3+6t^2+t+6)}{(1-t^3)(1-t^4)}$$

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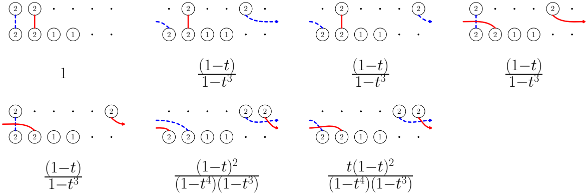

$$\Pr(2,2,1,1,0,0) = \frac{(1-t)(1-t^2)(t^4+t^3+6t^2+t+6)}{(1-t^3)(1-t^4)}$$

Corollary (Cantini–de Gier–Wheeler '15)

The Macdonald polynomial at $x_1 = \cdots = x_n = q = 1$ *equals:*

$$P_\lambda(1,\ldots,1;1,t) = \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q)$$

putting the “q” in the queue (Corteel–M–Williams ’18)

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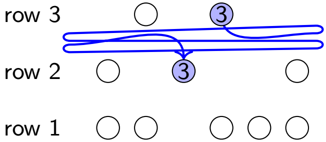
- A pairing (of type ℓ, from row r) that wraps around contributes $q^{\ell-r+1}$



$$(1-t)\sum_j t^{3j+2}q^{j+1} = \frac{qt^2(1-t)}{1-qt^3}$$

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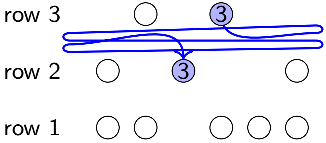

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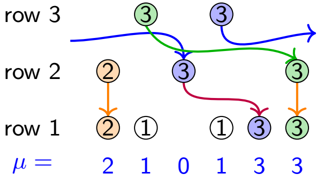

$$\frac{qt^2(1-t)}{1-qt^3} \cdot \frac{(1-t)}{1-qt^2} \cdot 1 \cdot \frac{t(1-t)}{1-q^2t^4} \cdot 1$$

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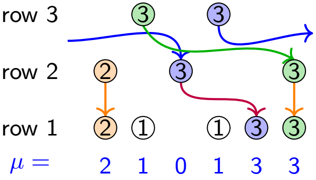
- Weight for each pairing is $t^{\text{skipped}} q^{(\ell-r+1)\delta_{\text{wrap}}} \frac{1-t}{1-q^{\ell-r+1}t^{\text{free}}}$
- Define the x-weight of a queue M to be $x^Q = \prod_j x_j^{\#}$ balls in col j



$$x^Q = x_1^2x_2^2x_3x_4^2x_5x_6^2$$

$$\frac{qt^2(1-t)}{1-qt^3} \cdot \frac{(1-t)}{1-qt^2} \cdot 1 \cdot \frac{t(1-t)}{1-q^2t^4} \cdot 1$$

$$\mathrm{wt}(Q)(X;q,t) = x^Q t^{\text{skipped}} \prod_{\text{pairings}} q^{(\ell-r+1)\delta_{\text{wrap}}} \frac{1-t}{1-q^{\ell-r+1}t^{\text{free}}}$$

$$= x_1^2x_2^2x_3x_4^2x_5x_6^2 \frac{qt^3(1-t)^4}{(1-q^2t^4)(1-qt^3)(1-qt^2)}$$

Theorem (CMW '18)

1. *For a partition μ,*

$$E_\mu(\mathbf{x}; q, t) = \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q)(\mathbf{x}; q, t)$$

   (in general, when μ is a composition, we get certain **permuted basement Macdonald polynomials $\mathbf{E}^{\sigma}_{\mathrm{sort}(\mu)}$***)*

2. *For a partition λ,*

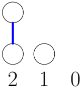
$$P_\lambda(\mathbf{x}; q, t) = \sum_{\mu} \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q)(\mathbf{x}; q, t)$$

   where μ is a composition that is a permutation of the parts of λ

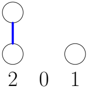
3. *(Corollary, Martin '18)*

$$\Pr(\mu) = \frac{1}{Z} \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q)(1, 1, \ldots, 1; 1, t)$$

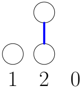
example for $P_{2,1}(x_1, x_2, x_3; q, t)$



$x_1^2x_2$



$x_1^2x_3$



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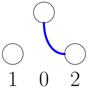
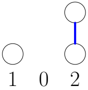


$\frac{(1-t)}{(1-qt^2)}x_1x_2x_3$



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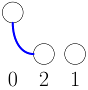

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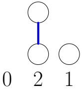
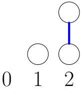


$x_2^2x_3$



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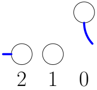
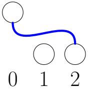


$\frac{q(1-t)}{(1-qt^2)}x_1x_2x_3$



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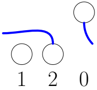

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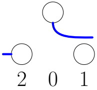

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$$P_{2,1}(x_1, x_2, x_3; q, t) = m_{(2,1)} + \frac{(2+t+q+2qt)(1-t)}{(1-qt^2)} m_{(1,1,1)}$$

our formula is "more compact" than other known formulas for P_λ!

Other formulas motivated by multiline queues

(Corteel–Haglund–M–Mason–Williams '19)

- a new, compact formula for the modified Macdonald polynomials $\widetilde{H}_\lambda(X; q, t)$:
    - $\widetilde{H}_\lambda$ is obtained via plethysm from a normalized form of $P_\lambda(X; q, t)$:

$$\widetilde{H}_\lambda(X; q, t) = t^{n(\lambda)} J_\lambda \left[\frac{X}{1 - t^{-1}}; q, t^{-1} \right]$$

    - our formula is inspired by the combinatorial interpretation of plethysm on multiline queues
- a new quasisymmetric Macdonald polynomial

quasisymmetric functions

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- now, $f \in$ QSym if for any composition α, the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of $x_{b_1}^{\alpha_1} x_{b_2}^{\alpha_2} \cdots x_{b_k}^{\alpha_k}$ for any sequence of integers $1 \leq b_1 < b_2 < \cdots < b_k$. (i.e. shift invariant)

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- Example:

$$x_1^3 x_2 + x_1^3 x_3 + x_2^3 x_3 \in \mathrm{QSym}_3$$

(this would be in Λ_3 if the terms $x_1 x_2^3$, $x_1 x_3^3$, and $x_2 x_3^3$ were also included)

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- give a useful toolbox for studying symmetric functions (e.g. Hopf algebras)

new quasisymmetric Macdonald polynomial

Definition (Corteel–Haglund–M–Mason–Williams)

Define $f_\mu = \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q)$.

Let γ be a strong composition (no zero parts), and define

$$G_\gamma(\mathbf{x}; q, t) = \sum_{\mu \,:\, \mu^+ = \gamma} f_\mu(\mathbf{x}; q, t)$$

where the sum is over (weak) compositions μ that compress to γ.

Examples:

$$G_{(2,1)}(x_1, x_2, x_3) = f_{2,1,0} + f_{2,0,1} + f_{0,2,1}, \qquad G_{(1,2)}(x_1, x_2, x_3) = f_{1,2,0} + f_{1,0,2} + f_{0,1,2}$$

and

$$G_{(2,1)}(\mathbf{x}; q, t) = M_{(2,1)} + \frac{(1-t)(1+q+qt)}{1-qt^2} M_{(1,1,1)}$$

$$G_{(1,2)}(\mathbf{x}; q, t) = M_{(1,2)} + \frac{(1-t)(1+t+qt)}{1-qt^2} M_{(1,1,1)}$$

Theorem (CHMMW, 2019)

- $G_\gamma(X; q, t) \in \mathrm{QSym}$

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- $G_\gamma(X; q, t) \in \mathrm{QSym}$
- *The Macdonald polynomial* P_λ *equals the sum* $\sum_\gamma G_\gamma$ *where* γ *is a permutation of the parts of* λ

Theorem (CHMMW, 2019)

- $G_\gamma(X; q, t) \in \mathrm{QSym}$
- *The Macdonald polynomial* P_λ *equals the sum* $\sum_\gamma G_\gamma$ *where* γ *is a permutation of the parts of* λ
- G_γ *is a* q, t*-generalization of the quasisymmetric Schur functions introduced by Haglund–Luoto–Mason–van Willigenburg '11:*

$$G_\gamma(X; 0, 0) = QS_\gamma(X).$$

Further remarks

- what are some nice properties and applications of G_γ?

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- what can G_γ tell us about the behavior of the ASEP, and vice versa?
- does q have any meaning in the ASEP world?

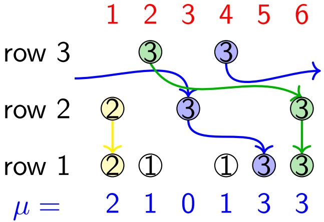
thank you!

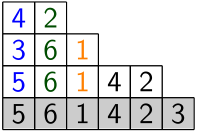
- From multiline queues to Macdonald polynomials (with Corteel and Williams), arXiv:1811.01024
- Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials (with Corteel, Haglund, Mason, and Williams), arXiv:2004.11907

natural bijection from multiline queues to **queue tableaux**

there is a natural bijection from multiline queues to tableaux of the flavor of the Haglund-Haiman-Loehr tableaux.

- let Q be a multiline queue of type μ, and let λ be the rearrangement of μ in decreasing order. Set T to be a Young tableau of shape λ' in French notation, with $\lambda_i + 1$ boxes in column i.
- fill column i of T with the column numbers of balls in the i'th string of balls, from top to bottom





4	2				
3	6	1			
5	6	1	4	2	
5	6	1	4	2	3

$$\lambda = (3, 3, 2, 1, 1, 0)$$

method of proof

- Recall $f_\mu(x_1, \ldots, x_n; q, t) = \sum_{Q \in \mathrm{MLQ}(\mu)} \mathrm{wt}(Q)$.
- E_μ are simultaneous eigenfunctions of Cherednik operators: certain products of Demazure–Luztig operators, which are generators for the affine Hecke algebra of type A_{n-1}:

$$(T_i - t)(T_i + 1) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ if } |i-j| > 1$$

$$T_i f = tf - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(f - s_i f),$$

$$Y_i = T_i \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1}, \qquad Y_i E_\mu = \phi_i(\mu) E_\mu$$

- We show that:

$$T_i f_\mu = \begin{cases} f_{s_i \mu}, & \text{when } \mu_i > \mu_{i+1}, \\ t f_\mu, & \text{when } \mu_i = \mu_{i+1} \end{cases}$$

and

$$f_{\omega\mu}(qx_n, x_1, \ldots, x_{n-1}) = q^{\mu_n} f_\mu(x_1, \ldots, x_n; q, t).$$

- This implies that when λ is a partition,

$$Y_i f_\lambda = y_i(\lambda) f_\lambda$$

and so f_λ coincides with E_λ.