CHAPTER 2

Complex Numbers

1. INTRODUCTION

You will probably recall using imaginary and complex numbers in algebra. The general solution of the quadratic equation

$$az^2 + bz + c = 0 \tag{1.1}$$

for the unknown z, is given by the *quadratic formula*

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.2}$$

If the *discriminant* $d = (b^2 - 4ac)$ is negative, we must take the square root of a negative number in order to find z. Since only non-negative numbers have real square roots, it is impossible to use (1.2) when $d < 0$ unless we introduce a new kind of number, called an imaginary number. We use the symbol $i = \sqrt{-1}$ with the understanding that $i^2 = -1$. Then

$$\sqrt{-16} = 4i, \quad \sqrt{-3} = i\sqrt{3}, \quad i^3 = -i$$

are imaginary numbers, but

$$i^2 = -1, \quad \sqrt{-2}\sqrt{-8} = i\sqrt{2} \cdot i\sqrt{8} = -4, \quad i^{4n} = 1$$

are real. In (1.2) we also need combinations of real and imaginary numbers.

Example. The solution of

$$z^2 - 2z + 2 = 0$$

is

$$z = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

We use the term *complex number* to mean any one of the whole set of numbers, real, imaginary, or combinations of the two like $1 \pm i$. Thus, $i + 5$, $17i$, 4, $3 + i\sqrt{5}$ are all examples of complex numbers.

Once the new kind of number is admitted into our number system, fascinating possibilities open up. Can we attach any meaning to marks like $\sin i$, $e^{i\pi}$, $\ln(1+i)$? We'll see later that we can and that, in fact, such expressions may turn up in problems in physics, chemistry, and engineering, as well as mathematics.

When people first considered taking square roots of negative numbers, they felt very uneasy about the problem. They thought that such numbers could not have any meaning or any connection with reality (hence the term "imaginary"). They certainly would not have believed that the new numbers could be of any practical use. Yet complex numbers are of great importance in a variety of applied fields; for example, the electrical engineer would, to say the least, be severely handicapped without them. The complex notation often simplifies setting up and solving vibration problems in either dynamical or electrical systems, and is useful in solving many differential equations which arise from problems in various branches of physics. (See Chapters 7 and 8.) In addition, there is a highly developed field of mathematics dealing with functions of a complex variable (see Chapter 14) which yields many useful methods for solving problems about fluid flow, elasticity, quantum mechanics, and other applied problems. Almost every field of either pure or applied mathematics makes some use of complex numbers.

▶ 2. REAL AND IMAGINARY PARTS OF A COMPLEX NUMBER

A complex number such as $5 + 3i$ is the sum of two terms. The real term (not containing i) is called the *real part* of the complex number. The *coefficient* of i in the other term is called the *imaginary part* of the complex number. In $5 + 3i$, 5 is the real part and 3 is the imaginary part. Notice carefully that the *imaginary part* of a complex number is *not imaginary*!

Either the real part or the imaginary part of a complex number may be zero. If the real part is zero, the complex number is called imaginary (or, for emphasis, *pure* imaginary). The zero real part is usually omitted; thus $0 + 5i$ is written just $5i$. If the imaginary part of the complex number is zero, the number is real. We write $7 + 0i$ as just 7. Complex numbers then include both real numbers and pure imaginary numbers as special cases.

In algebra a complex number is ordinarily written (as we have been doing) as a sum like $5 + 3i$. There is another very useful way of thinking of a complex number. As we have said, every complex number has a real part and an imaginary part (either of which may be zero). These are two *real* numbers, and we could, if we liked, agree to write $5+3i$ as $(5, 3)$. Any complex number could be written this way as a pair of real numbers, the real part first and then the imaginary part (which, you must remember, is real). This would not be a very convenient form for computation, but it suggests a very useful geometrical representation of a complex number which we shall now consider.

▶ 3. THE COMPLEX PLANE

In analytic geometry we plot the point $(5, 3)$ as shown in Figure 3.1. As we have seen, the symbol $(5, 3)$ could also mean the complex number $5+3i$. The point $(5, 3)$ may then be labeled either $(5, 3)$ or $5+3i$. Similarly, any complex number $x+iy$ (x and y real) can be represented by a point (x, y) in the (x, y) plane. Also any point (x, y) in the (x, y) plane can be labeled $x + iy$ as well as (x, y). When the (x, y)

plane is used in this way to plot complex numbers, it is called the *complex plane*. It is also sometimes called an *Argand diagram*. The x axis is called the real axis, and the y axis is called the imaginary axis (note, however, that you plot y and *not* iy).

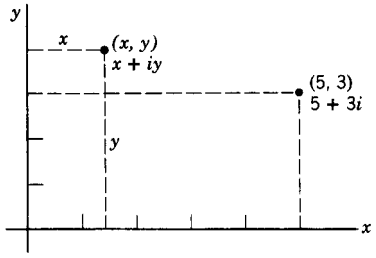


Figure 3.1

When a complex number is written in the form $x + iy$, we say that it is in *rectangular form* because x and y are the rectangular coordinates of the point representing the number in the complex plane. In analytic geometry, we can locate a point by giving its polar coordinates (r, θ) instead of its rectangular coordinates (x, y). There is a corresponding way to write any complex number. In Figure 3.2,

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{3.1}$$

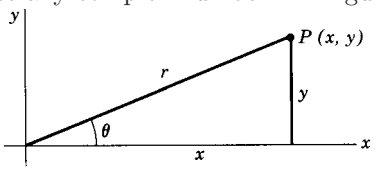


Figure 3.2

Then we have

$$\begin{aligned} x + iy &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned} \tag{3.2}$$

This last expression is called the *polar form* of the complex number. As we shall see (Sections 9 to 16), the expression $(\cos\theta + i\sin\theta)$ can be written as $e^{i\theta}$, so a convenient way to write the polar form of a complex number is

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{3.3}$$

The polar form $re^{i\theta}$ of a complex number is often simpler to use than the rectangular form.

▶ **Example.** In Figure 3.3 the point A could be labeled as $(1, \sqrt{3})$ or as $1 + i\sqrt{3}$. Similarly, using polar coordinates, the point A could be labeled with its (r, θ) values as $(2, \pi/3)$. Notice that r is always taken positive. Using (3.3) we have

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}.$$

This gives two more ways to label point A in Figure 3.3.

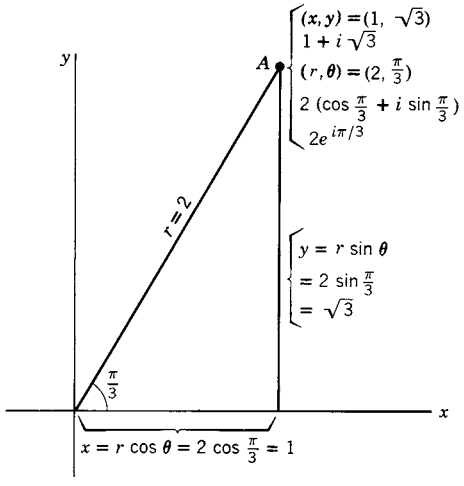


Figure 3.3

Radians and Degrees In Figure 3.3, the angle $\pi/3$ is in radians. Ever since you studied calculus, you have been expected to measure angles in radians and not degrees. Do you know why? You have learned that $(d/dx)\sin x = \cos x$. This formula is **not** correct—**unless** x is in radians. (Look up the derivation in your calculus book!) Many of the formulas you now know and use are correct *only* if you use radian measure; consequently that is what you are usually advised to do. However, it is sometimes convenient to do computations with complex numbers using degrees, so it is important to know when you can and when you cannot use degrees. You can use degrees to measure an angle and to add and subtract angles as long as the final step is to find the sine, cosine, or tangent of the resulting angle (with your calculator in degree mode). For example, in Figure 3.3, we can, if we like, say that $\theta = 60°$ instead of $\theta = \pi/3$. If we want to find $\sin(\pi/3 - \pi/4) = \sin(\pi/12) = 0.2588$ (calculator in radian mode), we can instead find $\sin(60° - 45°) = \sin 15° = 0.2588$ (calculator in degree mode). Note carefully that an angle is in radians unless the degree symbol is used; for example, in $\sin 2$, the 2 is 2 radians or about 115°.

In formulas, however, use radians. For example, in using infinite series, we say that $\sin\theta \cong \theta$ for very small θ. Try this on your calculator; you will find that it is true in radian mode but not in degree mode. As another example, consider $\int_0^1 dx/(1+x^2) = \arctan 1 = \pi/4 = 0.785$. Here $\arctan 1$ is *not* an angle; it is the numerical value of the integral, so the answer 45 (obtained from a calculator in degree mode) is wrong! Do not use degree mode in reading an arc tan (or arc sin or arc cos) *unless* you are finding an *angle* [for example, in Figure 3.2, $\theta = \arctan(y/x)$, and in Figure 3.3, $\theta = \arctan\sqrt{3} = \pi/3$ or 60°].

4. TERMINOLOGY AND NOTATION

Both i and j are used to represent $\sqrt{-1}$, j usually in any problem dealing with electricity since i is needed there for current. A physicist should be able to work with ease using either symbol. We shall for consistency use i throughout this book.

We often label a point with a single letter (for example, P in Figure 3.2 and A in Figure 3.3) even though it requires two coordinates to locate the point. If you have studied vectors, you will recall that a vector is represented by a single letter, say $\mathbf{v}$, although it has (in two dimensions) two components. It is customary to use a single letter for a complex number even though we realize that it is actually a pair of real numbers. Thus we write

$$z = x + iy = r(\cos\theta + i\,\sin\theta) = re^{i\theta}. \tag{4.1}$$

Here z is a complex number; x is the *real part* of the complex number z, and y is the *imaginary part* of z. The quantity r is called the *modulus* or *absolute value* of z, and θ is called the *angle* of z (or the *phase*, or the *argument*, or the *amplitude* of z). In symbols:

$$\begin{aligned} &\operatorname{Re} z = x, && |z| = \operatorname{mod} z = r = \sqrt{x^2+y^2}, \\ &\operatorname{Im} z = y\,(\textbf{not } iy), && \text{angle of } z = \theta. \end{aligned} \tag{4.2}$$

The values of θ should be found from a diagram rather than a formula, although we do sometimes write $\theta = \arctan(y/x)$. An example shows this clearly.

▶ **Example.** Write $z = -1 - i$ in polar form. Here we have $x = -1$, $y = -1$, $r = \sqrt{2}$ (Figure 4.1). There are an infinite number of values of θ,

$$\theta = \frac{5\pi}{4} + 2n\pi, \tag{4.3}$$

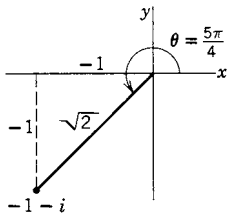


Figure 4.1

where n is any integer, positive or negative. The value $\theta = 5\pi/4$ is sometimes called the *principal angle* of the complex number $z = -1 - i$. Notice carefully, however, that this is not the same as the principal value $\pi/4$ of $\arctan 1$ as defined in calculus. The angle of a complex number must be in the same quadrant as the point representing the number. For our present work, any one of the values in (4.3) will do; here we would probably use either $5\pi/4$ or $-3\pi/4$. Then we have in our example

$$\begin{aligned} z = -1 - i &= \sqrt{2}\left[\cos\left(\frac{5\pi}{4} + 2n\pi\right) + i\sin\left(\frac{5\pi}{4} + 2n\pi\right)\right] \\ &= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\,e^{5i\pi/4}. \end{aligned}$$

[We could also write $z = \sqrt{2}\,(\cos 225^\circ + i\,\sin 225^\circ)$.]

The complex number $x - iy$, obtained by changing the sign of i in $z = x + iy$, is called the *complex conjugate* or simply the *conjugate* of z. We usually write the conjugate of $z = x + iy$ as $\bar{z} = x - iy$. Sometimes we use z^* instead of $\bar{z}$ (in fields such as statistics or quantum mechanics where the bar may be used to mean an average value). Notice carefully that the conjugate of $7i - 5$ is $-7i - 5$; that is, it is the i term whose sign is changed.

Complex numbers come in conjugate pairs; for example, the conjugate of $2 + 3i$ is $2 - 3i$ and the conjugate of $2-3i$ is $2+3i$. Such a pair of points in the complex plane are mirror images of each other with the x axis as the mirror (Figure 4.2). Then in polar form, z and $\bar{z}$ have the same r value, but their θ values are negatives of each other. If we write $z = r(\cos\theta + i\,\sin\theta)$, then

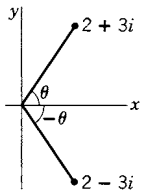


Figure 4.2

$$\bar{z} = r[\cos(-\theta) + i\,\sin(-\theta)] = r(\cos\theta - i\,\sin\theta) = re^{-i\theta}. \tag{4.4}$$

▶ PROBLEMS, SECTION 4

For each of the following numbers, first visualize where it is in the complex plane. With a little practice you can quickly find x, y, r, θ in your head for these simple problems. Then

plot the number and label it in five ways as in Figure 3.3. Also plot the complex conjugate of the number.

1. $1+i$ **2.** $i-1$ **3.** $1-i\sqrt{3}$

4. $-\sqrt{3}+i$ **5.** $2i$ **6.** $-4i$

7. -1 **8.** 3 **9.** $2i-2$

10. $2-2i$ **11.** $2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$

12. $4\left(\cos\dfrac{2\pi}{3}-i\sin\dfrac{2\pi}{3}\right)$ **13.** $\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}$

14. $2\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$ **15.** $\cos\pi-i\sin\pi$

16. $5(\cos 0+i\sin 0)$ **17.** $\sqrt{2}\,e^{-i\pi/4}$

18. $3\,e^{i\pi/2}$ **19.** $5(\cos 20^\circ+i\sin 20^\circ)$

20. $7(\cos 110^\circ-i\sin 110^0)$

5. COMPLEX ALGEBRA

A. Simplifying to $x+iy$ form

Any complex number can be written in the rectangular form $x+iy$. To add, subtract, and multiply complex numbers, remember that they follow the ordinary rules of algebra and that $i^2=-1$.

Example 1.

$$(1+i)^2=1+2i+i^2=1+2i-1=2i$$

To divide one complex number by another, first write the quotient as a fraction. Then reduce the fraction to rectangular form by multiplying numerator and denominator by the conjugate of the denominator; this makes the denominator real.

Example 2.

$$\frac{2+i}{3-i}=\frac{2+i}{3-i}\cdot\frac{3+i}{3+i}=\frac{6+5i+i^2}{9-i^2}=\frac{5+5i}{10}=\frac{1}{2}+\frac{1}{2}i.$$

It is sometimes easier to multiply or divide complex numbers in polar form.

Example 3. To find $(1+i)^2$ in polar form, we first sketch (or picture mentally) the point $(1,1)$. From Figure 5.1, we see that $r=\sqrt{2}$, and $\theta=\pi/4$, so $(1+i)=\sqrt{2}\,e^{i\pi/4}$. Then from Figure 5.2 we find the same result as in Example 1.

$$(1+i)^2=(\sqrt{2}\,e^{i\pi/4})^2=2\,e^{i\pi/2}=2i.$$

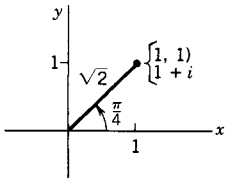


Figure 5.1

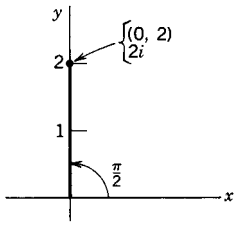


Figure 5.2

▶ **Example 4.** Write $1/[2(\cos 20^\circ + i \sin 20^\circ]$ in $x + iy$ form. Since $20^\circ = \pi/9$ radians,

$$\frac{1}{2(\cos 20^\circ + i \sin 20^\circ)} = \frac{1}{2(\cos \pi/9 + i \sin \pi/9)} = \frac{1}{2\, e^{i\pi/9}} = 0.5\, e^{-i\pi/9}$$
$$= 0.5(\cos \pi/9 - i \sin \pi/9) = 0.47 - 0.17i,$$

by calculator in radian mode. We obtain the same result leaving the angle in degrees and using a calculator in degree mode: $0.5(\cos 20^\circ - i \sin 20^\circ) = 0.47 - 0.17i$.

▶ PROBLEMS, SECTION 5

First simplify each of the following numbers to the $x + iy$ form or to the $re^{i\theta}$ form. Then plot the number in the complex plane.

1. $\dfrac{1}{1+i}$ **2.** $\dfrac{1}{i-1}$ **3.** i^4

4. $i^2 + 2i + 1$ **5.** $\left(i + \sqrt{3}\right)^2$ **6.** $\left(\dfrac{1+i}{1-i}\right)^2$

7. $\dfrac{3+i}{2+i}$ **8.** $1.6 - 2.7i$

9. $25\, e^{2i}$ *Careful!* The angle is 2 radians.

10. $\dfrac{3i-7}{i+4}$ *Careful! Not* $3 - 7i$

11. $17 - 12i$ **12.** $3(\cos 28^\circ + i \sin 28^\circ)$

13. $5\left(\cos \dfrac{2\pi}{5} + i \sin \dfrac{2\pi}{5}\right)$ **14.** $2.8e^{-i(1.1)}$

15. $\dfrac{5-2i}{5+2i}$ **16.** $\dfrac{1}{0.5(\cos 40^\circ + i \sin 40^\circ)}$

17. $(1.7 - 3.2i)^2$ **18.** $(0.64 + 0.77i)^4$

Find each of the following in rectangular $(a + bi)$ form if $z = 2 - 3i$; if $z = x + iy$.

19. z^{-1} **20.** $\dfrac{1}{z^2}$ **21.** $\dfrac{1}{z+1}$

22. $\dfrac{1}{z-i}$ **23.** $\dfrac{1+z}{1-z}$ **24.** $z/\bar{z}$

B. Complex Conjugate of a Complex Expression

It is easy to see that the conjugate of the sum of two complex numbers is the sum of the conjugates of the numbers. If

$$z_1 = x_1 + iy_1 \qquad \text{and} \qquad z_2 = x_2 + iy_2,$$

then

$$\bar{z}_1 + \bar{z}_2 = x_1 - iy_1 + x_2 - iy_2 = x_1 + x_2 - i(y_1 + y_2).$$

The conjugate of $(z_1 + z_2)$ is

$$\overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2).$$

Similarly, you can show that the conjugate of the difference (or product or quotient) of two complex numbers is equal to the difference (or product or quotient) of the conjugates of the numbers (Problem 25). In other words, you can get the conjugate of an expression containing i's by just changing the signs of all the i terms. We must watch out for hidden i's, however.

▶ **Example.** If

$$z = \frac{2-3i}{i+4}, \qquad \text{then} \qquad \bar{z} = \frac{2+3i}{-i+4}.$$

But if $z = f+ig$, where f and g are themselves complex, then the complex conjugate of z is $\bar{z} = \bar{f} - i\bar{g}$ (not $f - ig$).

▶ PROBLEMS, SECTION 5

25. Prove that the conjugate of the quotient of two complex numbers is the quotient of the conjugates. Also prove the corresponding statements for difference and product. *Hint:* It is easier to prove the statements about product and quotient using the polar coordinate $re^{i\theta}$ form; for the difference, it is easier to use the rectangular form $x + iy$.

C. Finding the Absolute Value of z

Recall that the definition of $|z|$ is $|z| = r = \sqrt{x^2+y^2}$ (positive square root!). Since $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$, or, in polar coordinates, $z\bar{z} = (re^{i\theta})(re^{-i\theta}) = r^2$, we see that $|z|^2 = z\bar{z}$, or $|z| = \sqrt{z\bar{z}}$. Note that $z\bar{z}$ is always real and ≥ 0, since x, y, and r are real. We have

(5.1) $$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}.$$

By Problem 25 and (5.1), the absolute value of a quotient of two complex numbers is the quotient of the absolute values (and a similar statement for product).

▶ **Example.**

$$\left|\frac{\sqrt{5}+3i}{1-i}\right| = \frac{|\sqrt{5}+3i|}{|1-i|} = \frac{\sqrt{14}}{\sqrt{2}} = \sqrt{7}.$$

▶ PROBLEMS, SECTION 5

Find the absolute value of each of the following using the discussion above. Try to do simple problems like these in your head—it saves time.

26. $\dfrac{2i-1}{i-2}$ **27.** $\dfrac{2+3i}{1-i}$ **28.** $\dfrac{z}{\bar{z}}$

29. $(1+2i)^3$ **30.** $\dfrac{3i}{i-\sqrt{3}}$ **31.** $\dfrac{5-2i}{5+2i}$

32. $(2-3i)^4$ **33.** $\dfrac{25}{3+4i}$ **34.** $\left(\dfrac{1+i}{1-i}\right)^5$

D. Complex Equations

In working with equations involving complex quantities, we must always remember that a complex number is actually a pair of real numbers. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. For example, $x + iy = 2 + 3i$ means $x = 2$ and $y = 3$. In other words, any equation involving complex numbers is really two equations involving real numbers.

▶ **Example.** Find x and y if

$$(x + iy)^2 = 2i. \tag{5.2}$$

Since $(x + iy)^2 = x^2 + 2ixy - y^2$, (5.2) is equivalent to the two real equations

$$\begin{aligned} x^2 - y^2 &= 0, \\ 2xy &= 2. \end{aligned}$$

From the first equation $y^2 = x^2$, we find $y = x$ or $y = -x$. Substituting these into the second equation gives

$$2x^2 = 2 \qquad \text{or} \qquad -2x^2 = 2.$$

Since x is real, x^2 cannot be negative. Thus we find only

$$x^2 = 1 \qquad \text{and} \qquad y = x,$$

that is,

$$x = y = 1 \qquad \text{and} \qquad x = y = -1.$$

▶ PROBLEMS, SECTION 5

Solve for all possible values of the real numbers x and y in the following equations.

35. $x + iy = 3i - 4$

36. $2ix + 3 = y - i$

37. $x + iy = 0$

38. $x + iy = 2i - 7$

39. $x + iy = y + ix$

40. $x + iy = 3i - ix$

41. $(2x - 3y - 5) + i(x + 2y + 1) = 0$

42. $(x + 2y + 3) + i(3x - y - 1) = 0$

43. $(x + iy)^2 = 2ix$

44. $x + iy = (1 - i)^2$

45. $(x + iy)^2 = (x - iy)^2$

46. $\dfrac{x + iy}{x - iy} = -i$

47. $(x + iy)^3 = -1$

48. $\dfrac{x + iy + 2 + 3i}{2x + 2iy - 3} = i + 2$

49. $|1 - (x + iy)| = x + iy$

50. $|x + iy| = y - ix$

E. Graphs

Using the graphical representation of the complex number z as the point (x, y) in a plane, we can give geometrical meaning to equations and inequalities involving z.

▶ **Example 1.** What is the curve made up of the points in the (x, y) plane satisfying the equation $|z| = 3$?

Since

$$|z| = \sqrt{x^2 + y^2},$$

the given equation is

$$\sqrt{x^2 + y^2} = 3 \quad \text{or} \quad x^2 + y^2 = 9.$$

Thus $|z| = 3$ is the equation of a circle of radius 3 with center at the origin. Such an equation might describe, for example, the path of an electron or of a satellite. (See Section F below.)

▶ **Example 2.**

(a) $|z - 1| = 2$. This is the circle $(x - 1)^2 + y^2 = 4$.

(b) $|z - 1| \leq 2$. This is the disk whose boundary is the circle in (a).

Note that we use "circle" to mean a curve and "disk" to mean an area. The interior of the disk is given by $|z - 1| < 2$.

▶ **Example 3.** (Angle of z) $= \pi/4$. This is the half-line $y = x$ with $x > 0$; this might be the path of a light ray starting at the origin.

▶ **Example 4.** $\operatorname{Re} z > \frac{1}{2}$. This is the half-plane $x > \frac{1}{2}$.

▶ PROBLEMS, SECTION 5

Describe geometrically the set of points in the complex plane satisfying the following equations.

51. $|z| = 2$

52. $\operatorname{Re} z = 0$

53. $|z - 1| = 1$

54. $|z - 1| < 1$

55. $z - \bar{z} = 5i$

56. angle of $z = \dfrac{\pi}{2}$

57. $\operatorname{Re}(z^2) = 4$

58. $\operatorname{Re} z > 2$

59. $|z + 3i| = 4$

60. $|z - 1 + i| = 2$

61. $\operatorname{Im} z < 0$

62. $|z + 1| + |z - 1| = 8$

63. $z^2 = \bar{z}^2$

64. $z^2 = -\bar{z}^2$

65. Show that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane. Use this result to identify the graphs in Problems 53, 54, 59, and 60 without computation.

F. Physical Applications

Problems in physics as well as geometry may often be simplified by using one complex equation instead of two real equations. See the following example and also Section 16.

▶ **Example.** A particle moves in the (x, y) plane so that its position (x, y) as a function of time t is given by

$$z = x + iy = \frac{i + 2t}{t - i}.$$

Find the magnitudes of its velocity and its acceleration as functions of t.

We *could* write z in $x + iy$ form and so find x and y as functions of t. It is easier to do the problem as follows. We define the complex velocity and complex acceleration by

$$\frac{dz}{dt} = \frac{dx}{dt} + i\,\frac{dy}{dt} \quad \text{and} \quad \frac{d^2z}{dt^2} = \frac{d^2x}{dt^2} + i\,\frac{d^2y}{dt^2}.$$

Then the magnitude v of the velocity is $v = \sqrt{(dx/dt)^2 + (dy/dt)^2} = |dz/dt|$, and similarly the magnitude a of the acceleration is $a = |d^2z/dt^2|$. Thus we have

$$\frac{dz}{dt} = \frac{2(t - i) - (i + 2t)}{(t - i)^2} = \frac{-3i}{(t - i)^2}.$$

$$v = \left|\frac{dz}{dt}\right| = \sqrt{\frac{-3i}{(t - i)^2} \cdot \frac{+3i}{(t + i)^2}} = \frac{3}{t^2 + 1},$$

$$\frac{d^2z}{dt^2} = \frac{(-3i)(-2)}{(t - i)^3} = \frac{6i}{(t - i)^3},$$

$$a = \left|\frac{d^2z}{dt^2}\right| = \frac{6}{(t^2 + 1)^{3/2}}.$$

Note carefully that all physical quantities (x, y, v, and a) are real; the complex expressions are used just for convenience in calculation.

▶ PROBLEMS, SECTION 5

66. Find x and y as functions of t for the example above, and verify for this case that v and a are correctly given by the method of the example.

67. Find v and a if $z = (1 - it)/(2t + i)$.

68. Find v and a if $z = \cos 2t + i \sin 2t$. Can you describe the motion?

▶ 6. COMPLEX INFINITE SERIES

In Chapter 1 we considered infinite series whose terms were real. We shall be very much interested in series with complex terms; let us reconsider our definitions and theorems for this case. The partial sums of a series of complex numbers will be complex numbers, say $S_n = X_n + iY_n$, where X_n and Y_n are real. Convergence is defined just as for real series: If S_n approaches a limit $S = X + iY$ as $n \to \infty$, we call the series convergent and call S its sum. This means that $X_n \to X$ and $Y_n \to Y$; in other words, the real and the imaginary parts of the series are each convergent series.

It is useful, just as for real series, to discuss absolute convergence first. It can be proved (Problem 1) that an absolutely convergent series converges. Absolute convergence means here, just as for real series, that the series of absolute values of the terms is a convergent series. Remember that $|z| = r = \sqrt{x^2 + y^2}$ is a positive number. Thus any of the tests given in Chapter 1 for convergence of series of positive terms may be used here to test a complex series for absolute convergence.

▶ **Example 1.** Test for convergence

$$1+\frac{1+i}{2}+\frac{(1+i)^2}{4}+\frac{(1+i)^3}{8}+\cdots+\frac{(1+i)^n}{2^n}+\cdots.$$

Using the ratio test, we find

$$\rho=\lim_{n\to\infty}\left|\frac{(1+i)^{n+1}}{2^{n+1}}\div\frac{(1+i)^n}{2^n}\right|=\lim_{n\to\infty}\left|\frac{1+i}{2}\right|=\left|\frac{1+i}{2}\right|=\frac{\sqrt{2}}{2}<1.$$

. Since $\rho<1$, the series is absolutely convergent and therefore convergent.

▶ **Example 2.** Test for convergence $\sum_1^\infty i^n/\sqrt{n}$. Here the ratio test gives 1 so we must try a different test. Let's write out a few terms of the series:

$$i-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{i}{\sqrt{5}}-\frac{1}{\sqrt{6}}\cdots.$$

We see that the real part of the series is

$$-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{6}}+\cdots=\sum_1^\infty\frac{(-1)^n}{\sqrt{2n}},$$

and the imaginary part of the series is

$$1-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}\cdots=\sum_0^\infty\frac{(-1)^n}{\sqrt{2n+1}}.$$

Verify that both these series satisfy the alternating series test for convergence. Thus, the original series converges.

▶ **Example 3.** Test for convergence $\sum_0^\infty z^n=\sum_0^\infty(re^{i\theta})^n=\sum_0^\infty r^ne^{in\theta}$. This is a geometric series with ratio $=z=re^{i\theta}$; it converges if and only if $|z|<1$. Recall that $|z|=r$. Thus, $\sum_0^\infty r^ne^{in\theta}$ converges if and only if $r<1$.

▶ PROBLEMS, SECTION 6

1. Prove that an absolutely convergent series of complex numbers converges. This means to prove that $\sum(a_n+ib_n)$ converges (a_n and b_n real) if $\sum\sqrt{a_n^2+b_n^2}$ converges. *Hint:* Convergence of $\sum(a_n+ib_n)$ means that $\sum a_n$ and $\sum b_n$ *both* converge. Compare $\sum|a_n|$ and $\sum|b_n|$ with $\sum\sqrt{a_n^2+b_n^2}$, and use Problem 7.9 of Chapter 1.

Test each of the following series for convergence.

2. $\sum(1+i)^n$ **3.** $\sum\frac{1}{(1+i)^n}$ **4.** $\sum\left(\frac{1-i}{1+i}\right)^n$

5. $\sum\left(\frac{1}{n^2}+\frac{i}{n}\right)$ **6.** $\sum\frac{1+i}{n^2}$ **7.** $\sum\frac{(i-1)^n}{n}$

8. $\sum e^{in\pi/6}$ **9.** $\sum\frac{i^n}{n}$ **10.** $\sum\left(\frac{1+i}{1-i\sqrt{3}}\right)^n$

11. $\sum\left(\frac{2+i}{3-4i}\right)^{2n}$ **12.** $\sum\frac{(3+2i)^n}{n!}$ **13.** $\sum\left(\frac{1+i}{2-i}\right)^n$

14. Prove that a series of complex terms diverges if $\rho>1$ ($\rho=$ ratio test limit). *Hint:* The nth term of a convergent series tends to zero.

▶7. COMPLEX POWER SERIES; DISK OF CONVERGENCE

In Chapter 1 we considered series of powers of x, $\sum a_n x^n$. We are now interested in series of powers of z,

$$\sum a_n z^n, \tag{7.1}$$

where $z = x + iy$, and the a_n are complex numbers. [Notice that (7.1) includes real series as a special case since $z = x$ if $y = 0$.] Here are some examples.

$$1 - z + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} + \cdots, \tag{7.2a}$$

$$1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \cdots, \tag{7.2b}$$

$$\sum_{n=0}^{\infty} \frac{(z+1-i)^n}{3^n\, n^2}. \tag{7.2c}$$

Let us use the ratio test to find for what z these series are absolutely convergent. For (7.2a), we have

$$\rho = \lim_{n\to\infty} \left| \frac{z \cdot n}{n+1} \right| = |z|.$$

The series converges if $\rho < 1$, that is, if $|z| < 1$, or $\sqrt{x^2 + y^2} < 1$. This is the interior of a disk of radius 1 with center at the origin in the complex plane. This disk is called the *disk of convergence* of the infinite series and the radius of the disk is called the *radius of convergence*. The disk of convergence replaces the interval of convergence which we had for real series. In fact (see Figure 7.1), the interval of convergence for the series $\sum(-x)^n/n$ is just the interval $(-1, 1)$ on the x axis contained within the disk of convergence of $\sum(-z)^n/n$, as it must be since x is the value of z when $y = 0$. For this reason we sometimes speak of the *radius* of convergence of a power series even though we are considering only real values of z. (Also see Chapter 14, Equations (2.5) and (2.6) and Figure 2.4.)

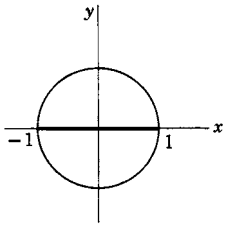


Figure 7.1

Next consider series (7.2b); here we have

$$\rho = \lim_{n\to\infty} \left| \frac{(iz)^{n+1}}{(n+1)!} \div \frac{(iz)^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{iz}{n+1} \right| = 0.$$

This is an example of a series which converges for all values of z. For series (7.2c), we have

$$\rho = \lim_{n\to\infty} \left| \frac{(z+1-i)}{3} \frac{n^2}{(n+1)^2} \right| = \left| \frac{z+1-i}{3} \right|.$$

Thus, this series converges for

$$|z + 1 - i| < 3, \text{ or } |z - (-1 + i)| < 3.$$

This is the interior of a disk (Figure 7.2) of radius 3 and center at $z = -1 + i$ (see Problem 5.65).

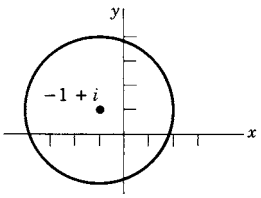


Figure 7.2

Just as for real series, if $\rho > 1$, the series diverges (Problem 6.14). For $\rho = 1$ (that is, on the boundary of the disk of convergence) the series may either converge or diverge. It may be difficult to find out which and we shall not in general need to consider the question.

The four theorems about power series (Chapter 1, Section 11) are true also for complex series (replace *interval* by *disk* of convergence). Also we can now state for Theorem 2 what the disk of convergence is for the quotient of two series of powers of z. Assume to start with that any common factor z has been cancelled. Let r_1 and r_2 be the radii of convergence of the numerator and denominator series. Find the closest point to the origin in the complex plane where the denominator is zero; call the distance from the origin to this point s. Then the quotient series converges at least inside the smallest of the three disks of radii r_1, r_2, and s, with center at the origin. (See Chapter 14, Section 2.)

▶ **Example.** Find the disk of convergence of the Maclaurin series for $(\sin z)/[z(1+z^2)]$.

We shall soon see that the series for $\sin z$ has the same form as the real series for $\sin x$ in Chapter 1. Using this fact we find (Problem 17)

$$\frac{\sin z}{z(1+z^2)} = 1 - \frac{7z^2}{6} + \frac{47z^4}{40} - \frac{5923z^6}{5040} + \cdots . \tag{7.3}$$

From (7.3) we can't find the radius of convergence, but let's use the theorem above. Let the numerator series be $(\sin z)/z$. By ratio test, the series for $(\sin z)/z$ converges for all z (if you like, $r_1 = \infty$). There is no r_2 since the denominator is not an infinite series. The denominator $1 + z^2$ is zero when $z = \pm i$, so $s = 1$. Then the series (7.3) converges inside a disk of radius 1 with center at the origin.

▶ PROBLEMS, SECTION 7

Find the disk of convergence for each of the following complex power series.

1. $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!}\cdots$ [equation (8.1)]

2. $z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \cdots$ **3.** $1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots$ **4.** $\displaystyle\sum_{n=0}^{\infty} z^n$

5. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$ **6.** $\displaystyle\sum_{n=1}^{\infty} n^2(3iz)^n$ **7.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{z^{2n}}{(2n+1)!}$ **9.** $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \frac{(iz)^n}{n^2}$

11. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3 z^n}{(3n)!}$ **12.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2 z^n}{(2n)!}$ **13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^n}{n}$

14. $\displaystyle\sum_{n=0}^{\infty} n(n+1)(z-2i)^n$ **15.** $\displaystyle\sum_{n=0}^{\infty} \frac{(z-2+i)^n}{2^n}$ **16.** $\displaystyle\sum_{n=1}^{\infty} 2^n (z+i-3)^{2n}$

17. Verify the series in (7.3) by computer. Also show that it can be written in the form

$$\sum_{n=0}^{\infty}(-1)^n z^{2n}\sum_{k=0}^{n}\frac{1}{(2k+1)!}.$$

Use this form to show by ratio test that the series converges in the disk $|z|<1$.

▶ 8. ELEMENTARY FUNCTIONS OF COMPLEX NUMBERS

The so-called elementary functions are powers and roots, trigonometric and inverse trigonometric functions, logarithmic and exponential functions, and combinations of these. All these you can compute or find in tables, as long as you want them as functions of real numbers. Now we want to find things like i^i, $\sin(1+i)$, or $\ln i$. These are not just curiosities for the amusement of the mathematically inclined, but may turn up to be evaluated in applied problems. To be sure, the values of experimental measurements are not imaginary. But the values of $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, angle of z, are real, and these are the quantities which have experimental meaning. Meanwhile, mathematical solutions of problems may involve manipulations of complex numbers before we arrive finally at a real answer to compare with experiment.

Polynomials and rational functions (quotients of polynomials) of z are easily evaluated.

▶ **Example.** If $f(z)=(z^2+1)/(z-3)$, we find $f(i-2)$ by substituting $z=i-2$:

$$f(i-2)=\frac{(i-2)^2+1}{i-2-3}=\frac{-4i+4}{i-5}\cdot\frac{-i-5}{-i-5}=\frac{8i-12}{13}.$$

Next we want to investigate the possible meaning of other functions of complex numbers. We should like to define expressions like e^z or $\sin z$ so that they will obey the familiar laws we know for the corresponding real expressions [for example, $\sin 2x=2\sin x\cos x$, or $(d/dx)e^x=e^x$]. We must, for consistency, define functions of complex numbers so that any equations involving them reduce to correct real equations when $z=x+iy$ becomes $z=x$, that is, when $y=0$. These requirements will be met if we define e^z by the power series

$$e^z=\sum_{0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots. \tag{8.1}$$

This series converges for all values of the complex number z (Problem 7.1) and therefore gives us the value of e^z for any z. If we put $z=x$ (x real), we get the familiar series for e^x.

It is easy to show, by multiplying the series (Problem 1), that

$$e^{z_1}\cdot e^{z_2}=e^{z_1+z_2}. \tag{8.2}$$

In Chapter 14 we shall consider in detail the meaning of derivatives with respect to a complex z. However, it is worth while for you to know that $(d/dz)z^n=nz^{n-1}$, and that, in fact, the other differentiation and integration formulas which you know

from elementary calculus hold also with x replaced by z. You can verify that $(d/dz)e^z = e^z$ when e^z is defined by (8.1) by differentiating (8.1) term by term (Problem 2). It can be shown that (8.1) is the only definition of e^z which preserves these familiar formulas. We now want to consider the consequences of this definition.

▶ PROBLEMS, SECTION 8

Show from the power series (8.1) that

1. $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

2. $\dfrac{d}{dz}e^z = e^z$

3. Find the power series for $e^x \cos x$ and for $e^x \sin x$ from the series for e^z in the following way: Write the series for e^z; put $z = x + iy$. Show that $e^z = e^x(\cos y + i \sin y)$; take real and imaginary parts of the equation, and put $y = x$.

▶ 9. EULER'S FORMULA

For real θ, we know from Chapter 1 the power series for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots, \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots. \end{aligned} \tag{9.1}$$

From our definition (8.1), we can write the series for e to any power, real or imaginary. We write the series for $e^{i\theta}$, where θ is real:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!}\cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\cdots\right). \end{aligned} \tag{9.2}$$

(The rearrangement of terms is justified because the series is absolutely convergent.) Now compare (9.1) and (9.2); the last line in (9.2) is just $\cos\theta + i\sin\theta$. We then have the very useful result we introduced in Section 3, known as Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{9.3}$$

Thus we have justified writing any complex number as we did in (4.1), namely

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{9.4}$$

Here are some examples of the use of (9.3) and (9.4). These problems can be done very quickly graphically or just by picturing them in your mind.

▶ **Examples.** Find the values of $2\,e^{i\pi/6}$, $e^{i\pi}$, $3\,e^{-i\pi/2}$, $e^{2n\pi i}$.

$2\,e^{i\pi/6}$ is $re^{i\theta}$ with $r = 2$, $\theta = \pi/6$. From Figure 9.1, $x = \sqrt{3}$, $y = 1$, $x + iy = \sqrt{3} + i$, so $2\,e^{i\pi/6} = \sqrt{3} + i$.

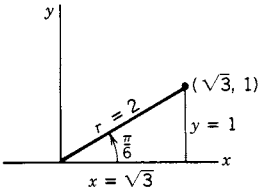


Figure 9.1

$e^{i\pi}$ is $re^{i\theta}$ with $r = 1$, $\theta = \pi$. From Figure 9.2, $x = -1$, $y = 0$, $x + iy = -1 + 0i$, so $e^{i\pi} = -1$. Note that $r = 1$ and $\theta = -\pi, \pm 3\pi, \pm 5\pi, \cdots$, give the same point, so $e^{-i\pi} = -1$, $e^{3\pi i} = -1$, and so on.

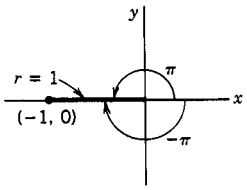


Figure 9.2

$3e^{-i\pi/2}$ is $re^{i\theta}$ with $r = 3$, $\theta = -\pi/2$. From Figure 9.3, $x = 0$, $y = -3$, so $3e^{-i\pi/2} = x + iy = 0 - 3i = -3i$.

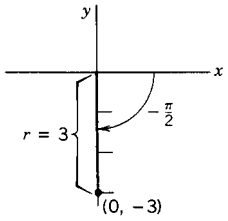


Figure 9.3

$e^{2n\pi i}$ is $re^{i\theta}$ with $r = 1$ and $\theta = 2n\pi = n(2\pi)$; that is, θ is an integral multiple of 2π. From Figure 9.4, $x = 1$, $y = 0$, so $e^{2n\pi i} = 1 + 0i = 1$.

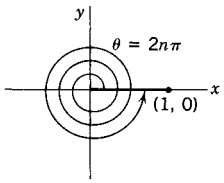


Figure 9.4

It is often convenient to use Euler's formula when we want to multiply or divide complex numbers. From (8.2) we obtain two familiar looking laws of exponents which are now valid for imaginary exponents:

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}, \qquad e^{i\theta_1} \div e^{i\theta_2} = e^{i(\theta_1-\theta_2)}. \tag{9.5}$$

Remembering that *any* complex number can be written in the form $re^{i\theta}$ by (9.4), we get

(9.6)
$$z_1 \cdot z_2 = r_1\, e^{i\theta_1} \cdot r_2\, e^{i\theta_2} = r_1\, r_2\, e^{i(\theta_1+\theta_2)},$$
$$z_1 \div z_2 = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}.$$

In words, to multiply two complex numbers, we multiply their absolute values and add their angles. To divide two complex numbers, we divide the absolute values and subtract the angles.

▶ **Example.** Evaluate $(1+i)^2/(1-i)$. From Figure 5.1 we have $1+i = \sqrt{2}\, e^{i\pi/4}$. We plot $1-i$ in Figure 9.5 and find $r = \sqrt{2}$, $\theta = -\pi/4$ (or $+7\pi/4$), so $1-i = \sqrt{2}\, e^{-i\pi/4}$. Then

$$\frac{(1+i)^2}{1-i} = \frac{(\sqrt{2}\, e^{i\pi/4})^2}{\sqrt{2}\, e^{-i\pi/4}} = \frac{2\, e^{i\pi/2}}{\sqrt{2}\, e^{-i\pi/4}} = \sqrt{2}\, e^{3i\pi/4}.$$

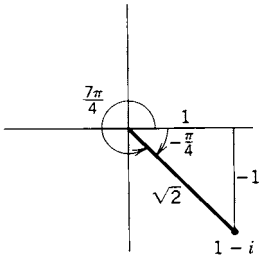


Figure 9.5

From Figure 9.6, we find $x = -1$, $y = 1$, so

$$\frac{(1+i)^2}{1-i} = x + iy = -1 + i.$$

We could use degrees in this problem. By (9.6), we find that the angle of $(1+i)^2/(1-i)$ is $2(45°) - (-45°) = 135°$ as in Figure 9.6.

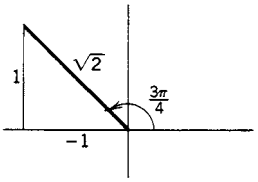


Figure 9.6

▶ PROBLEMS, SECTION 9

Express the following complex numbers in the $x + iy$ form. Try to visualize each complex number, using sketches as in the examples if necessary. The first twelve problems you should be able to do in your head (and maybe some of the others—try it!) Doing a problem quickly in your head saves time over using a computer. Remember that the point in doing problems like this is to gain skill in manipulating complex expressions, so a good study method is to do the problems by hand and use a computer to check your answers.

1. $e^{-i\pi/4}$ **2.** $e^{i\pi/2}$ **3.** $9\, e^{3\pi i/2}$

4. $e^{(1/3)(3+4\pi i)}$ **5.** $e^{5\pi i}$ **6.** $e^{-2\pi i} - e^{-4\pi i} + e^{-6\pi i}$

7. $3\, e^{2(1+i\pi)}$ **8.** $2\, e^{5\pi i/6}$ **9.** $2\, e^{-i\pi/2}$

10. $e^{i\pi} + e^{-i\pi}$ **11.** $\sqrt{2}\, e^{5i\pi/4}$ **12.** $4\, e^{-8i\pi/3}$

13. $\dfrac{\left(i-\sqrt{3}\right)^3}{1-i}$ **14.** $\left(1+i\sqrt{3}\right)^6$ **15.** $(1+i)^2 + (1+i)^4$

16. $\left(i-\sqrt{3}\right)\left(1+i\sqrt{3}\right)$ **17.** $\dfrac{1}{(1+i)^3}$ **18.** $\left(\dfrac{1+i}{1-i}\right)^4$

19. $(1-i)^8$ **20.** $\left(\dfrac{\sqrt{2}}{i-1}\right)^{10}$ **21.** $\left(\dfrac{1-i}{\sqrt{2}}\right)^{40}$

22. $\left(\dfrac{1-i}{\sqrt{2}}\right)^{42}$ **23.** $\dfrac{(1+i)^{48}}{(\sqrt{3}-i)^{25}}$ **24.** $\dfrac{(1-i\sqrt{3})^{21}}{(i-1)^{38}}$

25. $\left(\dfrac{i\sqrt{2}}{1+i}\right)^{12}$ **26.** $\left(\dfrac{2i}{i+\sqrt{3}}\right)^{19}$

27. Show that for any real y, $|e^{iy}| = 1$. Hence show that $|e^z| = e^x$ for every complex z.

28. Show that the absolute value of a product of two complex numbers is equal to the product of the absolute values. Also show that the absolute value of the quotient of two complex numbers is the quotient of the absolute values. *Hint:* Write the numbers in the $re^{i\theta}$ form.

Use Problems 27 and 28 to find the following absolute values. If you understand Problems 27 and 28 and equation (5.1), you should be able to do these in your head.

29. $|e^{i\pi/2}|$ **30.** $|e^{\sqrt{3}-i}|$ **31.** $|5\,e^{2\pi i/3}|$ **32.** $|3e^{2+4i}|$

33. $|2\,e^{3+i\pi}|$ **34.** $|4\,e^{2i-1}|$ **35.** $|3\,e^{5i}\cdot 7\,e^{-2i}|$ **36.** $|2\,e^{i\pi/6}|^2$

37. $\left|\dfrac{1+i}{1-i}\right|$ **38.** $\left|\dfrac{e^{i\pi}}{1+i}\right|$

▶ 10. POWERS AND ROOTS OF COMPLEX NUMBERS

Using the rules (9.6) for multiplication and division of complex numbers, we have

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \tag{10.1}$$

for any integral n. In words, to obtain the nth power of a complex number, we take the nth power of the modulus and multiply the angle by n. The case $r = 1$ is of particular interest. Then (10.1) becomes DeMoivre's theorem:

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{10.2}$$

You can use this equation to find the formulas for $\sin 2\theta$, $\cos 2\theta$, $\sin 3\theta$, etc. (Problems 27 and 28).

The nth root of z, $z^{1/n}$, means a complex number whose nth power is z. From (10.1) you can see that this is

$$z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right). \tag{10.3}$$

This formula must be used with care (see Examples 2 to 4 below).

Some examples will show how useful these formulas are.

▶ **Example 1.**

$$[\cos(\pi/10) + i\sin(\pi/10)]^{25} = (e^{i\pi/10})^{25} = e^{2\pi i}e^{i\pi/2} = 1\cdot i = i.$$

▶ **Example 2.** Find the cube roots of 8. We know that 2 is a cube root of 8, but there are also two complex cube roots of 8; let us see why. Plot the complex number 8 (that is, $x = 8$, $y = 0$) in the complex plane; the polar coordinates of the point are $r = 8$, and $\theta = 0$, or $360°, 720°, 1080°$, etc. (We can use either degrees or radians here; read the end of Section 3.) Now by equation (10.3), $z^{1/3} = r^{1/3}e^{i\theta/3}$; that is, to find the polar coordinates of the cube root of a number $re^{i\theta}$, we find the cube root of r and divide the angle by 3. Then the polar coordinates of $\sqrt[3]{8}$ are

$$\begin{aligned} (10.4) \qquad r = 2, \quad \theta = 0°, \quad 360°/3, \quad 720°/3, \quad 1080°/3 \cdots \\ = 0°, \quad 120°, \quad 240°, \quad 360° \cdots . \end{aligned}$$

We plot these points in Figure 10.1. Observe that the point $(2, 0°)$ and the point $(2, 360°)$ are the same. The points in (10.4) are all on a circle of radius 2 and are equally spaced $360°/3 = 120°$ apart. Starting with $\theta = 0$, if we add $120°$ repeatedly, we just repeat the three angles shown. Thus, there are exactly three cube roots for any number z, always on a circle of radius $\sqrt[3]{|z|}$ and spaced $120°$ apart.

Now to find the values of $\sqrt[3]{8}$ in rectangular form, we can read them from Figure 10.1, or we can calculate them from $z = r(\cos\theta + i\sin\theta)$ with $r = 2$ and $\theta = 0$, $120° = 2\pi/3$, $240° = 4\pi/3$. We can also use a computer to solve the equation $z^3 = 8$. By any of these methods we find

$$\sqrt[3]{8} = \{2, -1 + i\sqrt{3}, -1 - i\sqrt{3}\}.$$

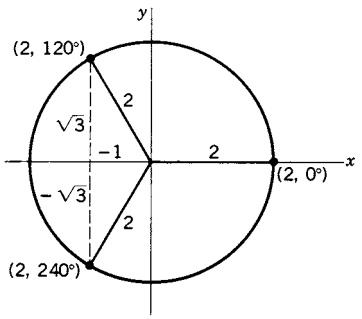


Figure 10.1

▶ **Example 3.** Find and plot all values of $\sqrt[4]{-64}$. From Figure 10.2 (or by visualizing a plot of -64), we see that the polar coordinates of -64 are $r = 64$, $\theta = \pi + 2k\pi$

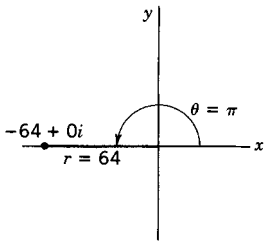


Figure 10.2

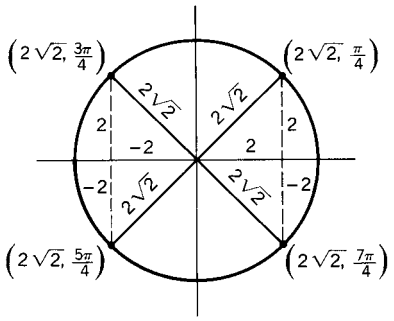


Figure 10.3

(where $k = 0, 1, 2, \cdots$). Then since $z^{1/4} = r^{1/4}e^{i\theta/4}$, the polar coordinates of $\sqrt[4]{-64}$ are

$$r = \sqrt[4]{64} = 2\sqrt{2},$$
$$\theta = \frac{\pi}{4}, \frac{\pi + 2\pi}{4}, \frac{\pi + 4\pi}{4}, \frac{\pi + 6\pi}{4}, \cdots = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

We plot these points in Figure 10.3. Observe that they are all on a circle of radius $2\sqrt{2}$, equally spaced $2\pi/4 = \pi/2$ apart. Starting with $\theta = \pi/4$, we add $\pi/2$

repeatedly, and find exactly 4 fourth roots. We can read the values of $\sqrt[4]{-64}$ in rectangular form from Figure 10.3:

$$\sqrt[4]{-64} = \pm 2 \pm 2i \quad \text{(all four combinations of } \pm \text{ signs)}$$

or we can calculate them as in Example 2, or we can solve the equation $z^4 = -64$ by computer.

▶ **Example 4.** Find and plot all values of $\sqrt[6]{-8i}$. The polar coordinates of $-8i$ are $r = 8$, $\theta = 270° + 360°k = 3\pi/2 + 2\pi k$. Then the polar coordinates of $\sqrt[6]{-8i}$ are

$$r = \sqrt{2}, \quad \theta = \frac{270° + 360°k}{6} = 45° + 60°k \quad \text{or} \quad \theta = \frac{\pi}{4} + \frac{\pi}{3}k. \tag{10.5}$$

In Figure 10.4, we sketch a circle of radius $\sqrt{2}$. On it we plot the point at 45° and then plot the rest of the 6 equally spaced points 60° apart. To find the roots in rectangular coordinates, we need to find all the values of $r(\cos\theta + i\sin\theta)$ with r and θ given by (10.5). We can do this one root at a time or more simply by using a computer to solve the equation $z^6 = -8i$. We find (see Problem 33)

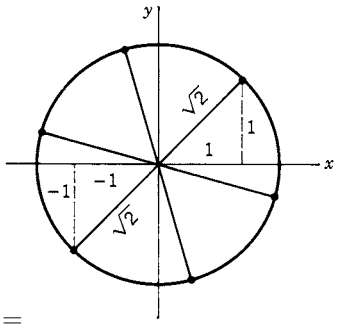


Figure 10.4

$$\pm\left\{1 + i, \quad \frac{\sqrt{3}+1}{2} - \frac{\sqrt{3}-1}{2}i, \quad \frac{\sqrt{3}-1}{2} - \frac{\sqrt{3}+1}{2}i\right\} =$$
$$\pm\{1 + i, \quad 1.366 - 0.366i, \quad 0.366 - 1.366i\}.$$

Summary In each of the preceding examples, our steps in finding $\sqrt[n]{re^{i\theta}}$ were:

(a) Find the polar coordinates of the roots: Take the nth root of r and divide $\theta + 2k\pi$ by n.

(b) Make a sketch: Draw a circle of radius $\sqrt[n]{r}$, plot the root with angle θ/n, and then plot the rest of the n roots around the circle equally spaced $2\pi/n$ apart. Note that we have now essentially solved the problem. From the sketch you can see the approximate rectangular coordinates of the roots and check your answers in (c). Since this sketch is quick and easy to do, it is worthwhile even if you use a computer to do part (c).

(c) Find the $x + iy$ coordinates of the roots by one of the methods in the examples. If you are using a computer, you may want to make a computer plot of the roots which should be a perfected copy of your sketch in (b).

▶ PROBLEMS, SECTION 10

Follow steps (a), (b), (c) above to find all the values of the indicated roots.

1.	$\sqrt[3]{1}$	**2.**	$\sqrt[3]{27}$	**3.**	$\sqrt[4]{1}$
4.	$\sqrt[4]{16}$	**5.**	$\sqrt[6]{1}$	**6.**	$\sqrt[6]{64}$
7.	$\sqrt[8]{16}$	**8.**	$\sqrt[8]{1}$	**9.**	$\sqrt[5]{1}$
10.	$\sqrt[5]{32}$	**11.**	$\sqrt[3]{-8}$	**12.**	$\sqrt[3]{-1}$

13. $\sqrt[4]{-4}$ **14.** $\sqrt[4]{-1}$ **15.** $\sqrt[6]{-64}$

16. $\sqrt[6]{-1}$ **17.** $\sqrt[5]{-1}$ **18.** $\sqrt{i}$

19. $\sqrt[3]{i}$ **20.** $\sqrt[3]{-8i}$ **21.** $\sqrt{2+2i\sqrt{3}}$

22. $\sqrt[3]{2i-2}$ **23.** $\sqrt[4]{8i\sqrt{3}-8}$ **24.** $\sqrt[8]{\dfrac{-1-i\sqrt{3}}{2}}$

25. $\sqrt[5]{-1-i}$ **26.** $\sqrt[5]{i}$

27. Using the fact that a complex equation is really two real equations, find the double angle formulas (for $\sin 2\theta$, $\cos 2\theta$) by using equation (10.2).

28. As in Problem 27, find the formulas for $\sin 3\theta$ and $\cos 3\theta$.

29. Show that the center of mass of three identical particles situated at the points z_1, z_2, z_3 is $(z_1 + z_2 + z_3)/3$.

30. Show that the sum of the three cube roots of 8 is zero.

31. Show that the sum of the n nth roots of any complex number is zero.

32. The three cube roots of $+1$ are often called 1, ω, and ω^2. Show that this is reasonable, that is, show that the cube roots of $+1$ are $+1$ and two other numbers, each of which is the square of the other.

33. Verify the results given for the roots in Example 4. You can find the exact values in terms of $\sqrt{3}$ by using trigonometric addition formulas or more easily by using a computer to solve $z^6 = -8i$. (You still may have to do a little work by hand to put the computer's solution into the given form.)

▶ 11. THE EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

Although we have already defined e^z by a power series (8.1), it is worth while to write it in another form. By (8.2) we can write

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y). \tag{11.1}$$

This is more convenient to use than the infinite series if we want values of e^z for given z. For example,

$$e^{2-i\pi} = e^2 e^{-i\pi} = e^2 \cdot (-1) = -e^2$$

from Figure 9.2.

We have already seen that there is a close relationship [Euler's formula (9.3)] between complex exponentials and trigonometric functions of real angles. It is useful to write this relation in another form. We write Euler's formula (9.3) as it

is, and also write it with θ replaced by $-\theta$. Remember that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Then we have

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta. \tag{11.2}$$

These two equations can be solved for $\sin\theta$ and $\cos\theta$. We get (Problem 2)

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{11.3}$$

These formulas are useful in evaluating integrals since products of exponentials are easier to integrate than products of sines and cosines. (See Problems 11 to 16, and Chapter 7, Section 5.)

So far we have discussed only trigonometric functions of real angles. We could define $\sin z$ and $\cos z$ for complex z by their power series as we did for e^z. We could then compare these series with the series for e^{iz} and derive Euler's formula and (11.3) with θ replaced by z. However, it is simpler to use the complex equations corresponding to (11.3) as our definitions for $\sin z$ and $\cos z$. We define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{11.4}$$

The rest of the trigonometric functions of z are defined in the usual way in terms of these; for example, $\tan z = \sin z/\cos z$.

▶ **Example 1.** $\cos i = \dfrac{e^{i\cdot i} + e^{-i\cdot i}}{2} = \dfrac{e^{-1} + e}{2} = 1.543\cdots$. (We will see in Section 15 that this expression is called the hyperbolic cosine of 1.)

▶ **Example 2.**

$$\sin\left(\frac{\pi}{2} + i\ln 2\right) = \frac{e^{i(\pi/2 + i\ln 2)} - e^{-i(\pi/2 + i\ln 2)}}{2i} = \frac{e^{i\pi/2}e^{-\ln 2} - e^{-i\pi/2}e^{\ln 2}}{2i} \qquad \text{by (8.2).}$$

From Figures 5.2 and 9.3, $e^{i\pi/2} = i$, and $e^{-i\pi/2} = -i$. By the definition of $\ln x$ [or see equations (13.1) and (13.2)], $e^{\ln 2} = 2$, so $e^{-\ln 2} = 1/e^{\ln 2} = 1/2$. Then

$$\sin\left(\frac{\pi}{2} + i\ln 2\right) = \frac{(i)(1/2) - (-i)(2)}{2i} = \frac{5}{4}.$$

Notice from both these examples that sines and cosines of complex numbers may be greater than 1. As we shall see (Section 15), although $|\sin x| \le 1$ and $|\cos x| \le 1$ for *real* x, when z is a complex number, $\sin z$ and $\cos z$ can have *any* value we like.

Using the definitions (11.4) of $\sin z$ and $\cos z$, you can show that the familiar trigonometric identities and calculus formulas hold when θ is replaced by z.

▶ **Example 3.** Prove that $\sin^2 z + \cos^2 z = 1$.

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} - 2 + e^{-2iz}}{-4},$$

$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} + 2 + e^{-2iz}}{4},$$

$$\sin^2 z + \cos^2 z = \frac{2}{4} + \frac{2}{4} = 1.$$

▶ **Example 4.** Using the definitions (11.4), verify that $(d/dz)\sin z = \cos z$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

$$\frac{d}{dz}\sin z = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

▶ PROBLEMS, SECTION 11

1. Define $\sin z$ and $\cos z$ by their power series. Write the power series for e^{iz}. By comparing these series obtain the definition (11.4) of $\sin z$ and $\cos z$.

2. Solve the equations $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$, for $\cos\theta$ and $\sin\theta$ and so obtain equations (11.3).

Find each of the following in rectangular form $x + iy$ and check your results by computer. Remember to save time by doing as much as you can in your head.

3. $e^{-(i\pi/4)+\ln 3}$ **4.** $e^{3\ln 2 - i\pi}$ **5.** $e^{(i\pi/4)+(\ln 2)/2}$

6. $\cos(i\ln 5)$ **7.** $\tan(i\ln 2)$ **8.** $\cos(\pi - 2i\ln 3)$

9. $\sin(\pi - i\ln 3)$ **10.** $\sin(i\ln i)$

In the following integrals express the sines and cosines in exponential form and then integrate to show that:

11. $\displaystyle\int_{-\pi}^{\pi} \cos 2x\, \cos 3x\, dx = 0$ **12.** $\displaystyle\int_{-\pi}^{\pi} \cos^2 3x\, dx = \pi$

13. $\displaystyle\int_{-\pi}^{\pi} \sin 2x\, \sin 3x\, dx = 0$ **14.** $\displaystyle\int_{0}^{2\pi} \sin^2 4x\, dx = \pi$

15. $\displaystyle\int_{-\pi}^{\pi} \sin 2x\, \cos 3x\, dx = 0$ **16.** $\displaystyle\int_{-\pi}^{\pi} \sin 3x\, \cos 4x\, dx = 0$

Evaluate $\int e^{(a+ib)x}dx$ and take real and imaginary parts to show that:

17. $\displaystyle\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\,\cos bx + b\,\sin bx)}{a^2 + b^2}$

18. $\displaystyle\int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\,\sin bx - b\,\cos bx)}{a^2 + b^2}$

▶ 12. HYPERBOLIC FUNCTIONS

Let us look at $\sin z$ and $\cos z$ for pure imaginary z, that is, $z = iy$:

$$\sin iy = \frac{e^{-y} - e^{y}}{2i} = i\,\frac{e^{y} - e^{-y}}{2}, \qquad \cos iy = \frac{e^{-y} + e^{y}}{2} = \frac{e^{y} + e^{-y}}{2}. \tag{12.1}$$

The real functions on the right have special names because these particular combinations of exponentials arise frequently in problems. They are called the hyperbolic sine (abbreviated sinh) and the hyperbolic cosine (abbreviated cosh). Their definitions for all z are

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}. \tag{12.2}$$

The other hyperbolic functions are named and defined in a similar way to parallel the trigonometric functions:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{1}{\tanh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}. \tag{12.3}$$

(See Problem 38 for the reason behind the term "hyperbolic" functions.)

We can write (12.1) as

$$\sin iy = i \sinh y, \qquad \cos iy = \cosh y. \tag{12.4}$$

Then we see that the hyperbolic functions of y are (except for one i factor) the trigonometric functions of iy. From (12.2) we can show that (12.4) holds with y replaced by z. Because of this relation between hyperbolic and trigonometric functions, the formulas for hyperbolic functions look very much like the corresponding trigonometric identities and calculus formulas. They are not identical, however.

▶ **Example.** You can prove the following formulas (see Problems 9, 10, 11 and 38).

$$\cosh^2 z - \sinh^2 z = 1 \qquad (\text{compare } \sin^2 z + \cos^2 z = 1),$$
$$\frac{d}{dz}\cosh z = \sinh z \qquad (\text{compare } \frac{d}{dz}\cos z = -\sin z).$$

▶ PROBLEMS, SECTION 12

Verify each of the following by using equations (11.4), (12.2), and (12.3).

1. $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

2. $\cos z = \cos x \cosh y - i \sin x \sinh y$

3. $\sinh z = \sinh x \cos y + i \cosh x \sin y$

4. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

5. $\sin 2z = 2 \sin z \cos z$

6. $\cos 2z = \cos^2 z - \sin^2 z$

7. $\sinh 2z = 2 \sinh z \cosh z$

8. $\cosh 2z = \cosh^2 z + \sinh^2 z$

9. $\dfrac{d}{dz} \cos z = -\sin z$

10. $\dfrac{d}{dz} \cosh z = \sinh z$

11. $\cosh^2 z - \sinh^2 z = 1$

12. $\cos^4 z + \sin^4 z = 1 - \dfrac{1}{2} \sin^2 2z$

13. $\cos 3z = 4 \cos^3 z - 3 \cos z$

14. $\sin iz = i \sinh z$

15. $\sinh iz = i \sin z$

16. $\tan iz = i \tanh z$

17. $\tanh iz = i \tan z$

18. $\tan z = \tan(x + iy) = \dfrac{\tan x + i \tanh y}{1 - i \tan x \tanh y}$

19. $\tanh z = \dfrac{\tanh x + i \tan y}{1 + i \tanh x \tan y}$

20. Show that $e^{nz} = (\cosh z + \sinh z)^n = \cosh nz + \sinh nz$. Use this and a similar equation for e^{-nz} to find formulas for $\cosh 3z$ and $\sinh 3z$ in terms of $\sinh z$ and $\cosh z$.

21. Use a computer to plot graphs of $\sinh x$, $\cosh x$, and $\tanh x$.

22. Using (12.2) and (8.1), find, in summation form, the power series for $\sinh x$ and $\cosh x$. Check the first few terms of your series by computer.

Find the real part, the imaginary part, and the absolute value of

23. $\cosh(ix)$

24. $\cos(ix)$

25. $\sin(x - iy)$

26. $\cosh(2 - 3i)$

27. $\sin(4 + 3i)$

28. $\tanh(1 - i\pi)$

Find each of the following in the $x + iy$ form and check your answers by computer.

29. $\cosh 2\pi i$

30. $\tanh \dfrac{3\pi i}{4}$

31. $\sinh\left(\ln 2 + \dfrac{i\pi}{3}\right)$

32. $\cosh\left(\dfrac{i\pi}{2} - \ln 3\right)$

33. $\tan i$

34. $\sin \dfrac{i\pi}{2}$

35. $\cosh(i\pi + 2)$

36. $\sinh\left(1 + \dfrac{i\pi}{2}\right)$

37. $\cos(i\pi)$

38. The functions $\sin t$, $\cos t$, $\cdots$, are called "circular functions" and the functions $\sinh t$, $\cosh t$, $\cdots$, are called "hyperbolic functions". To see a reason for this, show that $x = \cos t$, $y = \sin t$, satisfy the equation of a circle $x^2 + y^2 = 1$, while $x = \cosh t$, $y = \sinh t$, satisfy the equation of a hyperbola $x^2 - y^2 = 1$.

▶ 13. LOGARITHMS

In elementary mathematics you learned to find logarithms of positive numbers only; in fact, you may have been told that there were no logarithms of negative numbers. This is true if you use only real numbers, but it is not true when we allow complex numbers as answers. We shall now see how to find the logarithm of any complex number $z \neq 0$ (including negative real numbers as a special case). If

$$z = e^w, \tag{13.1}$$

then by definition

$$w = \ln z. \tag{13.2}$$

(We use ln for natural logarithms to avoid the cumbersome $\log_e$ and to avoid confusion with logarithms to the base 10.)

We can write the law of exponents (8.2), using the letters of (13.1), as

$$z_1 z_2 = e^{w_1} \cdot e^{w_2} = e^{w_1 + w_2}. \tag{13.3}$$

Taking logarithms of this equation, that is, using (13.1) and (13.2), we get

$$\ln z_1 z_2 = w_1 + w_2 = \ln z_1 + \ln z_2. \tag{13.4}$$

This is the familiar law for the logarithm of a product, justified now for complex numbers. We can then find the real and imaginary parts of the logarithm of a complex number from the equation

$$w = \ln z = \ln(re^{i\theta}) = \operatorname{Ln} r + \ln e^{i\theta} = \operatorname{Ln} r + i\theta, \tag{13.5}$$

where $\operatorname{Ln} r$ means the ordinary real logarithm to the base e of the real positive number r.

Since θ has an infinite number of values (all differing by multiples of 2π), a complex number has infinitely many logarithms, differing from each other by multiples of $2\pi i$. The *principal value* of $\ln z$ (often written as $\operatorname{Ln} z$) is the one using the principal value of θ, that is $0 \leq \theta < 2\pi$. (Some references use $-\pi < \theta \leq \pi$.)

▶ **Example 1.** Find $\ln(-1)$. From Figure 9.2, we see that the polar coordinates of the point $z = -1$ are $r = 1$ and $\theta = \pi$, $-\pi$, 3π, $\cdots$. Then,

$$\ln(-1) = \operatorname{Ln}(1) + i(\pi \pm 2n\pi) = i\pi, -i\pi, 3\pi i, \cdots.$$

▶ **Example 2.** Find $\ln(1+i)$. From Figure 5.1, for $z = 1 + i$, we find $r = \sqrt{2}$, and $\theta = \pi/4 \pm 2n\pi$. Then

$$\ln(1+i) = \operatorname{Ln}\sqrt{2} + i\left(\frac{\pi}{4} \pm 2n\pi\right) = 0.347\cdots + i\left(\frac{\pi}{4} \pm 2n\pi\right).$$

Even a positive real number now has infinitely many logarithms, since its angle can be taken as 0, 2π, -2π, etc. Only one of these logarithms is real, namely the principal value $\operatorname{Ln} r$ using the angle $\theta = 0$.

▶ 14. COMPLEX ROOTS AND POWERS

For real positive numbers, the equation $\ln a^b = b \ln a$ is equivalent to $a^b = e^{b \ln a}$. We define complex powers by the same formula with complex a and b. By definition, for complex a and b ($a \neq e$),

$$a^b = e^{b \ln a}. \tag{14.1}$$

[The case $a = e$ is excluded because we have already defined powers of e by (8.1).] Since $\ln a$ is multiple valued (because of the infinite number of values of θ), powers a^b are usually multiple valued, and unless you want just the principal value of $\ln z$ or of a^b you must use all values of θ. In the following examples we find all values of each complex power and write the answers in the $x + iy$ form.

▶ **Example 1.** Find all values of i^{-2i}. From Figure 5.2, and equation (13.5) we find $\ln i = \operatorname{Ln} 1 + i(\pi/2 \pm 2n\pi) = i(\pi/2 \pm 2n\pi)$ since $\operatorname{Ln} 1 = 0$. Then, by equation (14.1),

$$i^{-2i} = e^{-2i \ln i} = e^{-2i \cdot i(\pi/2 \pm 2n\pi)} = e^{\pi \pm 4n\pi} = e^{\pi}, e^{5\pi}, e^{-3\pi}, \cdots,$$

where $e^{\pi} = 23.14\cdots$. Note the infinite set of values of i^{-2i}, all real! Also read the end of Section 3, and note that here the final step is *not* to find sine or cosine of $\pi \pm 4n\pi$; thus, in finding $\ln i = i\theta$, we must *not* write θ in degrees.

▶ **Example 2.** Find all values of $i^{1/2}$. Using $\ln i$ from Example 1 we have $i^{1/2} = e^{(1/2) \ln i} = e^{i(\pi/4 + n\pi)} = e^{i\pi/4} e^{in\pi}$. Now $e^{in\pi} = +1$ when n is even (Fig. 9.4), and $e^{in\pi} = -1$ when n is odd (Fig. 9.2). Thus,

$$i^{1/2} = \pm e^{i\pi/4} = \pm \frac{1 + i}{\sqrt{2}}$$

using Figure 5.1. Notice that although $\ln i$ has an infinite set of values, we find just two values for $i^{1/2}$ as we should for a square root. (Compare the method of Section 10 which is easier for this problem.)

▶ **Example 3.** Find all values of $(1 + i)^{1-i}$. Using (14.1) and the value of $\ln(1 + i)$ from Example 2, Section 13, we have

$$\begin{aligned}
(1 + i)^{1-i} &= e^{(1-i) \ln(1+i)} = e^{(1-i)[\operatorname{Ln} \sqrt{2} + i(\pi/4 \pm 2n\pi)]} \\
&= e^{\operatorname{Ln} \sqrt{2}}\, e^{-i \operatorname{Ln} \sqrt{2}}\, e^{i\pi/4}\, e^{\pm 2n\pi i}\, e^{\pi/4}\, e^{\pm 2n\pi} \\
&= \sqrt{2}\, e^{i(\pi/4 - \operatorname{Ln} \sqrt{2})}\, e^{\pi/4}\, e^{\pm 2n\pi} \qquad (\text{since } e^{\pm 2n\pi i} = 1) \\
&= \sqrt{2}\, e^{\pi/4}\, e^{\pm 2n\pi} [\cos(\pi/4 - \operatorname{Ln} \sqrt{2}) + i \sin(\pi/4 - \operatorname{Ln} \sqrt{2})] \\
&\cong e^{\pm 2n\pi} (2.808 + 1.318i).
\end{aligned}$$

Now you may be wondering why not just do these problems by computer. The most important point is that it is useful for advanced work to have skill in manipulating complex expressions. A second point is that there may be several forms for an answer (see Section 15, Example 2) or there may be many answers (see examples

above), and your computer may not give you the one you want (see Problem 25). So to obtain needed skills, a good study method is to do problems by hand and compare with computer solutions.

▶ PROBLEMS, SECTION 14

Evaluate each of the following in $x + iy$ form, and compare with a computer solution.

1. $\ln(-e)$ **2.** $\ln(-i)$ **3.** $\ln(i + \sqrt{3})$

4. $\ln(i - 1)$ **5.** $\ln(-\sqrt{2} - i\sqrt{2})$ **6.** $\ln\left(\dfrac{1-i}{\sqrt{2}}\right)$

7. $\ln\left(\dfrac{1+i}{1-i}\right)$ **8.** $i^{2/3}$ **9.** $(-1)^i$

10. $i^{\ln i}$ **11.** 2^i **12.** i^{3+i}

13. $i^{2i/\pi}$ **14.** $(2i)^{1+i}$ **15.** $(-1)^{\sin i}$

16. $\left(\dfrac{1+i\sqrt{3}}{2}\right)^i$ **17.** $(i-1)^{i+1}$ **18.** $\cos(2i \ln i)$

19. $\cos(\pi + i \ln 2)$ **20.** $\sin\left(i \ln \dfrac{1-i}{1+i}\right)$ **21.** $\cos[i \ln(-1)]$

22. $\sin\left[i \ln\left(\dfrac{\sqrt{3}+i}{2}\right)\right]$ **23.** $\left(1 - \sqrt{2i}\right)^i$. *Hint:* Find $\sqrt{2i}$ first.

24. Show that $(a^b)^c$ can have more values than a^{bc}. As examples compare

(a) $[(-i)^{2+i}]^{2-i}$ and $(-i)^{(2+i)(2-i)} = (-i)^5$;

(b) $(i^i)^i$ and i^{-1}.

25. Use a computer to find the three solutions of the equation $x^3 - 3x - 1 = 0$. Find a way to show that the solutions can be written as $2\cos(\pi/9)$, $-2\cos(2\pi/9)$, $-2\cos(4\pi/9)$.

▶ 15. INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

We have already defined the trigonometric and hyperbolic functions of a complex number z. For example,

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{15.1}$$

defines $w = \cos z$; that is, for each complex number z, (15.1) gives us the complex number w. We now define the inverse cosine or $\arccos w$ by

$$z = \arccos w \qquad \text{if} \qquad w = \cos z. \tag{15.2}$$

The other inverse trigonometric and hyperbolic functions are defined similarly.

In dealing with real numbers, you know that $\sin x$ and $\cos x$ are never greater than 1. This is no longer true for $\sin z$ and $\cos z$ with z complex. To illustrate the method of finding inverse trigonometric (or inverse hyperbolic) functions, let's find $\arccos 2$.

▶ **Example 1.** We want z, where

$$z = \arccos 2 \quad \text{or} \quad \cos z = 2.$$

Then we have

$$\frac{e^{iz} + e^{-iz}}{2} = 2.$$

To simplify the algebra, let $u = e^{iz}$. Then $e^{-iz} = u^{-1}$, and the equation becomes

$$\frac{u + u^{-1}}{2} = 2.$$

Multiply by 2 and by u to get $u^2 + 1 = 4u$ or $u^2 - 4u + 1 = 0$. Solve this equation by the quadratic formula to find

$$u = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}, \quad \text{or} \quad e^{iz} = u = 2 \pm \sqrt{3}.$$

Take logarithms of both sides of this equation, and solve for z:

$$iz = \ln(2 \pm \sqrt{3}) = \operatorname{Ln}(2 \pm \sqrt{3}) + 2n\pi i,$$
$$\arccos 2 = z = 2n\pi - i \operatorname{Ln}(2 \pm \sqrt{3}\,) = 2n\pi \pm i \operatorname{Ln}(2 + \sqrt{3}\,)$$

since $\operatorname{Ln}(2 - \sqrt{3}) = -\operatorname{Ln}(2 + \sqrt{3})$.

It is instructive now to find $\cos z$ and see that it *is* 2. For $iz = \ln(2 \pm \sqrt{3})$, we have

$$e^{iz} = e^{\ln(2 \pm \sqrt{3})} = 2 \pm \sqrt{3},$$
$$e^{-iz} = \frac{1}{e^{iz}} = \frac{1}{2 \pm \sqrt{3}} = \frac{2 \mp \sqrt{3}}{4 - 3} = 2 \mp \sqrt{3}.$$

Then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{2 \pm \sqrt{3} + 2 \mp \sqrt{3}}{2} = \frac{4}{2} = 2,$$

as claimed.

By the same method, we can find all the inverse trigonometric and hyperbolic functions in terms of logarithms. (See Problems, Section 17.) Here is one more example.

▶ **Example 2.** In integral tables or from your computer you may find for the indefinite integral

$$\int \frac{dx}{\sqrt{x^2 + a^2}} \tag{15.3}$$

either

$$\sinh^{-1} \frac{x}{a} \quad \text{or} \quad \ln(x + \sqrt{x^2 + a^2}\,). \tag{15.4}$$

How are these related? Put

$$z = \sinh^{-1} \frac{x}{a} \quad \text{or} \quad \frac{x}{a} = \sinh z = \frac{e^z - e^{-z}}{2}. \tag{15.5}$$

We solve for z as in the previous example. Let $e^z = u$, $e^{-z} = 1/u$. Then

$$u - \frac{1}{u} = \frac{2x}{a},$$

$$au^2 - 2xu - a = 0,$$

$$e^z = u = \frac{2x \pm \sqrt{4x^2 + 4a^2}}{2a} = \frac{x \pm \sqrt{x^2 + a^2}}{a}. \tag{15.6}$$

For real integrals, that is, for real z, $e^z > 0$, so we must use the positive sign. Then, taking the logarithm of (15.6) we have

$$z = \ln(x + \sqrt{x^2 + a^2}\,) - \ln a. \tag{15.7}$$

Comparing (15.5) and (15.7) we see that the two answers in (15.4) differ only by the constant $\ln a$, which is a constant of integration.

▶ PROBLEMS, SECTION 15

Find each of the following in the $x + iy$ form and compare a computer solution.

1. $\arcsin 2$

2. $\arctan 2i$

3. $\cosh^{-1}(1/2)$

4. $\sinh^{-1}(i/2)$

5. $\arccos\left(i\sqrt{8}\right)$

6. $\tanh^{-1}(-i)$

7. $\arctan\left(i\sqrt{2}\right)$

8. $\arcsin(5/3)$

9. $\tanh^{-1}\left(i\sqrt{3}\right)$

10. $\arccos(5/4)$

11. $\sinh^{-1}\left(i/\sqrt{2}\right)$

12. $\cosh^{-1}\left(\sqrt{3}/2\right)$

13. $\cosh^{-1}(-1)$

14. $\arcsin(3i/4)$

15. $\arctan(2 + i)$

16. $\tanh^{-1}(1 - 2i)$

17. Show that $\tan z$ never takes the values $\pm i$. *Hint:* Try to solve the equation $\tan z = i$ and find that it leads to a contradiction.

18. Show that $\tanh z$ never takes the values ± 1.

▶ 16. SOME APPLICATIONS

Motion of a Particle We have already seen (end of Section 5) that the path of a particle in the (x, y) plane is given by $z = z(t)$. As another example of this, suppose $z = 1 + 3e^{2it}$. We see that

$$|z - 1| = |3\, e^{2it}| = 3. \tag{16.1}$$

Recall that $|z - 1|$ is the distance between the points z and 1; (16.1) says that this distance is 3. Thus the particle traverses a circle of radius 3, with center at $(1, 0)$. The magnitude of its velocity is $|dz/dt| = |6i\, e^{2it}| = 6$, so it moves around the circle at constant speed. (Also see Problem 2).

PROBLEMS, SECTION 16

1. Show that if the line through the origin and the point z is rotated 90° about the origin, it becomes the line through the origin and the point iz. This fact is sometimes expressed by saying that multiplying a complex number by i rotates it through 90°. Use this idea in the following problem. Let $z = a\,e^{i\omega t}$ be the displacement of a particle from the origin at time t. Show that the particle travels in a circle of radius a at velocity $v = a\omega$ and with acceleration of magnitude v^2/a directed toward the center of the circle.

In each of the following problems, z represents the displacement of a particle from the origin. Find (as functions of t) its speed and the magnitude of its acceleration, and describe the motion.

2. $z = 5e^{i\omega t}$, $\omega = \text{const.}$ *Hint:* See Problem 1.

3. $z = (1+i)e^{it}$.

4. $z = (1+i)t - (2+i)(1-t)$. *Hint:* Show that the particle moves along a straight line through the points $(1+i)$ and $(-2-i)$.

5. $z = z_1 t + z_2(1-t)$. *Hint:* See Problem 4; the straight line here is through the points z_1 and z_2.

Electricity In the theory of electric circuits, it is shown that if V_R is the voltage across a resistance R, and I is the current flowing through the resistor, then

$$V_R = IR \quad \text{(Ohm's law)}. \tag{16.2}$$

It is also known that the current and voltage across an inductance L are related by

$$V_L = L\frac{dI}{dt} \tag{16.3}$$

and the current and voltage across a capacitor are related by

$$\frac{dV_C}{dt} = \frac{I}{C}, \tag{16.4}$$

where C is the capacitance. Suppose the current I and voltage V in the circuit of Figure 16.1 vary with time so that I is given by

$$I = I_0 \sin \omega t. \tag{16.5}$$

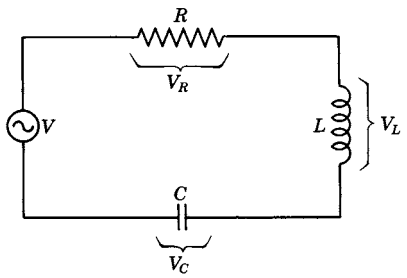


Figure 16.1

You can verify that the following voltages across R, L, and C are consistent with (16.2), (16.3), and (16.4):

$$V_R = RI_0 \sin \omega t, \tag{16.6}$$

$$V_L = \omega L I_0 \cos \omega t, \tag{16.7}$$

$$V_C = -\frac{1}{\omega C} I_0 \cos \omega t. \tag{16.8}$$

The total voltage

$$V = V_R + V_L + V_C \tag{16.9}$$

is then a complicated function. A simpler method of discussing a-c circuits uses complex quantities as follows. Instead of (16.5) we write

$$I = I_0\, e^{i\omega t}, \tag{16.10}$$

where it is understood that the actual physical current is given by the imaginary part of I in (16.10), that is, by (16.5). Note, by comparing (16.5) and (16.10), that the maximum value of I, namely I_0, is given in (16.10) by $|I|$. Now equations (16.6) to (16.9) become

$$V_R = RI_0\, e^{i\omega t} = RI, \tag{16.11}$$

$$V_L = i\omega LI_0\, e^{i\omega t} = i\omega LI, \tag{16.12}$$

$$V_C = \frac{1}{i\omega C} I_0\, e^{i\omega t} = \frac{1}{i\omega C} I, \tag{16.13}$$

$$V = V_R + V_L + V_C = \left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] I. \tag{16.14}$$

The complex quantity Z defined by

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right) \tag{16.15}$$

is called the (complex) impedance. Using it we can write (16.14) as

$$V = ZI \tag{16.16}$$

which looks much like Ohm's law. In fact, Z for an a-c circuit corresponds to R for a d-c circuit. The more complicated a-c circuit equations now take the same simple form as the d-c equations except that all quantities are complex. For example, the rules for combining resistances in series and in parallel hold for combining complex impedances (see Problems below).

▶ PROBLEMS, SECTION 16

In electricity we learn that the resistance of two resistors in series is $R_1 + R_2$ and the resistance of two resistors in parallel is $(R_1^{-1} + R_2^{-1})^{-1}$. Corresponding formulas hold for complex impedances. Find the impedance of Z_1 and Z_2 in series, and in parallel, given:

6. (a) $Z_1 = 2 + 3i$, $Z_2 = 1 - 5i$ (b) $Z_1 = 2\sqrt{3}\, e^{i\pi/6}$, $Z_2 = 2\, e^{2i\pi/3}$

7. (a) $Z_1 = 1 - i$, $Z_2 = 3i$ (b) $|Z_1| = 3.16$, $\theta_1 = 18.4^\circ$; $|Z_2| = 4.47$, $\theta_2 = 63.4^\circ$

8. Find the impedance of the circuit in Figure 16.2 (R and L in series, and then C in parallel with them). A circuit is said to be *in resonance* if Z is real; find ω in terms of R, L, and C at resonance.

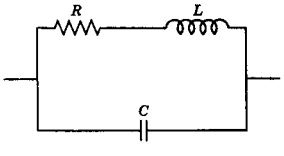


Figure 16.2

9. For the circuit in Figure 16.1:

(a) Find ω in terms of R, L, and C if the angle of Z is 45°.

(b) Find the resonant frequency ω (see Problem 8).

10. Repeat Problem 9 for a circuit consisting of R, L, and C, all in parallel.

Optics In optics we frequently need to combine a number of light waves (which can be represented by sine functions). Often each wave is "out of phase" with the preceding one by a fixed amount; this means that the waves can be written as $\sin t$, $\sin(t+\delta)$, $\sin(t+2\delta)$, and so on. Suppose we want to add all these sine functions together. An easy way to do it is to see that each sine is the imaginary part of a complex number, so what we want is the imaginary part of the series

$$e^{it} + e^{i(t+\delta)} + e^{i(t+2\delta)} + \cdots . \tag{16.17}$$

This is a geometric progression with first term e^{it} and ratio $e^{i\delta}$. If there are n waves to be combined, we want the sum of n terms of this progression, which is

$$\frac{e^{it}(1-e^{in\delta})}{1-e^{i\delta}}. \tag{16.18}$$

We can simplify this expression by writing

$$1 - e^{i\delta} = e^{i\delta/2}(e^{-i\delta/2} - e^{i\delta/2}) = -e^{i\delta/2}\cdot 2i\,\sin\frac{\delta}{2} \tag{16.19}$$

by (11.3). Substituting (16.19) and a similar formula for $(1-e^{in\delta})$ into (16.18), we get

$$\frac{e^{it}e^{in\delta/2}}{e^{i\delta/2}}\,\frac{\sin(n\delta/2)}{\sin(\delta/2)} = e^{i\{t+[(n-1)/2]\delta\}}\frac{\sin(n\delta/2)}{\sin(\delta/2)}. \tag{16.20}$$

The imaginary part of the series (16.17) which we wanted is then the imaginary part of (16.20), namely

$$\sin\left(t + \frac{n-1}{2}\,\delta\right)\sin\frac{n\delta}{2}\Big/\sin\frac{\delta}{2}.$$

▶ PROBLEMS, SECTION 16

11. Prove that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta},$$
$$\sin\theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}.$$

Hint: Use Euler's formula and the geometric progression formula.

12. In optics, the following expression needs to be evaluated in calculating the intensity of light transmitted through a film after multiple reflections at the surfaces of the film:

$$\left(\sum_{n=0}^{\infty} r^{2n}\cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n}\sin n\theta\right)^2.$$

Show that this is equal to $|\sum_{n=0}^{\infty} r^{2n}e^{in\theta}|^2$ and so evaluate it assuming $|r|<1$ (r is the fraction of light reflected each time).

Simple Harmonic Motion It is very convenient to use complex notation even for motion along a straight line. Think of a mass m attached to a spring and oscillating up and down (see Figure 16.3). Let y be the vertical displacement of the mass from its equilibrium position (the point at which it would hang at rest). Recall that the force on m due to the stretched or compressed spring is then $-ky$, where k is the spring constant, and the minus sign indicates that the force and displacement are in opposite directions. Then Newton's second law (force = mass times acceleration) gives

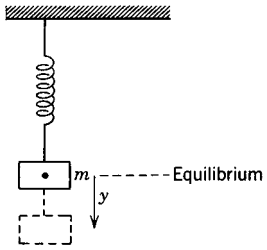


Figure 16.3

$$m\frac{d^2y}{dt^2} = -ky \quad \text{or} \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y = -\omega^2 y \quad \text{if} \quad \omega^2 = \frac{k}{m}. \tag{16.21}$$

Now we want a function $y(t)$ with the property that differentiating it twice just multiplies it by a constant. You can easily verify that this is true for exponentials, sines, and cosines (see problem 13). Just as in discussing electric circuits (see (16.10)), we may write a solution of (16.21) as

$$y = y_0\, e^{i\omega t} \tag{16.22}$$

with the understanding that the actual physical displacement is either the real or the imaginary part of (16.22). The constant $\omega = \sqrt{k/m}$ is called the angular frequency (see Chapter 7, Section 2). We will use this notation in Chapter 3, Section 12.

▶ PROBLEMS, SECTION 16

13. Verify that $e^{i\omega t}$, $e^{-i\omega t}$, $\cos\omega t$, and $\sin\omega t$ satisfy equation (16.21).

▶ 17. MISCELLANEOUS PROBLEMS

Find one or more values of each of the following complex expressions and compare with a computer solution.

1. $\left(\dfrac{1+i}{1-i}\right)^{2718}$ **2.** $\left(\dfrac{1+i\sqrt{3}}{\sqrt{2}+i\sqrt{2}}\right)^{50}$ **3.** $\sqrt[5]{-4-4i}$

4. $\sinh(1+i\pi/2)$ **5.** $\tanh(i\pi/4)$ **6.** $(-e)^{i\pi}$

7. $(-i)^i$ **8.** $\cos\left[2i\ln\dfrac{1-i}{1+i}\right]$ **9.** $\arcsin\left[\left(\dfrac{\sqrt{3}+i}{\sqrt{3}-i}\right)^{12}\right]$

10. $e^{2i\arctan(i\sqrt{3})}$ **11.** $e^{2\tanh^{-1}i}$ **12.** $e^{i\arcsin i}$

13. Find *real* x and y for which $|z+3| = 1 - iz$, where $z = x + iy$.

14. Find the disk of convergence of the series $\sum(z-2i)^n/n$.

15. For what z is the series $\sum z^{\ln n}$ absolutely convergent? *Hints:* Use equation (14.1). Also see Chapter 1, Problem 6.15.

16. Describe the set of points z for which $\mathrm{Re}(e^{i\pi/2}z) > 2$.

Verify the formulas in Problems 17 to 24.

17. $\arcsin z = -i \ln(iz \pm \sqrt{1 - z^2})$

18. $\arccos z = i \ln(z \pm \sqrt{z^2 - 1})$

19. $\arctan z = \dfrac{1}{2i} \ln \dfrac{1 + iz}{1 - iz}$

20. $\sinh^{-1} z = \ln(z \pm \sqrt{z^2 + 1})$

21. $\cosh^{-1} z = \ln(z \pm \sqrt{z^2 - 1}) = \pm \ln(z + \sqrt{z^2 - 1})$

22. $\tanh^{-1} z = \dfrac{1}{2} \ln \dfrac{1 + z}{1 - z}$

23. $\cos iz = \cosh z$

24. $\cosh iz = \cos z$

25. (a) Show that $\overline{\cos z} = \cos \bar{z}$.

(b) Is $\overline{\sin z} = \sin \bar{z}$?

(c) If $f(z) = 1 + iz$, is $\overline{f(z)} = f(\bar{z})$?

(d) If $f(z)$ is expanded in a power series with *real* coefficients, show that $\overline{f(z)} = f(\bar{z})$.

(e) Using part (d), verify, *without computing its value*, that $i[\sinh(1 + i) - \sinh(1 - i)]$ is real.

26. Find $\left| \dfrac{2e^{i\theta} - i}{ie^{i\theta} + 2} \right|$. *Hint:* See equation (5.1).

27. (a) Show that $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$ and that $\operatorname{Im} z = (1/2i)(z - \bar{z})$.

(b) Show that $|e^z|^2 = e^{2 \operatorname{Re} z}$.

(c) Use (b) to evaluate $|e^{(1+ix)^2(1-it) - |1+it|^2}|^2$ which occurs in quantum mechanics.

28. Evaluate the following absolute square of a complex number (which arises in a problem in quantum mechanics). Assume a and b are real. Express your answer in terms of a hyperbolic function.

$$\left| \frac{(a + bi)^2 e^b - (a - bi)^2 e^{-b}}{4abie^{-ia}} \right|^2$$

29. If $z = \dfrac{a}{b}$ and $\dfrac{1}{a + b} = \dfrac{1}{a} + \dfrac{1}{b}$, find z.

30. Write the series for $e^{x(1+i)}$. Write $1 + i$ in the $re^{i\theta}$ form and so obtain (easily) the powers of $(1 + i)$. Thus show, for example, that the $e^x \cos x$ series has no x^2 term, no x^6 term, etc., and a similar result for the $e^x \sin x$ series. Find (easily) a formula for the general term for each series.

31. Show that if a sequence of complex numbers tends to zero, then the sequence of absolute values tends to zero too, and vice versa. *Hint:* $a_n + ib_n \to 0$ means $a_n \to 0$ and $b_n \to 0$.

32. Use a series you know to show that $\displaystyle\sum_{n=0}^{\infty} \frac{(1 + i\pi)^n}{n!} = -e$.