Schur-positivity of some trees

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Plan

- ★ What is a Schur function?
- ★ What is a chromatic symmetric function?
- ★ Schur positivity of some chromatic symmetric functions.

Symmetric functions

Let $x = (x_1, x_2, ...)$ be an infinite set of variables and, let $n \in \mathbb{N}$. A symmetric function of degree n over the rational numbers $\mathbb{Q}$ is:

$$\mathrm{f(x)} = \sum_{\lambda \in \mathbf{G}} q_\lambda x^\lambda$$

- **G** is the set of all weak compositions of n (i.e $\lambda = (\lambda_1, \lambda_2, ...)$ s.t $\lambda_i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \lambda_i = n$)
- $q_\lambda \in \mathbb{Q}$
- $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2}..$
- $f(x_{\sigma(1)}, x_{\sigma(2)}, ...) = f(x_1, x_2, ...)$, for all permutations σ of $\mathbb{N}^*$.

Symmetric functions

★ We write by $\bigwedge^n$ the space of all symmetric functions of degree n.

Schur functions

- ★ Let $n \in \mathbb{N}^*$. A partition of n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p)$, s.t $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$ and $\sum_{i=1}^{p} \lambda_i = n$. λ_i are known as parts of λ and $\mathrm{len}(\lambda)$ is the number of its parts.
- ★ Let λ be a partition. An array $T = (T_{ij} > 0)$ is of shape λ if $1 \leq i \leq \mathrm{len}(\lambda)$ and $1 \leq j \leq \lambda_i$.
- ★ Let $T = (T_{ij})$ be an array of positive integers of shape λ. If the rows of T are weakly increasing and the columns are strictly increasing, then we say T is a semistandard young tableau (SSYT) of shape λ.

Schur functions

Example

If $\lambda = (7, 5, 3, 3, 2)$, an array of shape λ is of the form:

$$\begin{matrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} & T_{17} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & & \\ T_{31} & T_{32} & T_{33} & & & & \\ T_{41} & T_{42} & T_{43} & & & & \\ T_{51} & T_{52} & & & & & \end{matrix}$$

An example of SSYT of shape λ is:

$$\begin{matrix} 2 & 2 & 3 & 4 & 4 & 5 & 5 \\ 3 & 4 & 4 & 5 & 5 & & \\ 5 & 6 & 7 & & & & \\ 6 & 7 & 8 & & & & \\ 9 & 9 & & & & & \end{matrix}$$

Schur functions

★ Let T be a SSYT. The type of T is a sequence $\alpha = (\alpha_1, \alpha_2, ..)$ s.t α_i is the number of parts equals to i.

Example

For the SSYT of shape $\lambda = (7, 5, 3, 3, 2)$:

$$\begin{matrix} 2 & 2 & 3 & 4 & 4 & 5 & 5 \\ 3 & 4 & 4 & 5 & 5 & & \\ 5 & 6 & 7 & & & & \\ 6 & 7 & 8 & & & & \\ 9 & 9 & & & & & \end{matrix}$$

This SSYT has type $\alpha = (0, 2, 2, 4, 5, 2, 2, 1, 2)$

Schur functions

Let $n \in \mathbb{N}^*$, and let λ be a partition of n. The Schur function of shape λ or a Schur function of degree n is:

$$s_\lambda = \sum_{T_\alpha \in \mathbf{S}} x^\alpha$$

where, $\mathbf{S}$ is the set of all SSYT of shape λ and α is the type of the SSYT.

- ★ [2, Chapter 7] Schur function is a symmetric function.
- ★ [2, Chapter 7] The set of Schur functions of degree n form a basis of the vector space $\bigwedge^n$.

Chromatic symmetric function

Let $G = (V, E)$ be a finite simple graph.

- ★ A proper coloring of G is a function $k : V \to \mathbb{N}^*$ s.t $k(v_i) \neq k(v_j)$ for any $\{v_i, v_j\} \in E$.
- ★ The chromatic function of G is:

$$X_G(x) = \sum_{k \in \mathbb{K}(G)} x_{k(v_1)} x_{k(v_2)} \ldots x_{k(v_n)}$$

  where $\mathbb{K}(G)$ is the set of all proper colorings of G and $n = |V|$.
- ★ The chromatic function $X_G(x)$ of G is a symmetric function.
- ★ We say that $X_G(x)$ is Schur positive if we expand $X_G(x) = \sum_{s \in S} \alpha_s s$, we have $\alpha_s \geq 0$.

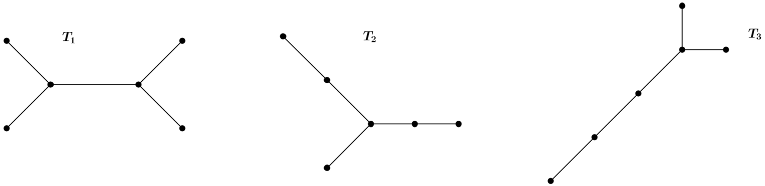
Our goal

Conjecture 42 of [1] (Dahlberg, She and van Willigenburg)

For every $n \geq 2$, there is a tree T on n vertices, one of which has degree $\lfloor \frac{n}{2} \rfloor$, such that the chromatic symmetric function of T is Schur positive.

Example

If n=6,



$X_{T_1}(x) = 32s[1,1,1,1,1,1] + 40s[2,1,1,1,1] + 18s[2,2,1,1] + 8s[2,2,2] + 16s[3,1,1,1] + 6s[3,2,1] + 2s[3,3] + 2s[4,1,1]$

$X_{T_2}(x) = 32s[1,1,1,1,1,1] + 36s[2,1,1,1,1] + 26s[2,2,1,1] + 4s[2,2,2] + 10s[3,1,1,1] + 9s[3,2,1] + 2s[3,3]$

$X_{T_3}(x) = 32s[1,1,1,1,1,1] + 36s[2,1,1,1,1] + 23s[2,2,1,1] + 13s[2,2,2] + 11s[3,1,1,1] + 7s[3,2,1] - s[3,3] + s[4,1,1] + s[4,2]$

Other definitions

- ★ A stable partition of G is a set partition $\pi : V = \bigcup_{i=1}^{k} \pi_i$, s.t π_i are independent in G (i.e no connection between each pairs of vertices in π_i).
- ★ Suppose $|\pi_i| \geq |\pi_{i+1}|$ for each $i \in [k-1]$. Clearly, $\lambda = (|\pi_1|, |\pi_2|, ..., |\pi_k|)$ is a partition of $|V|$ and we call λ the type of π.
- ★ Let $\mu = (\mu_1, \mu_2, ..., \mu_l)$ be a partition of $|V|$. λ dominates μ ($\mu \preceq \lambda$) , if $\sum_{i=1}^{i=m} \mu_i \leq \sum_{i=1}^{i=m} \lambda_i$ for all $m \in [k]$.

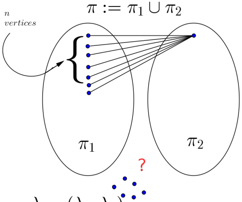
Lemma (Proposition 1.5 of [3])

If $X_G(x)$ is Schur positive and G admits a stable partition of type λ, then G admits a stable partition of type μ whenever $\mu \preceq \lambda$.

Corollary

Assume that $T = (V, E)$ is a tree on $2n$ vertices and $v \in V$ has degree n in T. If $X_T(x)$ is Schur positive, then every $x \in V$ that is neither v nor a neighbor of v is a leaf in T.

Proof.



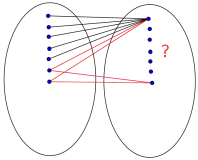
$\lambda = (\lambda_1, \lambda_2)$

$\lambda_1 \geq n$ and $\lambda_2 \geq 1$

$\mu = (n, n)$

Then $\mu \preceq \lambda \Rightarrow \mu = \lambda$

$\Rightarrow$



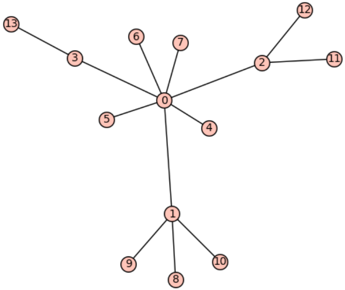
If one of the others is not leaf

⇒ there must be a cycle

For each partition $\nu = (\nu_1, \dots, \nu_t)$ of $n-1$, let $T(\nu)$ be a tree on $2n$ vertices in which one vertex v has exactly n neighbors $v_1, \dots, v_n$, and for $1 \leq i \leq t$, v_i has exactly ν_i neighbors other than v.

Example

Take $n = 7$ and $\nu = (3, 2, 1)$ a partition of 6. $T(\nu)$ tree on 14 vertices.



Corollary

If T is a tree on $2n$ vertices, one of which has degree n, and $X_T(x)$ is Schur positive, then there is some partition ν of $n-1$ such that T is isomorphic with $T(\nu)$.
ie. $X_T(x)$ is not Schur positive if T is not isomorphic with $T(\nu)$, for any ν.

Proposition

If ν is a partition of the integer nine, then $X_{T(\nu)}(x)$ is not Schur positive.
i.e If T is a tree on 20 vertices which contains a vertex of degree 10, then $X_T(x)$ is not Schur positive.

Conjecture 42 of [1] (Dahlberg, She and van Willigenburg)

For every $n \geq 2$, there is a tree T on n vertices, one of which has degree $\lfloor \frac{n}{2} \rfloor$, such that the chromatic symmetric function of T is Schur positive.

- Is there a tree T with a vertex of degree 10 s.t $X_T(x)$ is Schur positive?
  $\rightarrow$ **Yes!**
- **Generalization** Let $k \geq 3$ be an integer. Is there a tree T with a vertex of degree k s.t its chromatic function is Schur positive?

Merci beaucoup!

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