A tale of two polytopes 2:
the harmonic polytope

Based on arXiv:2006.0307
Joint work with Federico Ardila

Laura Escobar

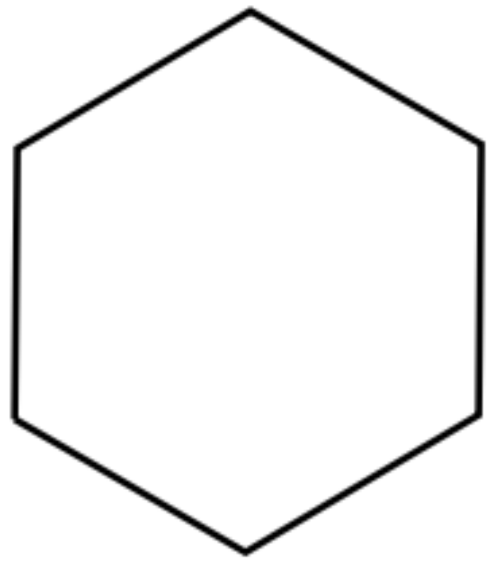
Washington University in St. Louis

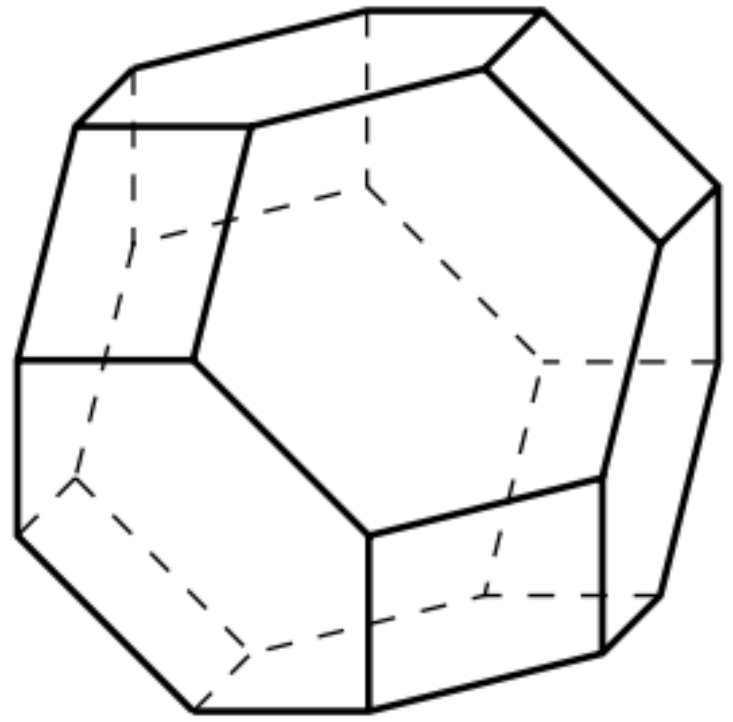
AlCoVe

June 15, 2020

This talk will be recorded

The harmonic polytope





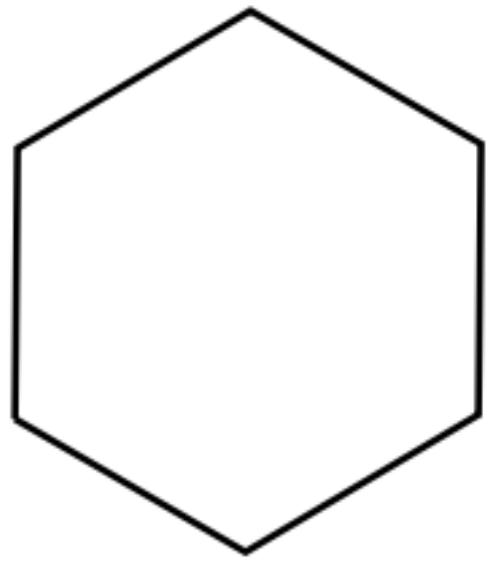
- The **permutohedron** is
  $\Pi_n = \text{conv}\{(z_1, \ldots, z_n) \mid z_1, \ldots, z_n$ is a permutation of $[n]\}$.

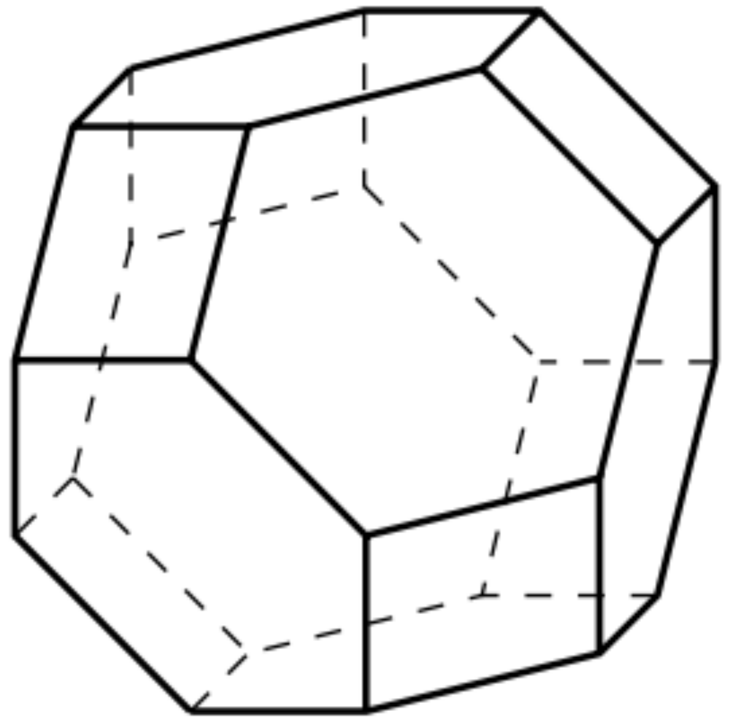
- Consider two copies of $\mathbb{R}^n$ with standard bases $\{\mathsf{e}_i \,:\, i \in [n]\}$ and $\{\mathsf{f}_i \,:\, i \in [n]\}$.

- Let D_n be the $(n-1)$-dimensional simplex $\text{conv}\left\{\mathsf{e}_i + \mathsf{f}_i \,:\, i \in [n]\right\} \subseteq \mathbb{R}^n \times \mathbb{R}^n$.

- The **harmonic polytope** is $H_{n,n} = D_n + (\Pi_n \times \Pi_n) \subset \mathbb{R}^n \times \mathbb{R}^n$.

The harmonic polytope





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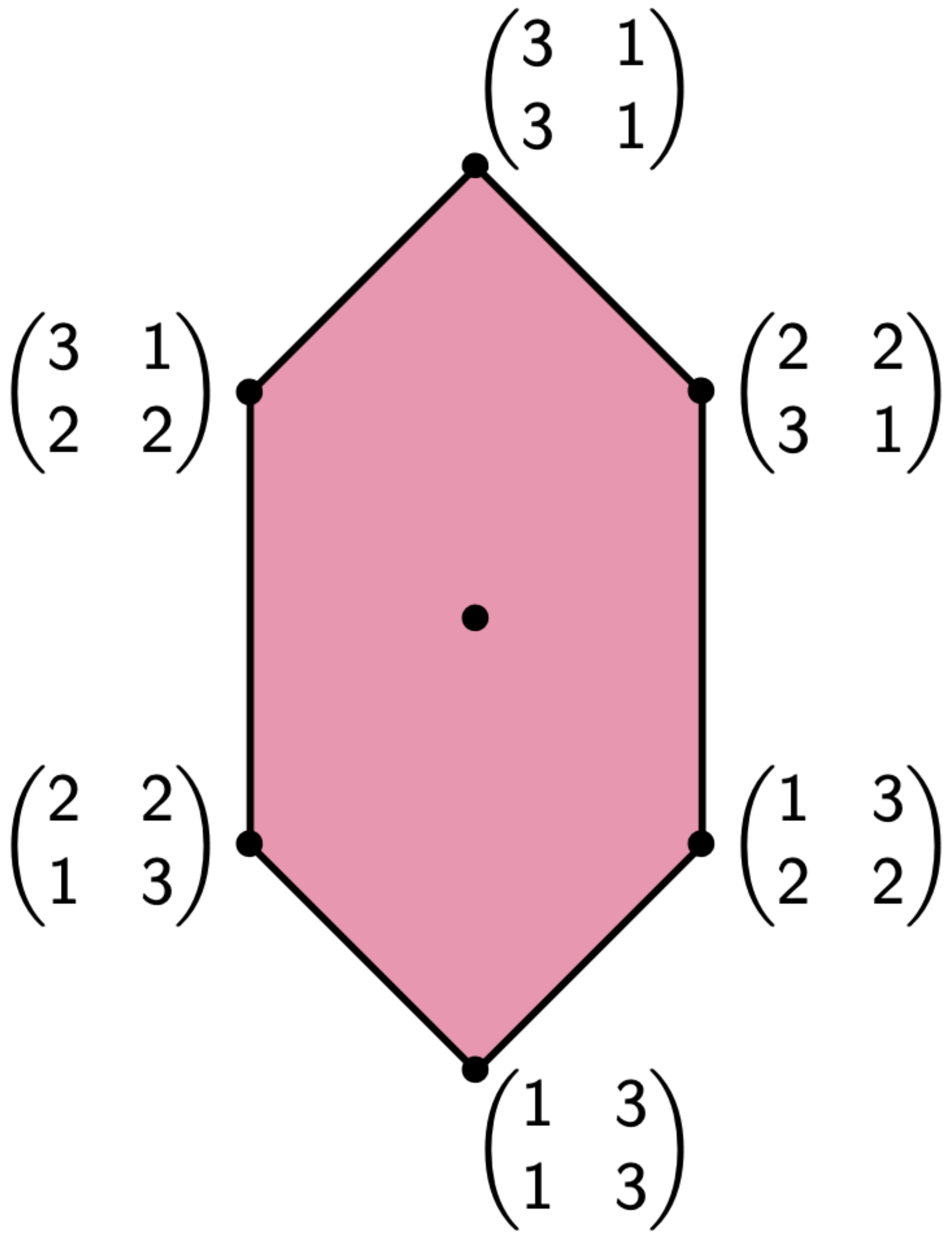

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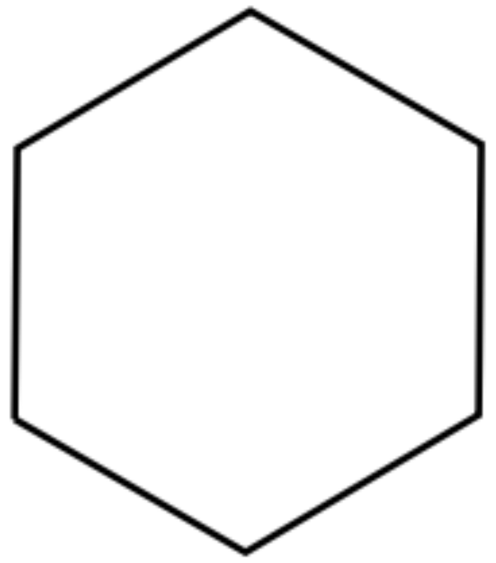

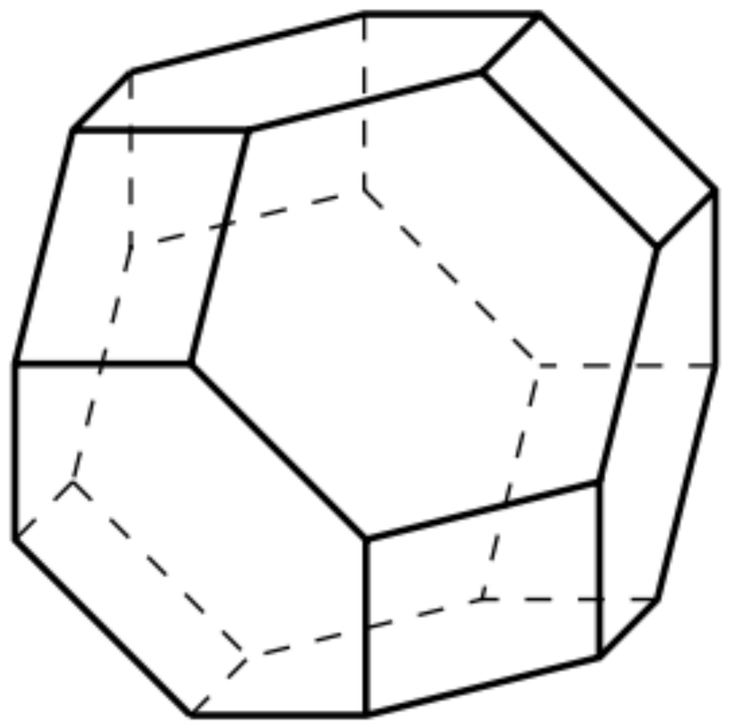

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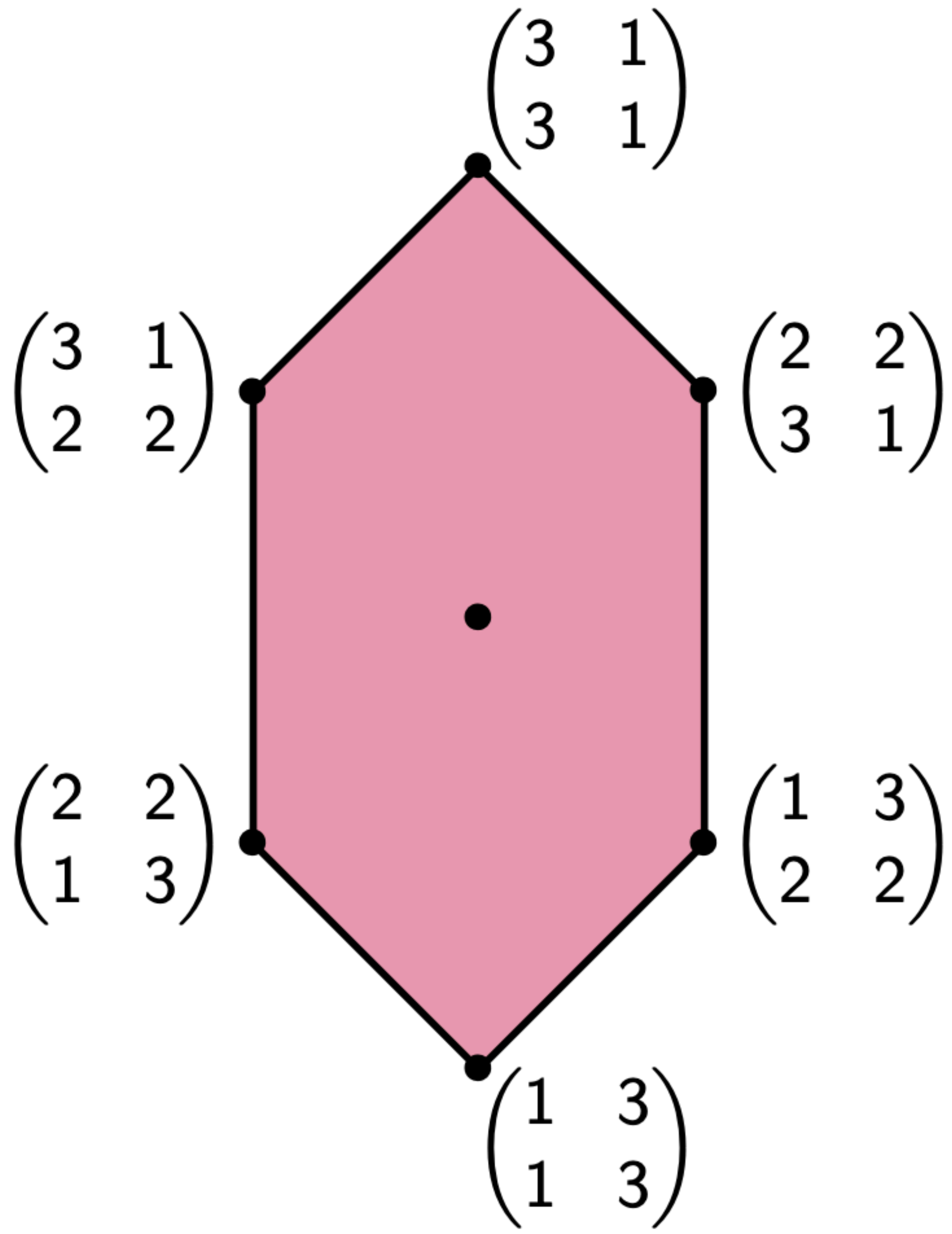
- $H_{n,n}$ is $(2n-2)$-dimensional.
- The harmonic polytope is a Minkowski summand of (a multiple of) the *bipermutohedron* of Ardila-Denham-Huh.



Faces of the harmonic polytope

- The faces of $H_{n,n}$ are in bijection with the harmonic triples on $[n]$.

- A **harmonic triple** $(K; \pi_1, \pi_2)$ on $[n]$ consists of $\varnothing \neq K \subseteq [n]$ and a pair of ordered set partitions π_1, π_2 of $[n]$ such that:

  - The restrictions $\pi_1|_K$ and $\pi_2|_K$ are opposite to each other, and

  - If $j \notin K$ appears in the same or a later block than $k \in K$ in one of the set partitions, then j must appear in an earlier block than k in the other set partition.

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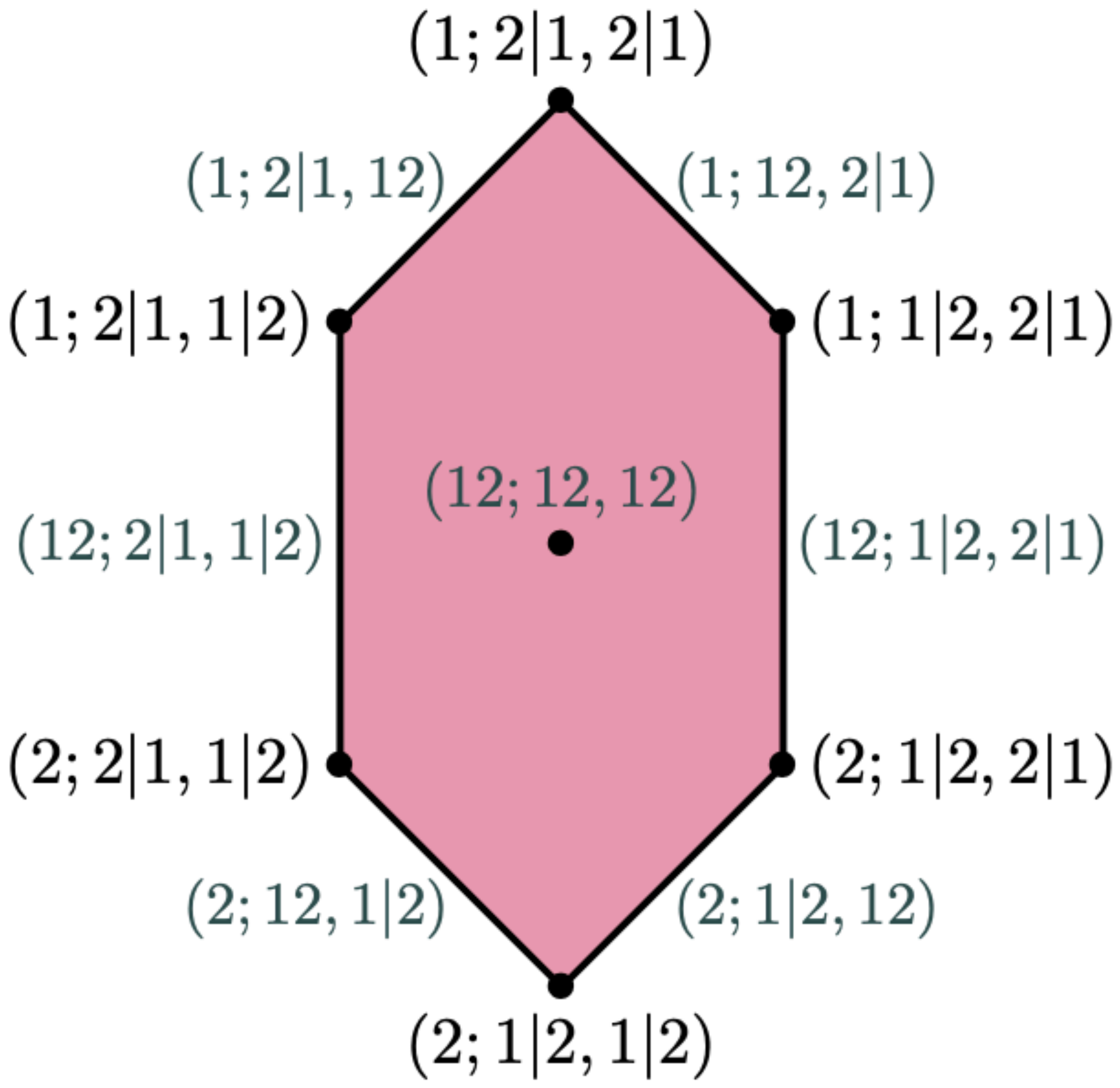

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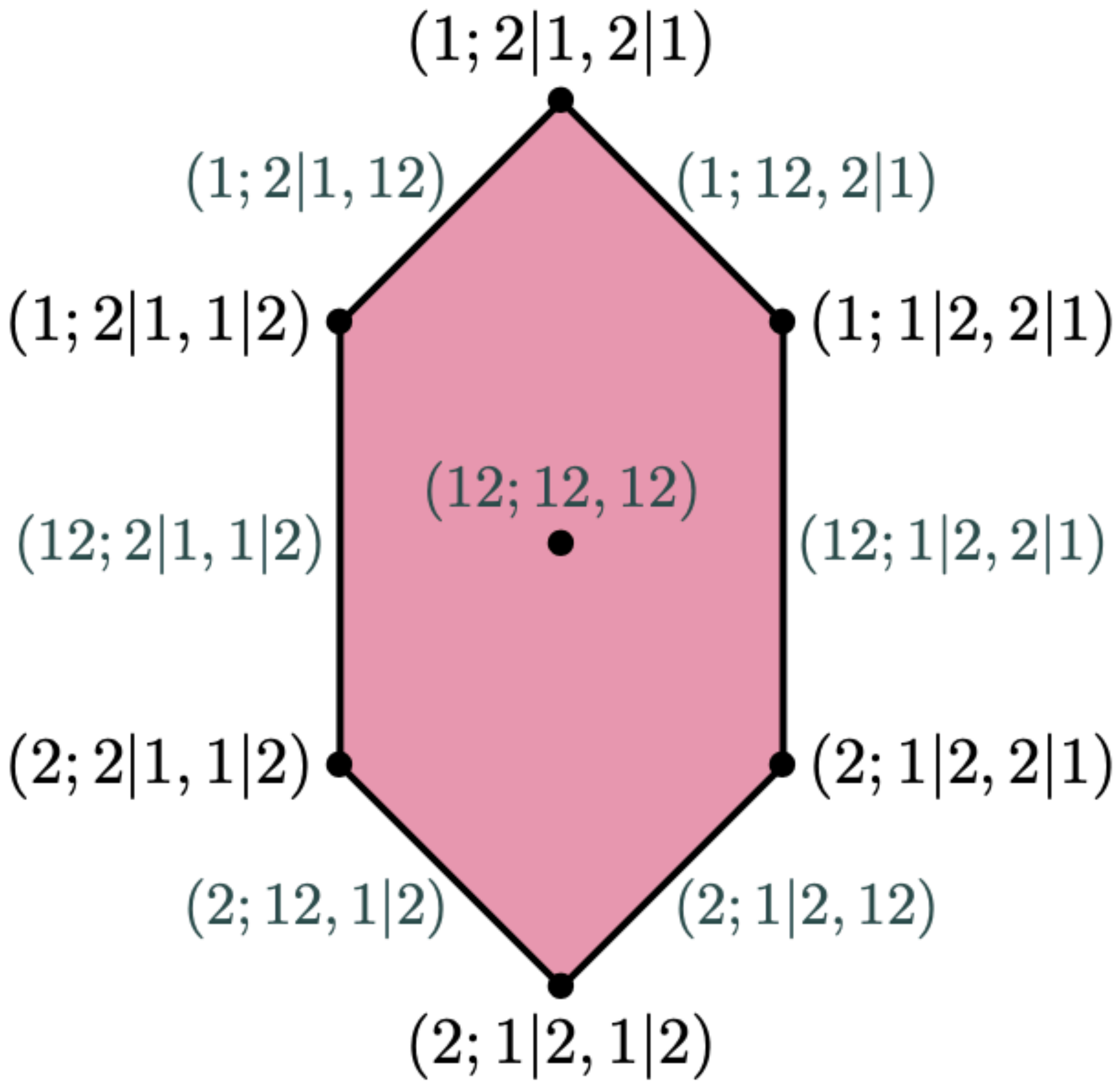
  - If $j \notin K$ appears in the same or a later block than $k \in K$ in one of the set partitions, then j must appear in an earlier block than k in the other set partition.
- The number of vertices of $H_{n,n}$ equals $(n!)^2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.
- The number of facets of $H_{n,n}$ equals $3^n - 3$.



What is the volume of the harmonic polytope?

Volumes of polytopes

- Let P be a d-dimensional polytope on an affine d-plane $L \subset \mathbb{R}^n$. The volume of P is measured on L and normalized appropriately.

- The **mixed volume** is the function such that for any collection of polytopes $P_1, \ldots, P_m \subset L$
  $\text{vol}(P_1 + \cdots + P_m) = \sum_{i_1,\ldots,i_d} \text{MV}(P_{i_1}, \ldots, P_{i_d})$.

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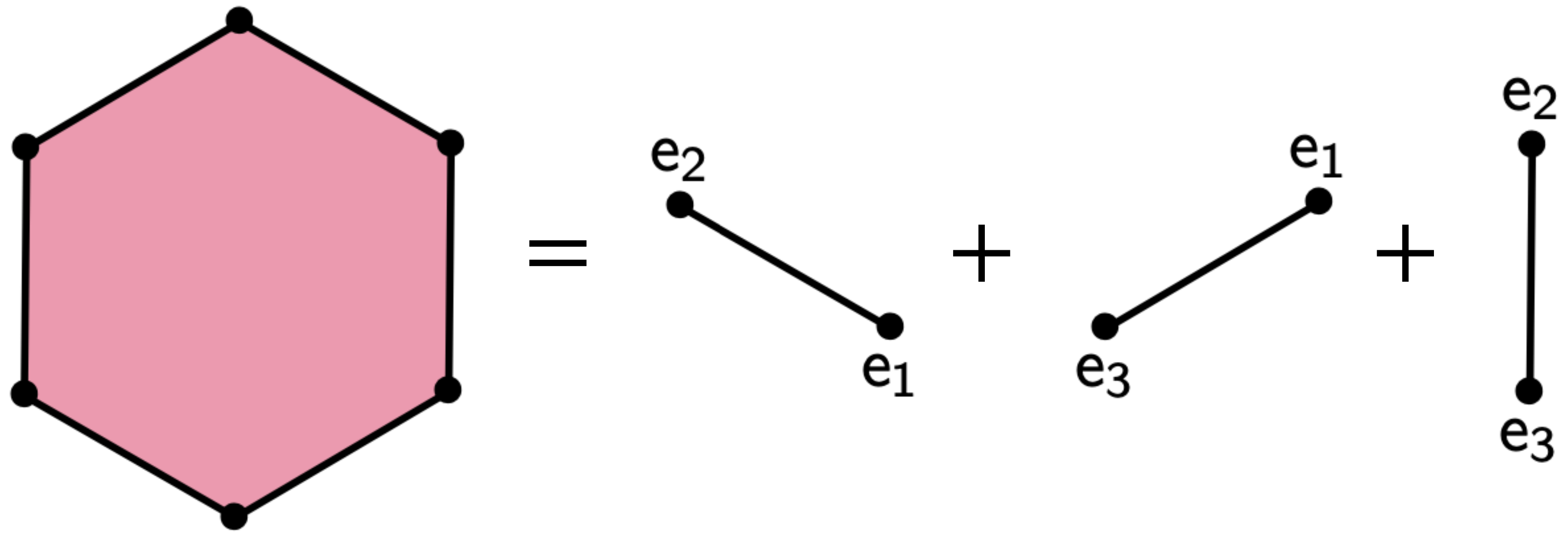
- $\Pi_3 =$ 

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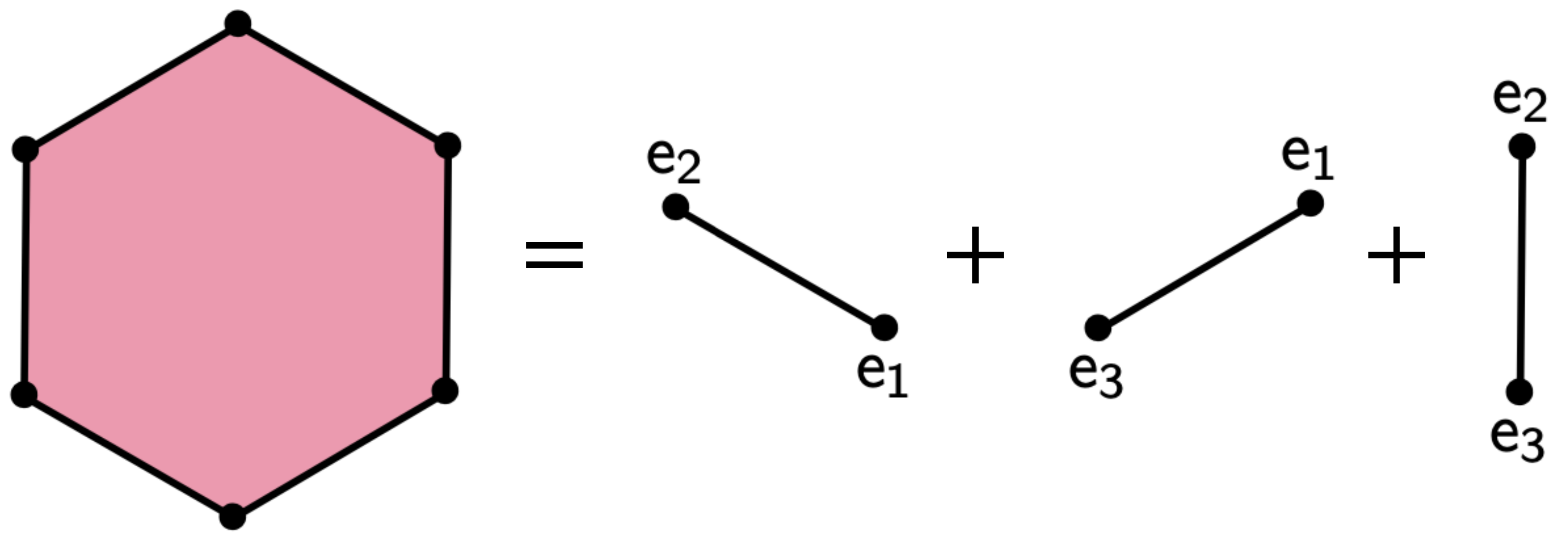
- $\Pi_3 =$ 

- vol$(\Pi_3) =$ MV$(\Delta_{12}, \Delta_{12}) +$ MV$(\Delta_{13}, \Delta_{13}) +$ MV$(\Delta_{23}, \Delta_{23}) + 2MV(\Delta_{12}, \Delta_{13}) + 2MV(\Delta_{12}, \Delta_{23}) + 2MV(\Delta_{13}, \Delta_{23})$

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- The permutohedron Π_n is a translation of the Minkowski sum $\sum_{i<j} \Delta_{ij}$, where $\Delta_{ij} := \text{conv}\{\mathsf{e}_i, \mathsf{e}_j\}$.

- Since $H_{n,n} = D_n + (\Pi_n \times \Pi_n)$, then $H_{n,n} = D_n + \sum_{i<j} \Delta_{ij} + \sum_{i<j} \Delta_{\bar{i}\bar{j}}$, where $\Delta_{\bar{i}\bar{j}} := \text{conv}\{\mathsf{f}_i, \mathsf{f}_j\}$.

- We compute $\text{vol}(H_{n,n})$ by evaluating the various $\text{MV}(G, \bar{G}) = \text{MV}(\Delta_{i_1 j_1}, \ldots, \Delta_{i_r j_r}, \Delta_{\bar{i}_1 \bar{j}_1}, \ldots, \Delta_{\bar{i}_s \bar{j}_s}, \underbrace{D_n, \ldots, D_n}_{2n-2-r-s \text{ times}})$.

Bernstein-Khovanskii-Kushnirenko Theorem

- **Bernstein-Khovanskii-Kushnirenko Theorem:** Let $A_1, \ldots, A_d \subset \mathbb{Z}^d$ be finite, $Q_i = \text{conv}(A_i)$, and $\lambda_{i,\alpha} \in \mathbb{C}$ be *sufficiently generic*. The number of solutions in $(\mathbb{C}^*)^d$ to the system

$$\left\{ \sum_{\alpha \in A_1} \lambda_{1,\alpha} x^\alpha = 0, \ldots, \sum_{\alpha \in A_d} \lambda_{d,\alpha} x^\alpha = 0 \right.$$

is finite and equals $d!\text{MV}(Q_1, \ldots, Q_d)$.

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- **Relevant Corollary 1:** if $A_1, \ldots, A_d \subset \mathbb{Z}^n$ and $Q_1, \ldots, Q_d$ all lie on the affine $(n-1)$-plane $\mathscr{L}_n$ the theorem above holds when counting solutions in $(\mathbb{C}^*)^n / \mathbb{C}^*$.

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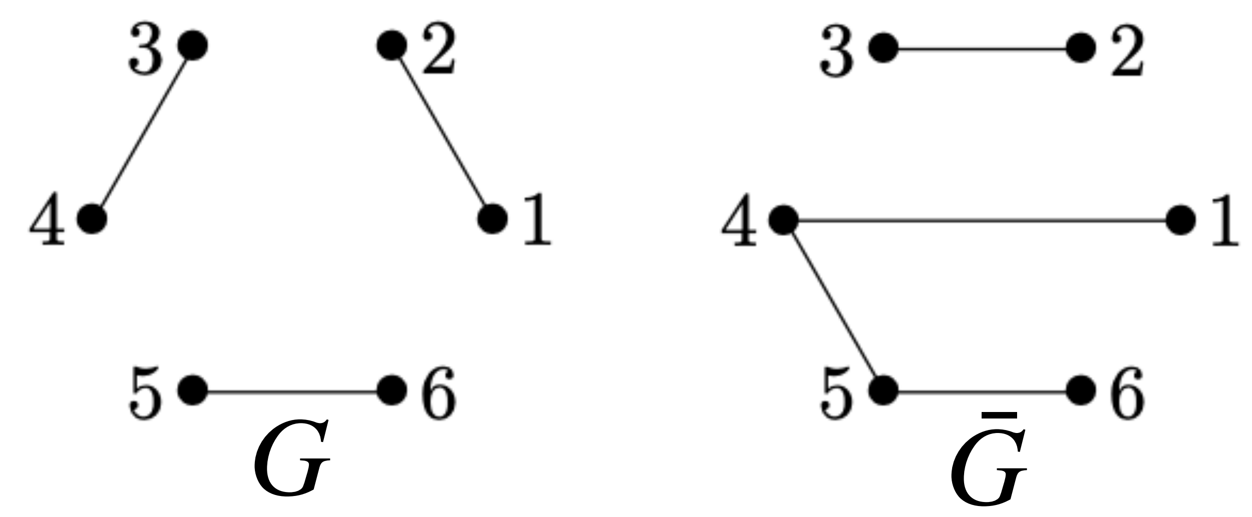
- **Relevant Corollary 2:** if $A_1, \ldots, A_d \subset \mathbb{Z}^n \times \mathbb{Z}^n$ and $Q_1, \ldots, Q_d$ all lie on the affine $(n-2)$-plane $\mathscr{L}_{n,n}$ the theorem above holds when counting solutions in $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n/\mathbb{C}^* \times \mathbb{C}^*$.

Towards the volume of the harmonic polytope



- Encode a sequence $i_1j_1, \ldots, i_rj_r$ as the edges of a graph G and $\bar{i}_1\bar{j}_1, \ldots, \bar{i}_s\bar{j}_s$ as the edges of a graph $\bar{G}$.
- Let $k = 2n - 2 - r - s$.
- Denote by $\mathrm{MV}(G, \bar{G}) = \mathrm{MV}(\Delta_{i_1j_1}, \ldots, \Delta_{i_rj_r}, \Delta_{\bar{i}_1\bar{j}_1}, \ldots, \Delta_{\bar{i}_s\bar{j}_s}, \underbrace{D_n, \ldots, D_n}_{k \text{ times}})$.
- The system associated to the mixed volume $\mathrm{MV}(G, \bar{G})$ is

$$\mathscr{E}(G, G') = \begin{cases} x_i = \lambda_{ij}x_j, \text{ for } ij \in E(G) & \nu_{11}x_1y_1 + \cdots + \nu_{1n}x_ny_n = 0 \\ y_i = \mu_{ij}y_j, \text{ for } \bar{i}\bar{j} \in E(\bar{G}) & \vdots \\ & \nu_{k1}x_1y_1 + \cdots + \nu_{kn}x_ny_n = 0 \end{cases}$$

An example:

- $10! \cdot \mathrm{MV}(G, \bar{G})$ is the number of solutions to

$$\mathscr{E}(G, \bar{G}) = \begin{cases} x_1 = \lambda_{12}\, x_2, & y_1 = \mu_{14}\, y_4, & \nu_{11}\, x_1 y_1 + \cdots + \nu_{16}\, x_6 y_6 = 0, \\ x_3 = \lambda_{34}\, x_4, & y_2 = \mu_{23}\, y_3, & \nu_{21}\, x_1 y_1 + \cdots + \nu_{26}\, x_6 y_6 = 0, \\ x_5 = \lambda_{56}\, x_6, & y_4 = \mu_{45}\, y_5, & \nu_{31}\, x_1 y_1 + \cdots + \nu_{36}\, x_6 y_6 = 0, \\ & y_5 = \mu_{56}\, y_6\,. & \end{cases}$$

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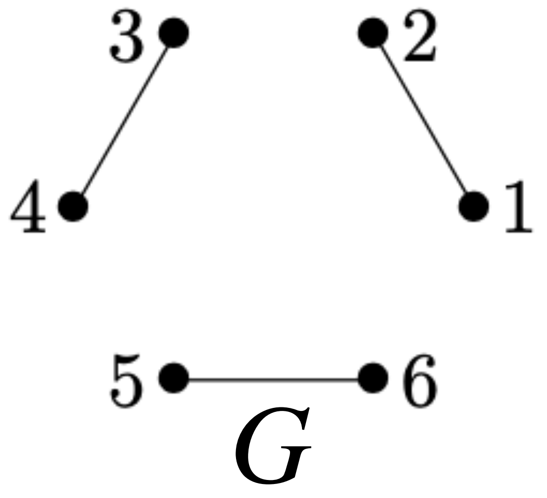

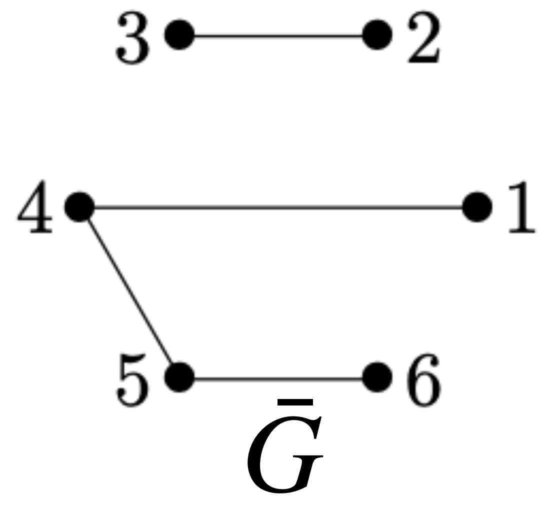

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- Append variables $x_{12}, x_{34}, x_{56}, y_{1456}, y_{23}$ and equations $x_{12} = x_1,\ x_{34} = x_3, x_{56} = x_5, y_{1456} = y_1, y_{23} = y_2$.

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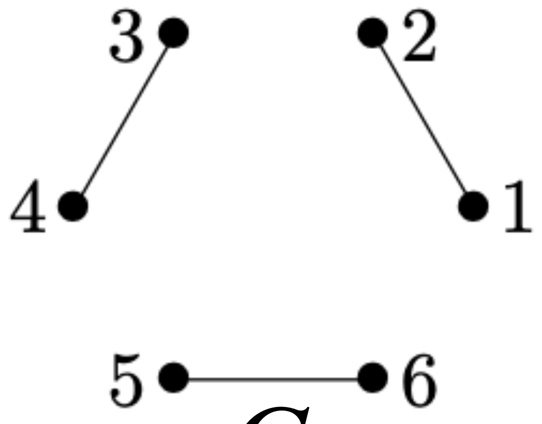
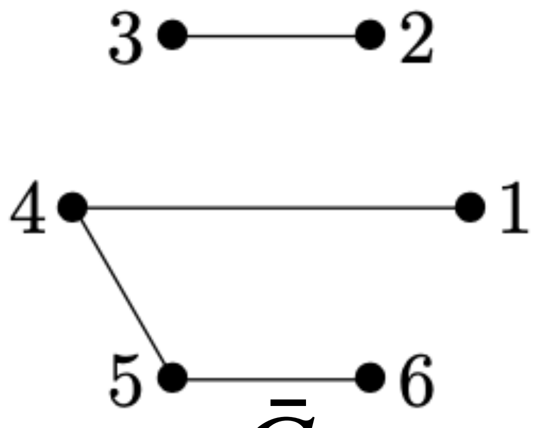


G



$\bar{G}$

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- Eliminating the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ we obtain

$$\mathscr{H}(G,\bar{G}) = \begin{cases} \eta_{11}\, x_{12}y_{1456} + \eta_{12}\, x_{12}y_{23} + \eta_{13}\, x_{34}y_{23} + \eta_{14}\, x_{34}y_{1456} + \eta_{15}\, x_{56}y_{1456} + \eta_{16}\, x_{56}y_{1456} = 0, \\ \eta_{21}\, x_{12}y_{1456} + \eta_{22}\, x_{12}y_{23} + \eta_{23}\, x_{34}y_{23} + \eta_{24}\, x_{34}y_{1456} + \eta_{25}\, x_{56}y_{1456} + \eta_{26}\, x_{56}y_{1456} = 0, \\ \eta_{31}\, x_{12}y_{1456} + \eta_{32}\, x_{12}y_{23} + \eta_{33}\, x_{34}y_{23} + \eta_{34}\, x_{34}y_{1456} + \eta_{35}\, x_{56}y_{1456} + \eta_{36}\, x_{56}y_{1456} = 0. \end{cases}$$

for some coefficients η_{ij} which are *sufficiently generic*.

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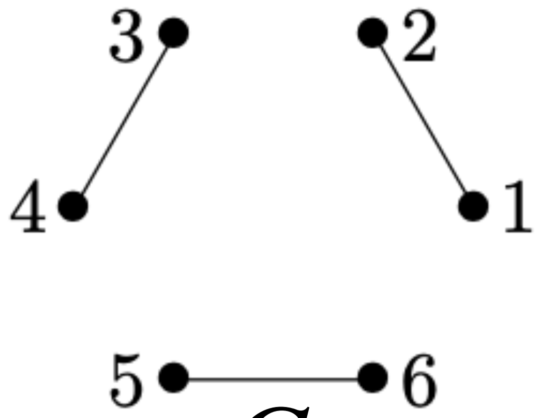
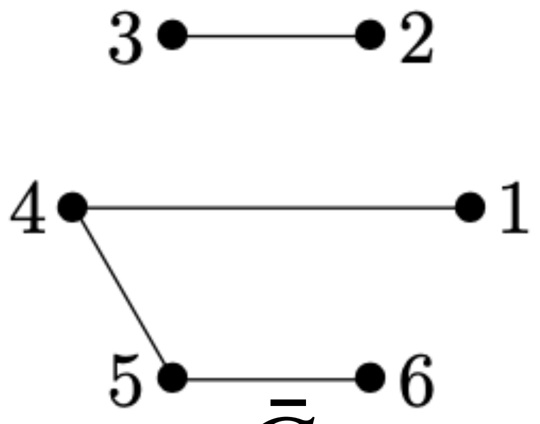


G



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- By the BKK Theorem, the number of solutions to $\mathscr{H}(G,\bar{G})$ is equal to 3! times the volume of $R = \mathrm{conv}\{\mathsf{e}_{12}+\mathsf{f}_{1456}, \mathsf{e}_{12}+\mathsf{f}_{23}, \mathsf{e}_{24}+\mathsf{f}_{23}, \mathsf{e}_{34}+\mathsf{f}_{1456}, \mathsf{e}_{56}+\mathsf{f}_{1456}\} \subset \mathbb{R}^3 \times \mathbb{R}^2$.

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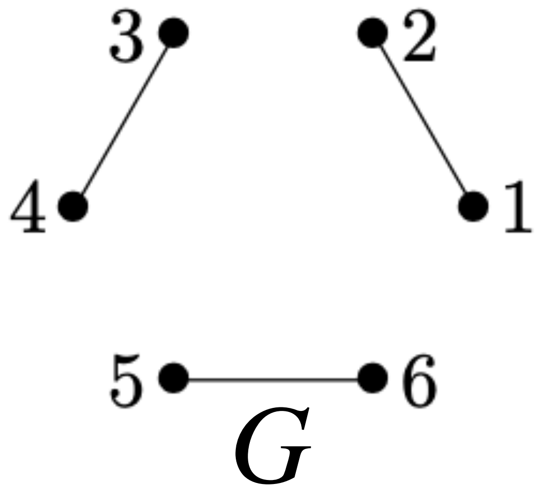

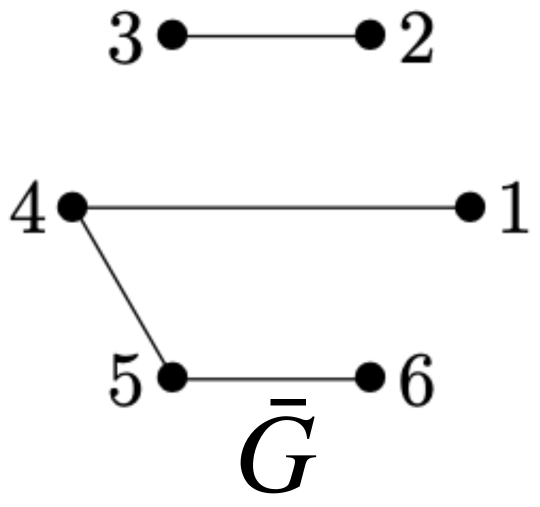

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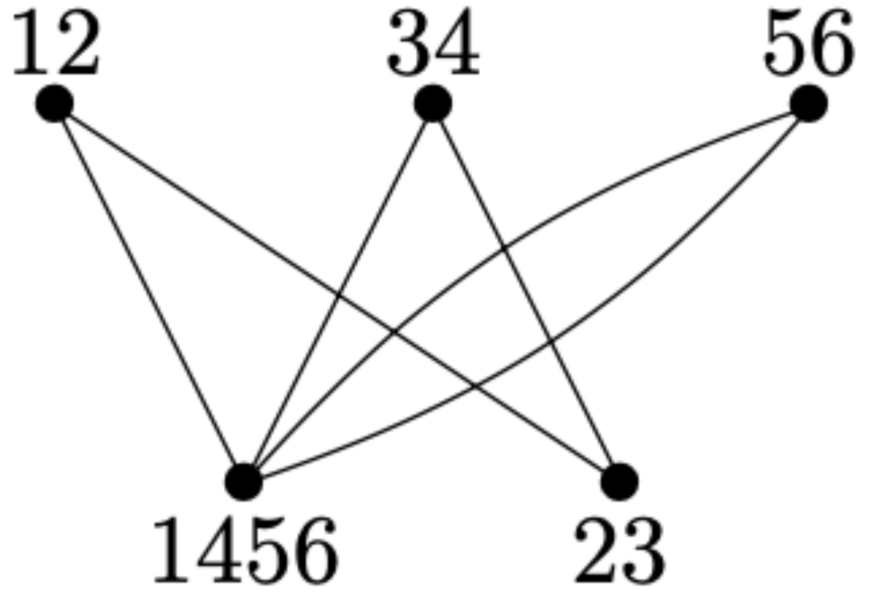
- R is the *edge polytope* R_Γ of the bipartite graph $\Gamma =$



The volume of an edge polytope

- Given a bipartite graph $\Gamma = (U \cup V, E)$ let $\Delta_V = \text{conv}\{\mathsf{e}_v \mid v \in V\} \subset \mathbb{R}^{|V|}$ and, given $u \in U$, let $\Delta_{\text{nbr}(u)} = \text{conv}\{\mathsf{e}_v \mid uv \in \Gamma\} \subset \mathbb{R}^{|V|}$.

- **Theorem (Postnikov, 2009):** $(|U| + |V| - 2)! \cdot \text{vol}(R_\Gamma) =$ number of integer points in P_Γ^-, where $P_\Gamma^- = \left(\sum_{u \in U} \Delta_{\text{nbr}(u)}\right) - \Delta_V$.

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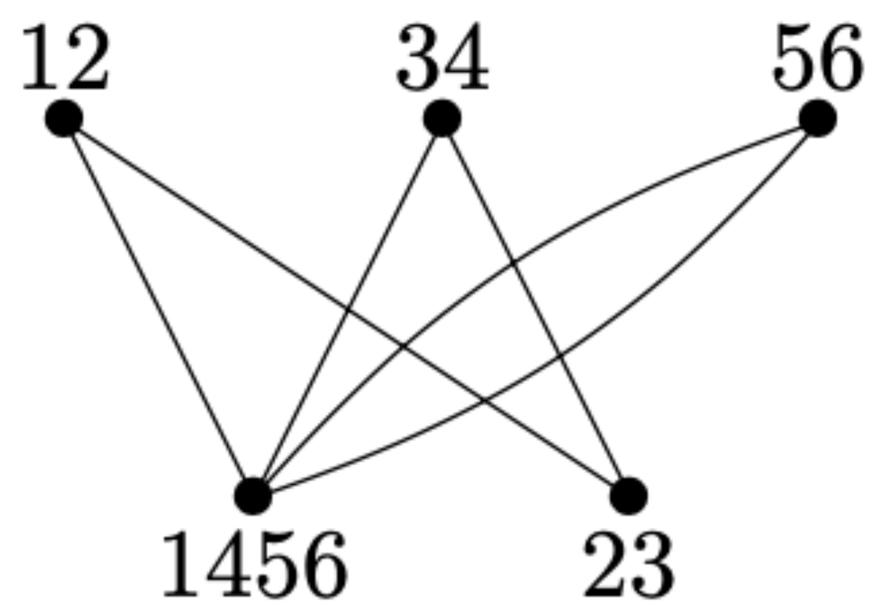
- Take Γ to be the graph on the right.

    - $P_\Gamma^- = 2\Delta_{1456,23} + \Delta_{1456} - \Delta_{1456,23} = \Delta_{1456,23} + \Delta_{1456}$ which contains two lattice points.

    - Number of solutions to $\mathscr{H}(G, \bar{G}) = 3! \cdot \text{vol}(R_\Gamma) = 2$.

    - Number of solutions to $\mathscr{E}(G, \bar{G}) = 10! \cdot \text{MV}(G, \bar{G}) = 2$.



The volume of the harmonic polytope

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- Let G and $\bar{G}$ be acyclic graphs with n vertices and Γ be the corresponding bipartite graph, having p and q vertices on each side of the bipartition. The following numbers are equal:
  - $(2n-2)! \cdot \mathsf{MV}(G, \bar{G})$.
  - The volume of the edge polytope R_Γ multiplied by $(p+q-2)!$.
  - The number of lattice points in $P_\Gamma^- \subset \mathbb{R}^q$.
- Furthermore, the numbers above are zero if and only if Γ is disconnected.

The volume of the harmonic polytope

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  - The number of lattice points in $P_\Gamma^- \subset \mathbb{R}^q$.
- Furthermore, the numbers above are zero if and only if Γ is disconnected.
- **Theorem (Ardila-E.):** vol$(H_{n,n}) = \sum_{\Gamma} \frac{i(P_\Gamma^-)}{(v(\Gamma)-2)!} \prod_{v \in V(\Gamma)} \deg(v)^{\deg(v)-2}$, summing over all connected bipartite multigraphs Γ on edge set $[n]$.
- vol$(H_{1,1}) = 1, \quad$ vol$(H_{2,2}) = 3, \quad$ vol$(H_{3,3}) = 33, \quad$ vol$(H_{4,4}) = 2848/3.$

Muchas Gracias!