On the Rational Generating function for intervals of Partitions

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Abstract: Suppose that $c_{k,n}$ is the average size of interval of partitions $[(0), \lambda]$, where λ runs through the set of all partitions of n with exactly k parts. We showed that the sequence of $c_{k,n}$ has polynomial growth.

Generating function $Q_k(x_1, \cdots, x_k, y)$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$, we define the monomial

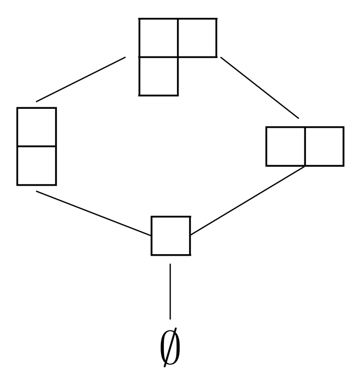
$$x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$$

and the polynomial

$$g_\lambda(y) = \sum_{\delta \leq \lambda} y^{|\delta|},$$

where $|\delta|$ is the sum of all parts of the partition δ.

Example 1: Let $\lambda = (2, 1)$. Then $x^\lambda = x_1^2 x_2$ and $g_\lambda(y) = 1 + y + 2y^2 + y^3$.



Remark. $g_\lambda(y)$ is the *Poincaré polynomial* $P_y(X_\lambda)$ of the Schubert variety X_λ of a Grassmannian.

Let $\mathbb{P}$ denote the set of all partitions. For $k \in \mathbb{N}$, let

$$Q_k = Q_k(x_1, x_2, \cdots, x_k, y) := \sum_{\substack{\lambda \in \mathbb{P} \\ l(\lambda) = k}} g_\lambda(y) x^\lambda,$$

where $l(\lambda)$ denotes the number of parts in λ.

Example 2:

$$\begin{aligned} Q_1 &= g_{(1)}(y)x_1 + g_{(2)}(y)x_1^2 + g_{(3)}(y)x_1^3 + \cdots \\ &= (1+y)x_1 + (1+y+y^2)x_1^2 + (1+y+y^2+y^3)x_1^3 + \cdots \end{aligned} \quad (1)$$

By rearranging the terms of the R.H.S. of (1), we get that

$$Q_1 = x_1 + \Big(x_1 y_1 + (x_1 y_1)^2 \Big) + (x_1 y_1)^3 + \cdots \Big) + x_1 Q_1,$$

which implies that

$$Q_1 = \frac{1}{(1 - x_1)(1 - x_1 y)} - 1.$$

Thus, we see that

Q_1 can be expressed as a rational function in x_1 and y.

For $n \geq k$, let

$$p_k = p_k(x_1, x_2, \cdots, x_n) := x_1 x_2 \cdots x_k.$$

We can express Q_2 in terms of Q_1.

$$Q_2 = \left(\frac{1}{1 - p_2} \right) \left(x_2 Q_1 + \left(\frac{p_2 y}{1 - p_1 y} \right) Q_1(p_2 y, y) + \left(\frac{p_2 y^2}{1 - p_2 y^2} \right) (1 + Q_1) \right)$$

In general, we have the following theorem.

Theorem 3: For $k \in \mathbb{N}$,

$$\begin{aligned} (1 - p_k) Q_k &= x_k Q_{k-1} \\ &+ \sum_{j=1}^{k} \left(\frac{p_k y^j}{1 - p_j y^j} \left(\sum_{i=0}^{j-1} Q_j \right) \right) Q_{k-j}(p_{j+1} y^j, x_{j+2}, \cdots x_k, y) \end{aligned}$$

Now, since Q_1 is a rational function, by using Theorem 3, we can prove by induction on k that

Corollary 4: For all $k \in \mathbb{N}$, Q_k is a rational function in $x_1, x_2, \cdots, x_k$, and y.

We call Q_k the rational generating function for interval of partitions.

Example 5:

$$Q_2(x_1, x_2, y) = \frac{x_1 x_2 + (x_1 x_2 - x_1^2 x_2 - x_1^2 x_2^2) y + (x_1 x_2 - x_1^2 x_2^2) y^2 + (x_1^3 x_2^3 - x_1^2 x_2^2) y^3}{(1 - x_1)(1 - x_1 x_2)(1 - x_1 y)(1 - x_1 x_2 y)(1 - x_1 x_2 y^2)}.$$

Asymptotic equivalence of the sequence of coefficients of $Q(x, \cdots, x, 1)$

We observe that $g_\lambda(1)$ is the size of the interval $[(0), \lambda]$, where by size of an interval of partitions we mean the number of partitions in that interval.

Define $\tilde{Q}_k(x) := Q_k(x, \cdots, x, 1)$ and consider the coefficient $q_{k,n}$ defined by

$$\tilde{Q}_k(x) = \sum_n q_{k,n} x^n.$$

It follows from the definition that

$q_{k,n}$ is the sum of the sizes of all intervals $[(0), \lambda]$ where λ is a partition of n into k parts.

Remark.

$$q_{k,n} = \sum_{\substack{\lambda \\ l(\lambda) = k}} \dim(H^*(X_\lambda))$$

Let $D_k = D_k(x_1, \cdots, x_2, y) := \prod_{j=1}^{k} \prod_{i=0}^{j} (1 - p_j y^i)$.

We see that, $(1 + Q_1) D_1 = 1$, which shows that $Q_1 D_1$ is a polynomial. Similarly, with the help of Theorem 3, one can show by induction on k that

Lemma 6: $D_k Q_k$ is a polynomial in $x_1, \cdots, x_k, y$.

Now, $D_k(x, \cdots, x, 1) = \prod_{j=1}^{k} (1 - x^j)^{j+1}$, which implies that

Lemma 7: All poles of $\tilde{Q}_k(x)$ are roots of unity.

Also, we have that

$$Q_1(x^i, 1) = \frac{2x^i - x^{2i}}{(1 - x^i)^2}, \quad (2)$$

and from Theorem 3, we see that

$$\begin{aligned} &(1 - x^{k+i-1}) Q_k(x^i, x, \cdots, x, 1) \\ &= x Q_{k-1}(x^i, x, \cdots, x, 1) \\ &+ \sum_{j=1}^{k} \left(\frac{x^{k+i-1}}{1 - x^{j+i-1}} \left(\sum_{l=0}^{j-1} Q_l(x^i, x \cdots, x, 1) \right) Q_{k-j}(x^{i+j}, x, \cdots, x, 1) \right). \end{aligned} \quad (3)$$

By using (2) and (3) one can show that

Theorem 8: 1 is the pole of $\tilde{Q}_k(x)$ of the largest order which is $2k$.

For an asymptotic equivalence of the sequence of $q_{k,n}$, we only need to consider the pole at $x = 1$ since it corresponds to the fastest exponential growth[2]. Therefore,

$$\tilde{Q}_k(x) \sim \frac{C_1}{(1 - x)^{2k}},$$

where $C_1 = \lim_{x \to 1} (1 - x)^{2k} \tilde{Q}_k(x)$. Hence

$$q_{k,n} \sim C_2(n^{2k-1}), \text{ for some constant } C_2. \quad (4)$$

Our main result

Let $P_{k,n}$ be the number of partition of n with exactly k parts.

Then by Erdős-Lehner theorem[1],

$$P_{k,n} \sim C_3(n^{k-1}), \text{ for some constant } C_3 \quad (5)$$

Let $T_k(x) := \sum_{n \geq k} c_{k,n} x^n$, where $c_{k,n} = \frac{q_{k,n}}{P_{k,n}} =$ the average size of interval $[(0), \lambda]$, where λ runs through the set of all partitions of n with exactly k parts.

Now by using (4) and (5), we conclude that

$$c_{k,n} \sim C(n^k), \text{ for some constant } C.$$

Thus we see that

Theorem 9: The sequence of $c_{k,n}$ has polynomial growth and $c_{k,n} = \mathcal{O}(n^k)$.

References

[1] George E. Andrews, The Theory of Partitions, Cambridge University Press, 1984.

[2] Philippe Flajolet and Robert Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009.