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APPROXIMATING RATIONAL POINTS ON TORIC VARIETIES

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Abstract. Given a smooth projective variety X over a number field k and $P \in X(k)$, the first author conjectured that in a precise sense, any sequence that approximates P sufficiently well must lie on a rational curve. We prove this conjecture for smooth split toric surfaces conditional on Vojta's conjecture. More generally, we show that if X is a $\mathbb{Q}$-factorial terminal split toric variety of arbitrary dimension, then P is better approximated by points on a rational curve than by any Zariski dense sequence.

Contents

1. Introduction

In Dirichlet's 1842 Approximation Theorem, he showed that for every irrational number x, there exist infinitely many rational numbers $\frac{a}{b}$ in reduced form satisfying the equation $|x-\frac{a}{b}|<\frac{1}{b^2}$. His result can be rephrased as follows. For a point $x \in \mathbb{R}$ the *approximation exponent* τ_x of x is the unique extended real number $\tau_x \in (0,\infty]$ such that the inequality

$$\left|x-\frac{a}{b}\right| \leq \frac{1}{b^{\tau_x+\delta}}$$

has only finitely many solutions $\frac{a}{b} \in \mathbb{Q}$ in reduced form whenever $\delta > 0$, and has infinitely many solutions whenever $\delta < 0$. The approximation exponent measures a certain tension between our ability to closely approximate x by rational numbers (the distance term $|x-\frac{a}{b}|$) and the complexity (the $\frac{1}{b}$ term) of the number required to make this approximation. In this notation, Dirichlet's theorem then states $\tau_x \geq 2$ for irrational x. In 1844, Liouville [Li44] proved that if $x \in \mathbb{R}$ is algebraic of degree d over $\mathbb{Q}$, then $\tau_x \leq d$. This upper bound was subsequently improved by Thue [Th09]

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in 1909, Siegel [Si21] in 1921, and independently by Dyson [Dy47] and Gelfand in 1947, leading finally to Roth's famous 1955 theorem [Ro55] that $\tau_x \leq 2$ for all algebraic $x \in \mathbb{R}$. Therefore, Dirichlet's Theorem and Roth's Theorem together show that $\tau_x = 2$ for all irrational and algebraic x.

McKinnon and Roth [MR15] generalized τ_x to arbitrary projective varieties X over a number field k by replacing the function $|x - \frac{a}{b}|$ by a distance function $\operatorname{dist}_v(x, \cdot)$ depending on a place v of k, and measuring the complexity of a point via a height function $H_D(\cdot)$ depending on an ample divisor D. An essential change, however, is that they moved the exponent τ_x from the height to the distance; this was done to make their generalized exponents behave better with respect to changes in D. Given any sequence $\{x_i\}$ approximating x, one then obtains an associated *approximation constant* $\alpha_{x,\{x_i\}}(D)$, see Section 2 for the precise definition. The constant $\alpha_x(D)$ is defined to be the infimum of $\alpha_{x,\{x_i\}}(D)$ over all choices of sequences $\{x_i\}$; if one restricts attention only to sequences contained in a subvariety $Z \subseteq X$, then the resulting infimum is denoted by $\alpha_{x,Z}(D)$.

The focus of our paper is a conjecture introduced by the first author in 2007:

Conjecture 1.1 ([McK07, Conjecture 2.7])**.** *Let X be an algebraic variety defined over a number field k, and D any ample divisor on X. Let $P \in X(k)$ and assume that there is a rational curve defined over k passing through P. Then there exists a curve $C \subseteq X$ (necessarily rational) for which $\alpha_{P,C}(D) = \alpha_P(D)$.*

This conjecture is known in some special cases, primarily in dimension 2: it was shown for split rational surfaces of Picard rank at most four in [McK07], cubic surfaces in [MR16], and blow-ups of the n-th Hirzebruch surface at special configurations of at most $2n$ points in [Ca19]. The conjecture was also verified in [Hu18] for smooth projective split toric varieties X with torus T when $P \in T(k)$ and the pseudo-effective cone $\overline{\operatorname{Eff}}(X)$ is simplicial. Unfortunately, this is a rather restrictive condition: it is equivalent to the combinatorial hypothesis that there exists a maximal cone σ in the fan of X such that every ray outside σ is a negative linear combination of the rays of σ, see [Hu18, Lemma 6.2]. In particular, all of the aforementioned results still leave open the case of smooth split toric surfaces even if one requires $P \in T(k)$.

In this work, we considerably extend the list of cases where Conjecture 1.1 is known: we prove it not only for all smooth split toric surfaces X and arbitrary $P \in X(k)$ conditional on Vojta's Conjecture, but we also obtain approximation results more generally for $\mathbb{Q}$-factorial terminal singularities on projective split toric varieties of arbitrary dimension.

The starting point for our work is a new class of points that we now introduce.

Definition 1.2. Let X be a $\mathbb{Q}$-Gorenstein algebraic variety defined over a number field k. We say X is *canonically bounded at* $P \in X(k)$ if $\alpha_{P,\{x_i\}}(-K_X) \geq \dim X$ for all Zariski dense sequences $\{x_i\}$.

Canonical boundedness is a highly natural notion. Indeed, we show that under mild hypotheses, every point on a smooth variety is conjecturally canonically bounded:

Proposition 1.3. *Let X be a smooth projective variety over a number field k, and let $P \in X(k)$ be a k-rational point. Assume that there is an ample divisor A on X for which $\alpha_P(A)$ is finite. Then Vojta's Main Conjecture implies that X is canonically bounded at P.*

Remark 1.4. The condition that $\alpha_P(A)$ is finite is a mild hypothesis. For example, if there exists a rational curve defined over k passing through P, then $\alpha_P(A)$ is finite for every ample divisor A. In particular, the conditions of Proposition 1.3 are true for every split toric variety over k.

Our first main result is that Conjecture 1.1 holds for split toric surfaces in the presence of the canonical boundedness condition:

Theorem 1.5. *Let X be a split toric surface over a number field k and let $P \in X(k)$ be a smooth point that is canonically bounded in the minimal resolution of X. Then Conjecture* 1.1 *holds at P for every nef divisor D on X.*

Remark 1.6. Notice that for split toric surfaces, Theorem 1.5 is stronger than Conjecture 1.1 in the sense that the theorem holds for all nef divisors while the conjecture is only stated for ample divisors. We do not expect Conjecture 1.1 to be true in general for nef divisors.

In fact, Theorem 1.5 follows from a much more general theorem which we prove for all higher dimensional split toric varieties. Given a split toric variety X over a number field k, we say $f\colon \widetilde{X} \to X$ is a *terminal resolution* if it is a proper birational toric morphism defined over k and $\widetilde{X}$ is $\mathbb{Q}$-factorial, projective, and has at worst terminal singularities.

Theorem 1.7. *Let X be a split toric variety over a number field k and let $P \in X(k)$. Suppose $f\colon \widetilde{X} \to X$ is a terminal resolution which is an isomorphism at P, and that P is canonically bounded in $\widetilde{X}$.*

Then for all $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisors D on X, there exists an irreducible rational curve C through P such that C is unibranch at P and

$$\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\}$.

Remark 1.8. Theorem 1.7 says there exists a curve C whose α value is smaller than that of every Zariski dense sequence. Notice that this does not imply Conjecture 1.1 in higher dimensions since it is possible that there exists a subvariety Z with $1 < \dim Z < \dim X$ for which $\alpha_{P,Z}(D) < \alpha_{P,C}(D)$. However, when X is a surface, no such Z can exist. Hence, Theorem 1.7 implies Conjecture 1.1 for surfaces, i.e. Theorem 1.7 implies Theorem 1.5.

Remark 1.9. A subtle point here is that the curve C we construct in the proof of Theorem 1.7 need not satisfy $\alpha_{P,C}(D) = \alpha_P(D)$, even for surfaces. That is, we show $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$, and for surfaces, this is enough to guarantee the existence of some auxiliary curve C' with $\alpha_{P,C'}(D) = \alpha_P(D)$, but C' may not equal C. Indeed, our construction of C is independent of the number field k, but in Section 8 we show that for $P = [1 : 1 : 1] \in \mathbb{P}(4, 7, 13)$, the value of $\alpha_P(D)$ depends on k. In particular, any proof of Theorem 1.7 without assuming *a priori* that P is canonically bounded must include an explanation for the subtle fact that certain curves such as C' are contained in the Zariski closed locus of exceptions to the canonical boundedness condition provided by Vojta's Conjecture.

Finally, combining Proposition 1.3 and Theorem 1.7 yields:

Theorem 1.10. *Let X be a split toric variety over a number field k and assume Vojta's Main Conjecture holds for some projective toric (strong) resolution of singularities of X. Then for all smooth points $P \in X(k)$ and all $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisors D on X, there exists an irreducible rational curve C through P such that C is unibranch at P and*

$$\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\}$.

2. Key properties of the approximation constant α_P

In this section, we collect the relevant facts we need about the approximation constant. For a more detailed discussion of α, see [MR15]. Proofs of all of the facts below can be found in [MR16].

Definition 2.1. Let X be a projective variety over a number field k, let $P \in X(\overline{k})$, and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. For any sequence $\{x_i\} \subset X(k)$ of distinct points with $\mathrm{dist}_v(P, x_i) \to 0$, which we denote by $\{x_i\} \to P$, we set

$$A(\{x_i\}, D) = \{\gamma \in \mathbb{R} \mid \mathrm{dist}_v(P, x_i)^{\gamma} H_D(x_i) \text{ is bounded from above}\}.$$

Remark 2.2. It follows immediately from the definition that if $A(\{x_i\}, D)$ is nonempty then it is an interval unbounded to the right, i.e., if $\gamma \in A(\{x_i\}, D)$ then $\gamma + \delta \in A(\{x_i\}, D)$ for any $\delta > 0$.

Remark 2.3. Note that the height $H_D(x)$ is well defined even if D is a $\mathbb{Q}$-divisor, by $H_D(x) = H_{mD}(x)^{1/m}$.

Definition 2.4. With hypotheses as in Definition 2.1, if $A(\{x_i\}, D)$ is empty we set $\alpha_{P,\{x_i\}}(D) = \infty$. Otherwise we set $\alpha_{P,\{x_i\}}(D)$ to be the infimum of $A(\{x_i\}, D)$. We call $\alpha_{P,\{x_i\}}(D)$ the *approximation constant* of $\{x_i\}$ with respect to D.

Remark 2.5. If $\{x_i'\}$ is a subsequence of $\{x_i\}$ then $A(\{x_i\}, D) \subseteq A(\{x_i'\}, D)$. In particular, $\alpha_{P,\{x_i'\}}(D) \leq \alpha_{P,\{x_i\}}(D)$, so we may freely replace a sequence with a subsequence when trying to establish lower bounds.

As $i \to \infty$ we have $\mathrm{dist}_v(P, x_i) \to 0$. We thus expect that $\mathrm{dist}_v(P, x_i)^{\gamma} H_D(x_i)$ goes to 0 for large γ and to ∞ for small γ. The number $\alpha_{P,\{x_i\}}(D)$ marks the transition point between these two behaviours.

Definition 2.6. Let k be a number field, X a projective variety over k, D a $\mathbb{Q}$-Cartier divisor on X, and $P \in X(\overline{k})$. Then $\alpha_P(D)$ is defined to be the infimum of all $\alpha_{P,\{x_i\}}(D)$ as we range over sequences of distinct points $\{x_i\} \subset X(k)$ converging to P. If no such sequence exists then set $\alpha_P(D) = \infty$.

To expand upon the connection between α_x and the usual approximation exponent τ_x as defined in the Introduction, suppose that D is an ample $\mathbb{Q}$-divisor on X. We may define an approximation constant $\tau_P(D)$ by simply extending the definition on $\mathbb{P}^1$, namely by defining $\tau_P(D)$ to be the unique extended real number $\tau_P(D) \in [0, \infty]$ such that the inequality

$$\mathrm{dist}_v(P, Q) < \frac{1}{H_D(Q)^{\tau_P(D)+\delta}}$$

has only finitely many solutions $Q \in X(k)$ whenever $\delta > 0$ and has infinitely many solutions $Q \in X(k)$ whenever $\delta < 0$. Then [MR16, Proposition 2.11] implies that $\alpha_P(D) = \frac{1}{\tau_P(D)}$. In particular the theorem of Liouville becomes $\alpha_P(\mathcal{O}_{\mathbb{P}^1}(1)) \geq \frac{1}{d}$ for $P \in \mathbb{R}$ of degree d over $\mathbb{Q}$, and it is this type of lower bound that we wish to generalize to arbitrary varieties. We use the reciprocal of τ because α behaves more naturally when we vary D (see, for example, Proposition 2.9 of [MR16] for more details).

We will need one further property of α_P. By Theorem 2.8 of [MR16] (see also Theorem 2.16 of [MR15]), we have:

Theorem 2.7. *Let C be an irreducible k-rational curve and $\varphi\colon \mathbb{P}^1 \to C$ its normalization map. Then for any ample $\mathbb{Q}$-divisor D on C, and any $P \in C(\overline{k})$ we have the equality:*

$$\alpha_{P,C}(D) = \min_{Q \in \varphi^{-1}(P)} \frac{d}{r_Q m_Q}$$

where $d = \deg(D)$, m_Q is the multiplicity of the branch of C through Q corresponding to Q, and

$$r_Q = \begin{cases} 0 & \textit{if } \kappa(Q) \not\subseteq k_v \\ 1 & \textit{if } \kappa(Q) = k \\ 2 & \textit{otherwise.} \end{cases}$$

We are primarily interested in the case where the curve C is unibranch at P, so there is only one point $Q \in \varphi^{-1}(P)$ which necessarily has $r_Q = 1$. Thus, we have the following result.

Theorem 2.8. *Let X be a variety defined over a number field k, and let C be an irreducible rational curve on X, with C also defined over k. Let P be a k-rational, unibranch point of C, and let D be a $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisor on X. Then*

$$\alpha_{P,C}(D) = \frac{1}{m} C \cdot D,$$

where m is the multiplicity of P on C.

3. Vojta's Main Conjecture and canonical boundedness

The goal in this section is to show that Vojta's Main Conjecture implies every point of a smooth projective variety is canonically bounded, i.e. we prove Proposition 1.3. We turn to the proof after recalling for the reader's convenience the statement of the conjecture [Vo87].

Conjecture 3.1 (Vojta's Main). *Let X be a smooth algebraic variety defined over a number field k, with canonical divisor K. Let S be a finite set of places of k. Let A be a big divisor on X, and let D be a normal crossings divisor on X. Choose height functions h_K and h_A for K and A, respectively, and define a proximity function $m_S(D,P) = \sum_{v \in S} h_{D,v}(P)$ for D with respect to S, where $h_{D,v}$ is a local height function for D at v. Choose any $\epsilon > 0$. Then there exists a nonempty Zariski open set $U = U(\epsilon) \subseteq X$ such that for every k-rational point $Q \in U(k)$, we have the following inequality:*

$$m_S(D,Q) + h_K(Q) \leq \epsilon h_A(Q). \tag{3.2}$$

Remark 3.3. There is often an $O(1)$ included on the right side of (3.2) to account for the uncertainty in the choice of height functions. We omit this term, preferring instead to choose the Zariski open set $U(\epsilon)$ small enough to exclude the degenerate set of points that do not satisfy the inequality as written.

We now turn to Proposition 1.3.

Proof of Proposition 1.3. Let $\dim X = n$ and fix a place v of k. Let $S = \{v\}$, A any very ample divisor on X, and D the union of any n normal crossings divisors that intersect properly and transversely at P. We claim that there is a constant C such that for all $Q \in X(k)$, we have:

$$m_S(D,Q) \geq -n \log \mathrm{dist}_v(P,Q) + C \tag{3.4}$$

To see this, note that the divisor D has multiplicity at least n at P, by construction. Thus, if $\phi\colon Y \to X$ is the blowup of X at P, the divisor $\phi^*D - nE$ is effective, where E is the exceptional divisor of ϕ. This implies that $m_S(\phi^*D - nE, Q)$ is bounded below independently of Q (see [Vo87, Lemma 1.3.3.(b)]), and so

$$m_S(\phi^*D,Q) \geq m_S(nE,Q) + C$$

for some constant C independent of Q. Equation (3.4) then follows from Lemma 1.3.3.(d) of [Vo87].

Fix any $\epsilon > 0$. If Q satisfies inequality (3.2), then

$$\mathrm{dist}_v(P,Q)^n H_{-K_X}(Q) \geq C' H_A(Q)^{-\epsilon}. \tag{3.5}$$

for some positive constant C', independent of Q.

By hypothesis, there is some ample divisor A for which $\alpha_P(A)$ is finite. Choose a real number a such that $\alpha_P(A) < a$. By definition of $\alpha_P(A)$, for any k-rational point Q on X, we know that $\mathrm{dist}_v(P,Q)^a H_A(Q)$ is bounded above, independently of Q. We therefore deduce that

$$\mathrm{dist}_v(P,Q)^{a\epsilon} H_A(Q)^{\epsilon} \leq \kappa \tag{3.6}$$

for some positive constant κ depending on ϵ but not Q. Therefore if Q satisfies inequality (3.2), then combining inequalities (3.5) and (3.6), we obtain

$$\mathrm{dist}_v(P,Q)^{n-a\epsilon} H_{-K_X}(Q) \geq C'\kappa^{-1}. \tag{3.7}$$

In particular, if $\{x_i\}$ is a sequence satisfying $\alpha_{P,\{x_i\}}(-K_X) < n$, then choosing ϵ sufficiently small, we see $\{x_i\}$ must be eventually contained in the complement of the set $U(\epsilon)$ from Vojta's Main Conjecture. So, $\{x_i\}$ must be contained in a finite union of proper subvarieties, as desired. ☐

4. PRELIMINARY REDUCTIONS IN THE PROOF OF THEOREM 1.7

For the remainder of the paper, we fix a number field k a place v of k, and a v-adic distance function dist_v which we will denote by dist. We begin by reducing Theorem 1.7 to the case where X is $\mathbb{Q}$-factorial with terminal singularities itself.

Proposition 4.1. *Let X be a split toric variety defined over a number field k, let $P \in X(k)$, and let D be a $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisor on X. Suppose $f\colon \widetilde{X} \to X$ is a toric proper birational map which is an isomorphism at P, and that there is an irreducible rational curve $C \subseteq \widetilde{X}$ through $f^{-1}(P)$ such that C is unibranch at $f^{-1}(P)$ and*

$$\alpha_{f^{-1}(P),C}(f^*D) \leq \alpha_{f^{-1}(P),\{x_i\}}(f^*D)$$

for all Zariski dense sequences $\{x_i\}$. *Then the curve* $f(C)$ *is an irreducible rational curve that is unibranch at* P *and satisfies*

$$\alpha_{P,f(C)}(D) \leq \alpha_{P,\{x_i\}}(D).$$

for all Zariski dense sequences $\{x_i\}$.

Proof. Irreducibility of $f(C)$ follows from that of C. Moreover, since f is an isomorphism at P, the fact that C is rational and unibranch at $f^{-1}(P)$ immediately implies that $f(C)$ is rational and unibranch at P. Lastly, applying Corollary 8.6 of [MR15] to the subset of X on which f is an isomorphism implies that $\alpha_{P,f(C)}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$. ☐

By Proposition 4.1, to prove Theorem 1.7, we can assume that our split toric variety X is projective, $\mathbb{Q}$-factorial, and has at worst terminal singularities. Thus, it remains to prove the following theorem, which is slightly more general.

Theorem 4.2. *Let* X *be a projective terminal* $\mathbb{Q}$*-factorial split toric variety over a number field* k. *Let* $P \in X(k)$ *and* D *be a nef* $\mathbb{Q}$*-divisor on* X. *If* P *is canonically bounded, then there exists an irreducible curve* C *through* P *which is unibranch at* P *and*

$$\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\}$ *on* X. *Moreover, if* $X \not\cong \mathbb{P}^n$, *then we can choose* C *so that* $-K_X \cdot C \leq \dim X$.

We prove Theorem 4.2 using an induction argument via the Minimal Model Program (MMP). In order to explain this, we begin with several preliminary results.

Lemma 4.3. *Let* X *be a* $\mathbb{Q}$*-factorial algebraic variety over a number field* k *which is canonically bounded at* $P \in X(k)$, *Let* $a \in \mathbb{Q}_{\geq 0}$ *and* D *be a nef* $\mathbb{Q}$*-divisor on* X *such that* $D + aK_X$ *is also nef. Suppose* C *is an irreducible rational curve through* P *which is unibranch at* P, $-K_X \cdot C \leq \dim X$, *and*

$$\alpha_{P,C}(D + aK_X) \leq \alpha_{P,\{x_i\}}(D + aK_X)$$

for all Zariski dense sequences $\{x_i\}$ *on* X. *Then for all Zariski dense sequences* $\{x_i\}$ *on* X, *we have*

$$\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$$

as well.

Proof. Since C is unibranch at P, Theorem 2.8 gives us that $\alpha_{P,C}(F) = \frac{1}{m}C \cdot F$ for every nef $\mathbb{Q}$-divisor F, where m is the multiplicity of C at P. In particular,

$$\alpha_{P,C}(D) = \frac{1}{m}C \cdot D = \frac{1}{m}C \cdot (D + aK_X) - \frac{a}{m}K_X \cdot C \leq \alpha_{P,C}(D + aK_X) + a\dim X.$$

Using the defining property of C and the fact that X is canonically bounded at P, we see

$$\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D + aK_X) + a\alpha_{P,\{x_i\}}(-K_X).$$

Lastly, concavity of α, shown in [MR15, Proposition 2.14.(b)], yields

$$\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D),$$

proving the desired result for D. ☐

Now, let X be a projective $\mathbb{Q}$-factorial split toric variety over a number field k which is canonically bounded at $P \in X(k)$, and let D be a nef $\mathbb{Q}$-divisor on X. Since X is toric, the Mori cone $\overline{\mathrm{NE}}(X)$ is polyhedral. Let $C_0, \dots, C_\ell$ be the torus-invariant curves generating the K_X-negative extremal rays, and set

$$a = \min_i \frac{D \cdot C_i}{-K_X \cdot C_i}; \tag{4.4}$$

without loss of generality, $a = \frac{D \cdot C_0}{-K_X \cdot C_0}$. By construction, $D + aK_X$ intersects non-negatively with every extremal ray of $\overline{\mathrm{NE}}(X)$, so $D + aK_X$ is nef. By Lemma 4.3, to prove Theorem 4.2 for D, it then suffices to prove the theorem for $D + aK_X$.

The advantage to working with $D+aK_X$ as opposed to D is that $C_0 \cdot (D+aK_X) = 0$. Let $\pi\colon X \to Y$ be the extremal contraction corresponding to the ray $\mathbb{R}_{\geq 0} C_0$. If π is either a Mori fiber space or a divisorial contraction, then there is a nef $\mathbb{Q}$-divisor D' on Y for which $D + aK_X = \pi^* D'$. If π is a flipping contraction, then let $\psi\colon X \dashrightarrow X'$ denote the associated elementary flip. By [CLS11, Lemma 15.5.7], we have a commutative diagram

$$\begin{tikzcd}
 & X^* \arrow[dl, "\Phi"'] \arrow[dr, "\Phi'"] & \\
X \arrow[rr, dashed, "\psi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\
 & Y &
\end{tikzcd} \tag{4.5}$$

such that X^* is a common star subdivision of X, X', and Y, the maps Φ and Φ' are isomorphisms away from the exceptional locus $\mathrm{Exc}(\psi)$, and if D^* denotes the torus-invariant divisor on X^* corresponding to the newly inserted ray, then

$$\Phi^* F = {\Phi'}^* F' - (F \cdot C_0) D^* \tag{4.6}$$

for all $\mathbb{Q}$-divisors F on X where $F' = \psi_* F$. Letting $D' := \psi_*(D + aK_X)$, equation (4.6) tells us $\Phi^*(D + aK_X) = {\Phi'}^* D'$. As Φ and Φ' are proper and surjective, the fact that $D + aK_X$ is nef implies $\Phi^*(D + aK_X)$ is nef, which in turn implies D' is nef.

To unify notation among these three cases, we denote by $\psi\colon X \dashrightarrow X'$ the elementary MMP step corresponding to the ray $\mathbb{R}_{\geq 0} C_0$, i.e. if π is a Mori fiber space or a divisorial contraction, we let $X' := Y$ and $\psi := \pi$; if on the other hand, π is a flipping contraction, we let ψ be the associated elementary flip. We have therefore shown that in all three cases, there is a nef $\mathbb{Q}$-divisor D' on X' for which $D + aK_X = \psi^* D'$. If P is not in the exceptional locus, then we would like to apply an inductive strategy to deduce the theorem for $(X, P, D + aK_X)$ from that of $(X', \psi(P), D')$. Proposition 4.8 will allow us to do so.

Lemma 4.7. *Let $\pi\colon X \to Y$ be a surjective birational morphism of projective $\mathbb{Q}$-factorial varieties over a number field k. Let $P \in X(k)$ be a point which is not in the exceptional locus* $\mathrm{Exc}(\pi)$ *and let D' be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Suppose either that $\{x_i\}$ is a Zariski dense sequence on X converging to P and let $x_i' := \pi(x_i)$, or suppose $\{x_i'\}$ is a Zariski dense sequence on X' converging to $\pi(P)$ and let $x_i := \pi^{-1}(x_i')$ whenever $x_i' \notin \pi(\mathrm{Exc}(\pi))$. Then $\alpha_{P,\{x_i\}}(\pi^* D') = \alpha_{\pi(P),\{x_i'\}}(D')$.*

Proof. If $\{x_i\}$ is a Zariski dense sequence on X converging to P, then only finitely many of the $x_i \in \operatorname{Exc}(\pi)$; similarly if $\{x'_i\}$ is a Zariski dense sequence on X' converging to $\pi(P)$, then only finitely many of the $x'_i \in \pi(\operatorname{Exc}(\pi))$. Since the value of α for a sequence is unchanged by removing finitely many elements from the sequence, we may assume $x_i \notin \operatorname{Exc}(\pi)$ and $x'_i \notin \pi(\operatorname{Exc}(\pi))$ for all i. Then $H_{\pi^* D}(x_i) = H_{D'}(x'_i)$. Moreover, the proof of [MR15, Proposition 2.4] applied to $X \smallsetminus \operatorname{Exc}(\pi)$ shows that the distance functions $\operatorname{dist}(P, \cdot)$ and $\operatorname{dist}(\pi(P), \pi(\cdot))$ differ only by a multiplicative factor bounded independently of P; note that the cited proposition is stated for only projective varieties, but the proof reduces immediately to compact neighbourhoods of a point. Therefore, it follows directly from the definition of α that $\alpha_{P,\{x_i\}}(\pi^* D') = \alpha_{\pi(P),\{x'_i\}}(D')$. ☐

Proposition 4.8. *Let X be a projective terminal $\mathbb{Q}$-factorial split toric variety over a number field k and let $\psi\colon X \dashrightarrow X'$ be a birational elementary MMP step. If $P \in X(k) \smallsetminus \operatorname{Exc}(\psi)$ is a canonically bounded point of X, then $\psi(P)$ is a canonically bounded point of X'.*

Proof. We first consider the case where ψ is a divisorial contraction. Let $E \subset X$ be the exceptional divisor and let

$$\psi^*(-K_{X'}) = -K_X + aE.$$

Since X' has terminal singularities, $a > 0$. Given any Zariski dense sequence $\{x'_i\}$ converging to $\psi(P)$, letting $\{x_i\}$ be as in Lemma 4.7, we find

$$\alpha_{\psi(P),\{x'_i\}}(-K_{X'}) = \alpha_{P,\{x_i\}}(\psi^*(-K_{X'})) = \alpha_{P,\{x_i\}}(-K_X + aE).$$

By concavity of α, shown in [MR15, Proposition 2.14.(b)], we see

$$\begin{aligned}\alpha_{\psi(P),\{x'_i\}}(-K_{X'}) = \alpha_{P,\{x_i\}}(-K_X + aE) &\geq \alpha_{P,\{x_i\}}(-K_X) + a\alpha_{P,\{x_i\}}(E) \\ &> \alpha_{P,\{x_i\}}(-K_X) \geq \dim X = \dim X'\end{aligned}$$

where the last line follows from the previous one by the effectiveness of E and the fact that $P \notin E$.

We next handle the case where $\psi\colon X \dashrightarrow X'$ is an elementary flip. Let C_0 be the generator of the K_X-negative ray corresponding to ψ. Let Φ, Φ', and X^* be as in diagram (4.5). Then applying equation (4.6) with $F = -K_X$, we have

$$\Phi'^*(-K_{X'}) = \Phi^*(-K_X) + (-K_X.C_0)D^*.$$

Since $-K_X.C_0 > 0$, Lemma 4.7 tells us that for any Zariski dense sequence $\{x_i\}$ on X' converging to $\psi(P)$, we have

$$\begin{aligned}\alpha_{P,\{x_i\}}(-K_{X'}) &= \alpha_{P,\{x_i\}}(\Phi'^*(-K_{X'})) > \alpha_{P,\{x_i\}}(\Phi^*(-K_X)) \\ &= \alpha_{P,\{x_i\}}(-K_X) \geq \dim X = \dim X'\end{aligned}$$

where for ease of notation, P and x_i are used to denote points on any of X, X', or X^*. It follows that $\psi(P)$ is a canonically bounded point of X'. ☐

In light of Proposition 4.8 and the discussion beforehand, we employ the following method to prove Theorem 4.2. Let X_1 be a projective terminal $\mathbb{Q}$-factorial split toric variety, D_1 a nef $\mathbb{Q}$-divisor on X_1, and $P_1 \in X(k)$ a canonically bounded point. Let a_1 be as in equation (4.4). Then $D_1 + a_1 K_{X_1} \in \mathcal{R}_1^\perp$ for some K_{X_1}-negative extremal ray $\mathcal{R}_1$ of $\overline{\mathrm{NE}}(X_1)$. Let $\psi_1\colon X_1 \dashrightarrow X_2$ be the associated elementary

MMP step, and let D_2 be the nef $\mathbb{Q}$-divisor on X_2 such that $D_1 + a_1 K_{X_1} = \psi_1^* D_2$. Proceeding in this manner, we arrive at the following data: we have a sequence

$$X_1 \overset{\psi_1}{\dashrightarrow} X_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_m}{\dashrightarrow} X_{m+1}$$

of elementary MMP steps and a sequence of points $P_i \in X_i(k)$ such that $P_i \notin \operatorname{Exc}(\psi_i)$ and $P_{i+1} = \psi_i(P_i)$ for $1 \leq i < m$, and $P_m \in \operatorname{Exc}(\psi_m)$. Furthermore, for $1 \leq i \leq m$, we have a nef $\mathbb{Q}$-divisor D_i on X_i and a real number $a_i \geq 0$ such that $D_i + a_i K_{X_i} = \psi_i^* D_{i+1}$ is a nef $\mathbb{Q}$-divisor perpendicular to the K_{X_i}-negative extremal ray corresponding to ψ_i.

Applying Proposition 4.8 repeatedly, we see that P_i is a canonically bounded point of X_i for $1 \leq i \leq m$. By Lemma 4.3, Theorem 4.2 for the triple (X_i, P_i, D_i) follows from that of $(X_i, P_i, D_i + a_i K_{X_i})$. So, to prove Theorem 4.2 for the triple (X_1, P_1, D_1), it suffices to show the result for (X_m, P_m, D_m) and additionally show that the case of $(X_i, P_i, D_i + a_i K_{X_i})$ follows from that of $(X_{i+1}, P_{i+1}, D_{i+1})$. In other words, we have reduced to proving Propositions 4.9 and 4.10.

Proposition 4.9. *Let X be a projective terminal $\mathbb{Q}$-factorial split toric variety over a number field k and let $\psi\colon X \dashrightarrow X'$ be a birational elementary MMP step corresponding to the extremal ray $\mathcal{R}$. Let $D \in \operatorname{Nef}(X) \cap \mathcal{R}^\perp$ and $D' \in \operatorname{Nef}(X')$ be $\mathbb{Q}$-divisors such that $D = \psi^* D'$. If $P \in X(k) \smallsetminus \operatorname{Exc}(\psi)$ is canonically bounded and Theorem 4.2 holds for $(X', \psi(P), D')$ then it holds for (X, P, D).*

Proposition 4.10. *Let X be a projective terminal $\mathbb{Q}$-factorial split toric variety over a number field k and let $\psi\colon X \dashrightarrow X'$ be an elementary MMP step corresponding to the extremal ray $\mathcal{R}$. If $D \in \operatorname{Nef}(X) \cap \mathcal{R}^\perp$ be a $\mathbb{Q}$-divisor and $P \in X(k) \cap \operatorname{Exc}(\psi)$ is canonically bounded, then Theorem 4.2 holds for (X, P, D).*

5. Induction step: P is not in the exceptional locus

In this section, we prove Proposition 4.9. We assume throughout that X is a projective terminal $\mathbb{Q}$-factorial split toric variety over a number field k, $P \in X(k) \smallsetminus \operatorname{Exc}(\psi)$ is a canonically bounded point, and $\psi\colon X \dashrightarrow X'$ is a birational elementary MMP step corresponding to the contraction of the extremal ray $\mathcal{R}$. We let C_0 be the generator of $\mathcal{R}$, $D \in \operatorname{Nef}(X) \cap \mathcal{R}^\perp$, and $D' \in \operatorname{Nef}(X')$ be $\mathbb{Q}$-divisors such that $D = \psi^* D'$. We handle the case where ψ is a divisorial contraction in Section 5.1 and the case where ψ is a flip in Section 5.2.

5.1. The case of divisorial contractions.

Throughout this subsection, we assume $\psi\colon X \to X'$ is a divisorial contraction and let $E \subset X$ be the exceptional divisor. We first handle the case where $X' \simeq \mathbb{P}^n$.

Lemma 5.1. *If $X' \simeq \mathbb{P}^n$, then there is a smooth irreducible curve C through P such that $-K_X \cdot C \leq \dim X$ and $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$ on X.*

Proof. We may assume $X' = \mathbb{P}^n$ and let $Z \subset X'$ be the locus along which ψ is the blow-up. Let ℓ be a line in $\mathbb{P}^n$ that contains both P and at least one point of Z. Letting C be the strict transform of ℓ, we have $C.E \geq 1$. Since $K_X = \psi^* K_{\mathbb{P}^n} + rE$ with $r = \operatorname{codim}(Z) - 1$, we have

$$-K_X \cdot C = -K_{\mathbb{P}^n} \cdot \psi_* C - rE \cdot C = -K_{\mathbb{P}^n} \cdot \ell - rE \cdot C \leq n + 1 - r \leq n.$$

Next, let $\{x_i\}$ be a Zariski dense sequence on X converging to P. By Lemma 4.7, $\alpha_{P,\{x_i\}}(D) = \alpha_{\psi(P),\{\psi(x_i)\}}(D')$. If $d = \deg(D')$, then Lemma 2.13 and Proposition 2.14(a) of [MR15] show

$$\alpha_{P,\{x_i\}}(D) = \alpha_{\psi(P),\{\psi(x_i)\}}(D') = d\alpha_{\psi(P),\{\psi(x_i)\}}(\mathcal{O}(1)) \geq d.$$

On the other hand, since C is smooth at P, we have

$$\alpha_{P,C}(D) = C \cdot D = \ell \cdot D' = d,$$

proving $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$. ☐

Having dispensed with the case where X' is isomorphic to $\mathbb{P}^n$, we can assume that there is a rational irreducible curve $C' \subseteq X'$ through $\psi(P)$ which is unibranch at $\psi(P)$ such that $-K_{X'} \cdot C' \leq \dim X' = \dim X$ and

$$\alpha_{\psi(P),C'}(D') \leq \alpha_{P,\{x_i'\}}(D')$$

for all Zariski dense sequences $\{x_i'\}$ on X'. To prove Proposition 4.9 in the case of divisorial contractions, it remains to show the following.

Lemma 5.2. *Let $C \subseteq X$ be the strict transform of C'. Then C is unibranch at P, $-K_X \cdot C \leq \dim X$, and $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$ on X.*

Proof. Since X has terminal singularities, $K_X = \psi^* K_{X'} + rE$ with $r > 0$. Then

$$-K_X \cdot C = (-\psi^* K_{X'} - rE) \cdot C = -K_{X'} \cdot C' - rE \cdot C \leq \dim X - rE \cdot C \leq \dim X,$$

where the last inequality follows because E is effective, C is irreducible, and C is not contained in E.

Next, let m be the multiplicity of C at P. Since P is not in the exceptional locus, m is also the multiplicity of C' at $\psi(P)$. Applying Theorem 2.8 and using that C and C' are unibranch at P and $\psi(P)$ respectively, we find

$$\alpha_{P,C}(D) = \frac{1}{m} C \cdot D = \frac{1}{m} C \cdot \psi^* D' = \frac{1}{m} C' \cdot D' = \alpha_{\psi(P),C'}(D').$$

Now if $\{x_i\}$ is a Zariski dense sequence on X converging to P, then Lemma 4.7 shows $\alpha_{P,\{x_i\}}(D) = \alpha_{\psi(P),\{\psi(x_i)\}}(D')$. By the defining property of C', we see $\alpha_{\psi(P),C'}(D') \leq \alpha_{\psi(P),\{\psi(x_i)\}}(D')$, which proves $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$. ☐

5.2. **The case of flips.** In this subsection, we handle the case where $\psi\colon X \dashrightarrow X'$ is an elementary flip. Since ψ is an isomorphism in codimension 1, the Picard numbers of X and X' are equal. Since the Picard number of X must be at least 2, we see then that $X' \not\simeq \mathbb{P}^n$. So we may assume there is a rational irreducible curve $C' \subseteq X'$ through $\psi(P)$ which is unibranch at $\psi(P)$ such that $-K_{X'} \cdot C' \leq \dim X' = \dim X$ and $\alpha_{\psi(P),C'}(D') \leq \alpha_{P,\{x_i'\}}(D')$ for all Zariski dense sequences $\{x_i'\}$ on X'. Let X^*, Φ, and Φ' be as in diagram (4.5). It then suffices to prove the following.

Lemma 5.3. *Let $\widetilde{C'} \subset X^*$ be the strict transform of C' and $C = \Phi(\widetilde{C'})$. Then C is rational, irreducible, and unibranch at P, $-K_X \cdot C \leq \dim X$, and $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$ on X.*

Proof. Since Φ and Φ' are isomorphisms away from $\mathrm{Exc}(\psi)$, and C' is rational and irreducible, it follows that C is as well. Moreover, since C' is unibranch at $P' := \psi(P)$ and $P \notin \mathrm{Exc}(\psi)$, we see C is unibranch at P.

Next, we see $\Phi'^{*}D' \cdot \widetilde{C'} = D' \cdot \Phi'_{*}\widetilde{C'} = D' \cdot C'$ and similarly $\Phi^{*}D \cdot \widetilde{C'} = D \cdot C$. Since $D \cdot C_0 = 0$, equation (4.6) tells us $\Phi^{*}D = \Phi'^{*}D'$. Let m be the multiplicity of C at P. Since m is also the multiplicity of C' at P', we see from Theorem 2.8 that

$$\alpha_{P,C}(D) = \frac{1}{m} D \cdot C = \frac{1}{m} D' \cdot C' = \alpha_{P',C'}(D').$$

Again applying (4.6), we find

$$K_X \cdot C = \Phi^{*}K_X \cdot \widetilde{C'} = (\Phi'^{*}K_{X'} - (K_X \cdot C_0)D^{*}) \cdot \widetilde{C'} = K_{X'} \cdot C' - (K_X \cdot C_0)(D^{*} \cdot \widetilde{C'}).$$

Recall that C_0 generates a K_X-negative ray. Since $\widetilde{C'}$ is irreducible and not contained in the effective divisor D^{*}, we have $D^{*} \cdot \widetilde{C'} \geq 0$. By hypothesis, $-K_{X'} \cdot C' \leq \dim X$, so we find $-K_X \cdot C \leq \dim X$.

It remains to show that $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$ on X converging to P. Since $P \notin \operatorname{Exc}(\psi)$, only finitely many of the $x_i \in \operatorname{Exc}(\psi)$. So, removing these finitely many terms, we may assume $x_i \notin \operatorname{Exc}(\psi)$ for all i. Let $P^{*} := \Phi^{-1}(P)$, $x_i^{*} := \Phi^{-1}(x_i)$, and $x_i' := \Phi'(x_i^{*})$. Then two applications of Lemma 4.7 show

$$\alpha_{P,\{x_i\}}(D) = \alpha_{P^{*},\{x_i^{*}\}}(\Phi^{*}D) = \alpha_{P^{*},\{x_i^{*}\}}(\Phi'^{*}D') = \alpha_{P',\{x_i'\}}(D').$$

It follows that $\alpha_{P,C}(D) = \alpha_{P',C'}(D') \leq \alpha_{P',\{x_i'\}}(D') = \alpha_{P,\{x_i\}}(D)$. □

6. RESULTS ON FAKE WEIGHTED PROJECTIVE SPACES

The analysis in this section is by far the most involved. Our goal is to prove the following result which forms a crucial step in the proof of Proposition 4.10.

Proposition 6.1. *Let W be a fake weighted projective space with torus T, and let $P \in W(k)$. Then there is a unibranch rational curve $C \subseteq W$ through P satisfying the following properties:*

- (a) *There is a T-orbit closure $Z \subseteq W$ and a 1-parameter subgroup $C_0 \subseteq T_Z$ of the torus of Z such that C is the translate of the closure of C_0 by a $T(k)$-point,*
- (b) $-K_W \cdot C \leq 1 + \dim W$,
- (c) *If W has terminal singularities and is not isomorphic to projective space, then C can be chosen to additionally satisfy $-K_W \cdot C \leq \dim W$.*

Recall that every fake weighted projective space W admits a canonical toric cover $f\colon W' \to W$ which is étale in codimension 1 and such that W' is a weighted projective space, see e.g. [Bu02]. Moreover, there is a subgroup scheme $G = \prod_{i=1}^{\ell} \mu_{r_i}$ of the torus T' of W' such that under the induced action of G, we have $W = W'/G$ and f is the quotient map. The morphism f is referred to as the *universal covering in codimension 1*, and is constructed explicitly as follows. Let $v_0, \dots, v_n \in N$ be the primitive generators for the rays of the fan of W. There exist relatively prime positive integers $a_0, \dots, a_n$ such that $\sum a_i v_i = 0$ in N. The map f corresponds to the finite index inclusion $\iota\colon N' \hookrightarrow N$, where N' is the lattice generated by the v_i.

We begin by reducing Proposition 6.1 to a subclass of fake weighted projective spaces.

Lemma 6.2. *If Proposition* 6.1(a) *and* (b) *hold for all weighted projective spaces, then they hold for all fake weighted projective spaces.*

Furthermore, suppose Proposition 6.1 *holds for*

(a) *weighted projective spaces, and*

(b) *fake weighted projective spaces of the form* $\mathbb{P}^n/\mu_p$*, where* p *is prime and the quotient map* $\mathbb{P}^n \to \mathbb{P}^n/\mu_p$ *is the universal covering in codimension 1.*

Then Proposition 6.1 *holds for all fake weighted projective spaces.*

Proof. Let W be a fake weighted projective space. We define a finite surjective toric morphism $g\colon W' \to W$ which is étale in codimension 1 as follows. If the universal covering in codimension 1 of W is not isomorphic to projective space, then we take $g\colon W' \to W$ to be the universal covering in codimension 1. If, on the other hand, $f\colon \mathbb{P}^n \to W$ is the universal covering in codimension 1 realizing W as $\mathbb{P}^n/G$, then choose a prime p and a subgroup scheme $\mu_p \subseteq G$. The map f then factors as $\mathbb{P}^n \to W' := \mathbb{P}^n/\mu_p \xrightarrow{g} W$. Since f is finite surjective and étale in codimension 1, the map g is as well. Since in either case $g\colon W' \to W$ is toric finite surjective, the induced map on lattices $N' \to N$ is a finite index inclusion which induces a bijection between the cones in the fans Σ_W and $\Sigma_{W'}$. Let T and T' denote the tori of W and W', respectively. By [KM98, Proposition 5.20(3)], if W has terminal singularities, then W' does as well.

Given $P \in W(k)$, let $Y \subseteq W$ be the T-orbit closure for which $P \in T_Y(k)$. It suffices to prove the lemma for the identity of the torus T_Y. Indeed suppose C is the desired curve for the identity of the torus T_Y. Note that T acts on Y via the quotient map $\eta\colon T \to T_Y$ which is split and hence surjective on k-points. So, choosing $Q \in T(k)$ with $\eta(Q) = P$, we see that the Q-translate of C contains P and satisfies the properties of Proposition 6.1.

It remains to handle the case where $P \in T_Y(k)$ is the identity of the torus. Since Y corresponds to a cone $\sigma \in \Sigma_W$, by considering σ as a cone of $\Sigma_{W'}$ on the coarser lattice N', we obtain a T'-orbit closure $Y' \subseteq W'$ and a toric map $g|_{Y'}\colon Y' \to Y$. In particular, choosing $P \in T_{Y'}(k)$ to be the identity of the torus, we see $g(P') = P$. By hypothesis, there is a T'-orbit closure $Z' \subseteq W'$ and a 1-parameter subgroup $C'_0 \subseteq T_{Z'}$ such that its closure $C' \subseteq W'$ contains P' and satisfies $-K_{W'} \cdot C' \leq 1 + \dim W' = 1 + \dim W$ or $-K_{W'} \cdot C' \leq \dim W$, depending on whether W' has terminal singularities. As above, we have a T-orbit closure $Z \subseteq W$ corresponding to Z' and a toric map $g|_{Z'}\colon Z' \to Z$. Since C' is the translate of the closure of C'_0 by a $T'(k)$-point, its image $C := f(C')$ is the translate of the closure of the 1-parameter subgroup of $f(C'_0) \subseteq T_Z$ by a $T(k)$-point. In particular, C is unibranch and contains P. Since g is étale in codimension 1, we have $g^*K_W = K_{W'}$. Letting d denote the degree of $g|_{C'}\colon C' \to C$, we find

$$-K_W \cdot C = \frac{1}{d}(-K_W) \cdot g_*C' = \frac{1}{d}(-K_{W'}) \cdot C' \leq -K_{W'} \cdot C',$$

thereby yielding the desired bound for $-K_W \cdot C$. □

Lemma 6.3 provides a bound that is useful throughout the rest of this section.

Lemma 6.3. *If* W *is a weighted projective space and* $P \in W(k)$*. Then there is a curve* $C \subseteq W$ *through* P *satisfying property* (a) *of Proposition* 6.1*, and such that* $D \cdot C \leq 1$ *for all torus-invariant divisors* D *on* W*.*

Proof. Let $v_0, \ldots, v_n \in N$ be the primitive generators for the rays of Σ_W, and let $a_0, \ldots, a_n$ be relatively prime positive integers with $\sum a_i v_i = 0$ in N. Without loss of generality $a_0 = \max(a_i)$. Since W is a weighted projective space, N is the lattice spanned by the v_i. Let D_i be the torus-invariant divisor corresponding to v_i. We prove the result by inducting on dimension.

We first handle the base case where $\dim W = 1$, i.e. $W = \mathbb{P}^1$. Then choosing $C = W$, we find $C \cdot D_i = \deg(D_i) = 1$.

Next, we handle the case where $P \in T$ or where P is in the torus T_{D_0} of D_0. Let C be the closure of the 1-parameter subgroup corresponding to the lattice point $v_0 \in N$. Let ϕ denote the unique function $\phi\colon N_{\mathbb{R}} \to \mathbb{R}$ which is linear on all maximal cones subject to the condition $\phi(v_0) = 1$ and $\phi(v_i) = 0$ for $i \neq 0$. Then

$$D_0 \cdot C = \phi(v_0) + \phi(-v_0).$$

Since $-v_0 = \sum_{i>0} \frac{a_i}{a_0} v_i$ is in the maximal cone generated by $v_1, \ldots, v_n$, we see $\phi(-v_0) = 0$ and so $D_0 \cdot C = 1$. Furthermore, since $\frac{1}{a_0} D_0$ and $\frac{1}{a_i} D_i$ are linearly equivalent for all i, and $a_i \leq a_0$, we find

$$D_i \cdot C = \frac{a_i}{a_0} D_0 \cdot C \leq 1.$$

Note that C contains both the identity of T and the identity of T_{D_0}. Thus, if $P \in T$ or $P \in T_{D_0}$, a suitable T-translate of C contains P.

It remains to handle the case where $P \in D_j$ for some $j \neq 0$. Now, D_j is a weighted projective space of dimension $\dim W - 1$; its lattice is given by $\overline{N} := N/\mathbb{Z}v_j$ and its torus-invariant divisors D'_i correspond to the ray spanned by v_i in $\overline{N}$ for $i \neq j$. By induction, there exists a curve $C \subseteq D_j$ which is the translate of the closure of a 1-parameter subgroup in a T_{D_j}-orbit closure by a $T_{D_j}(k)$-point; since the quotient map $T \to T_{D_j}$ is surjective on k-points, C is also the translate of the closure of a 1-parameter subgroup in a T-orbit closure by a $T(k)$-point. By construction $D'_i \cdot C \leq 1$ for all $i \neq j$. Letting $m_{ij} \geq 1$ denote the multiplicity of the cone $\langle v_i, v_j \rangle$ in N, we have from [Fu93, p. 100] that

$$D_i \cdot C = \frac{1}{m_{ij}} D'_i \cdot C \leq 1$$

for $i \neq j$. To handle the case of D_j, we apply the same technique as above:

$$D_j \cdot C = \frac{a_j}{a_0} D_0 \cdot C \leq D'_0 \cdot C \leq 1.$$

This completes the proof of the result. ☐

Applying Lemmas 6.2 and 6.3, we are able to handle many cases of Proposition 6.1.

Corollary 6.4. *The following are true:*

(1) *Proposition* 6.1 *holds for weighted projective spaces.*
(2) *Proposition* 6.1(a) *and* (b) *hold for all fake weighted projective spaces.*

Proof. By Lemma 6.2, statement (2) follows from statement (1).

Let W be a weighted projective space. We let $n = \dim W$ and again denote by $v_0, \ldots, v_n \in N$ the primitive generators for the rays of Σ_W. Let $a_0, \ldots, a_n$ be relatively prime positive integers with $\sum a_i v_i = 0$ in N. Without loss of generality $a_0 = \max(a_i)$.

By Lemma 6.3, there is a curve $C \subseteq W$ through P satisfying property (a) of Proposition 6.1, and such that $D_i \cdot C \leq 1$ for all i. Since $\frac{1}{a_0} D_0$ and $\frac{1}{a_i} D_i$ are linearly equivalent for all i, and $D_0 \cdot C \leq 1$, we see

$$-K_W \cdot C = \frac{1}{a_0}(\sum_{i=0}^{n} a_i) D_0 \cdot C \leq \frac{1}{a_0}\sum_{i=0}^{n} a_i.$$

As $a_0 = \max(a_i)$, we see $\frac{1}{a_0}\sum_{i=0}^{n} a_i \leq n+1$, thereby proving Proposition 6.1(b) for weighted projective spaces.

It remains to prove that if W has terminal singularities and is not isomorphic to $\mathbb{P}^n$, then $\frac{1}{a_0}\sum_{i=0}^{n} a_i \leq n$. For ease, of notation, let $h = \sum_{i=0}^{n} a_i$. By [Ka13, Proposition 2.3], we see

$$\sum_{i=0}^{n} \left\{ \frac{a_i \kappa}{h} \right\} \leq n-1 \tag{6.5}$$

for all $2 \leq \kappa \leq h-2$, where $\{x\} = x - \lfloor x \rfloor$. Since $W \not\cong \mathbb{P}^n$, we know each $a_i \geq 1$ and $a_0 \geq 2$; in particular, we can choose $\kappa = n$.

Now, if $\frac{1}{a_0}\sum_{i=0}^{n} a_i > n$, then $\frac{na_0}{h} < 1$, and so $\lfloor \frac{na_i}{h} \rfloor = 0$ for all i. As a result,

$$\sum_{i=0}^{n} \left\{ \frac{na_i}{h} \right\} = \sum_{i=0}^{n} \frac{na_i}{h} = n,$$

contradicting (6.5). ☐

In light of Lemma 6.2 and Corollary 6.4, to finish the proof of Proposition 6.1, it remains to handle the case where W has terminal singularities and is of the form given in Lemma 6.2(b). We first handle the case where $P \in T$ through Lemma 6.6 and Corollary 6.7.

Lemma 6.6. *Let $W = \mathbb{P}^n/\mu_r$ be a fake weighted projective space where the quotient map $\mathbb{P}^n \to W$ is the universal covering in codimension 1. Then there is a standard affine patch $x_j \neq 0$ of $\mathbb{P}^n$ on which the action of $\zeta \in \mu_r$ is given by*

$$[\zeta^{w_0}x_0 : \ldots : \zeta^{w_{j-1}}x_{j-1} : x_j : \zeta^{w_{j+1}}x_{j+1} : \ldots : \zeta^{w_n}x_n]$$

such that $w_i \leq \frac{rn}{n+1}$ for all i.

Proof. In what follows, we will denote by $M(k)$ the unique element of $\{0, 1, \ldots, r-1\}$ that is congruent to k modulo r.

First, note that we may reorder the coordinates of $\mathbb{P}^n$ so that the action of μ_r on $\mathbb{P}^n$ globally is given by

$$[x_0 : \zeta^{w_1}x_1 : \ldots : \zeta^{w_n}x_n],$$

where the w_i are positive integers satisfying $r := w_0 > w_1 \geq \ldots \geq w_n \geq w_{n+1} := 0$. For $0 \leq j \leq n$, if we identify the j-th affine patch $x_j = 1$ with $\mathbb{A}^n$, the action of $\zeta \in \mu_r$ is given by

$$\left(\zeta^{w_0 - w_j}x_0, \ldots, \zeta^{w_{j-1}-w_j}x_{j-1}, \zeta^{w_{j+1}-w_j}x_{j+1}, \ldots, \zeta^{w_n - w_j}x_n\right).$$

Next, notice that $\sum_{j=0}^{n} M(w_j - w_{j+1}) = (w_0 - w_1) + \cdots + (w_{n-1} - w_n) + (w_n - w_{n+1}) = r$. So by the Pigeonhole Principle, there is some j for which

$$w_j - w_{j+1} = M(w_j - w_{j+1}) \geq \frac{r}{n+1}.$$

In particular, $w_j \geq w_j - w_{j+1} \geq \frac{r}{n+1}$. Furthermore, $w_j > w_{j+1}$ since otherwise $r = 0$, a contradiction.

On the j-th affine patch, the weights of the μ_r-action are given by $M(w_i - w_j)$ for $i \neq j$. If $i < j$, then $M(w_i - w_j) = w_i - w_j$; since $w_i < r$ and $w_j \geq \frac{r}{n+1}$, we find $M(w_i - w_j) < \frac{rn}{n+1}$. If $i > j$, then since $w_j > w_{j+1} \geq w_i$, we find $M(w_i - w_j) = r + w_i - w_j \leq r + w_{j+1} - w_j \leq \frac{rn}{n+1}$, as desired. $\square$

Corollary 6.7. *Proposition* 6.1 *holds for fake weighted projective spaces W of the form given in Lemma* 6.2(b) *whenever $P \in T$. In fact, the stronger conclusion $-K_W \cdot C \leq \dim W$ holds even if W does not have terminal singularities.*

Proof. Let $W = \mathbb{P}^n/\mu_p$ be a fake weighted projective space where the quotient map $f\colon \mathbb{P}^n \to W$ is the universal covering in codimension 1. By Lemma 6.6, after permuting coordinates, we may assume that on the standard affine patch $x_0 \neq 1$, $\zeta \in \mu_p$ acts by $(\zeta^{w_1}x_1, \dots, \zeta^{w_n}x_n)$ with

$$w_n \leq \cdots \leq w_1 \leq \frac{np}{n+1}.$$

Since the restriction of $f\colon \mathbb{P}^n \to W$ to the torus $T = \mathbb{G}_m^n \subseteq W$ is a μ_p-torsor, giving a 1-parameter subgroup $\mathbb{G}_m \to T$ is equivalent to giving a diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "\gamma"] \arrow[d, "\beta"'] & \mathbb{G}_m^n \\
\mathbb{G}_m &
\end{tikzcd}$$

where β is a μ_p-torsor and γ is a μ_p-equivariant map. In particular, we can take β and γ to be the maps $\beta(t) = t^p$ and $\gamma(t) = (t^{w_1}, \dots, t^{w_n})$. Let $C \subseteq W$ be the closure of the 1-parameter subgroup defined by the diagram, and let $C' \subseteq \mathbb{P}^n$ be the closure of the 1-parameter subgroup defined by γ. We then have $f(C') = C$, and since β is a degree p map, we see $f_*C' = pC$. As $w_1 = \max(w_i)$, we have $-K_{\mathbb{P}^n} \cdot C' = (n+1)w_1$. Since f is étale in codimension 1, $f^*K_W = K_{\mathbb{P}^n}$ and so

$$-K_W \cdot C = \frac{1}{p}(-K_W) \cdot f_*C' = \frac{1}{p}(-K_{\mathbb{P}^n}) \cdot C' = \frac{n+1}{p} w_1 \leq n.$$

Thus, translating C by $P \in T(k)$, we obtain our desired curve. $\square$

We now turn to the case where P lives on the boundary of W, which is handled in Lemma 6.8 and Corollary 6.9.

Lemma 6.8. *Let p be a prime and $W = \mathbb{P}^n/\mu_p$ a fake weighted projective space such that the quotient map $\mathbb{P}^n \to W$ is the universal covering in codimension 1. If $D \subseteq W$ is a torus-invariant divisor, then either $D \simeq \mathbb{P}(1, \dots, 1, p, \dots, p)$, or $D \simeq \mathbb{P}^{n-1}/\mu_p$ is a fake weighted projective space such that the quotient map $\mathbb{P}^{n-1} \to D$ is the universal covering in codimension 1.*

Furthermore, if D is a weighted projective space, then there is a torus-invariant divisor $D' \neq D$ such that the cone in the fan Σ_W corresponding to $D \cap D'$ has multiplicity strictly greater than 1.

Proof. Let $v_0, \dots, v_n \in N$ be the primitive generators for the rays of Σ_W and let $N' = \mathbb{Z}v_0 + \cdots + \mathbb{Z}v_n$. By hypothesis, $[N : N'] = p$ and $\sum_{i=0}^n v_i = 0$. The fan for D_n lives on the lattice $\overline{N} := N/\mathbb{Z}v_n$; its rays are generated by the images $\overline{v}_i \in \overline{N}$ of the v_i for $0 \leq i < n$. Let $b_i \in \mathbb{Z}^+$ and $\overline{v}'_i \in \overline{N}$ be the primitive lattice point such that $\overline{v}_i = b_i\overline{v}'_i$. Letting $\overline{N}' := N'/\mathbb{Z}v_n$, we see the induced map $N/N' \to \overline{N}/\overline{N}'$ is an isomorphism, and hence

$$[\overline{N} : \overline{N}'] = p.$$

Let $\overline{N}'_0 := \mathbb{Z}\overline{v}'_0 + \cdots + \mathbb{Z}\overline{v}'_{n-1}$, and note that the universal covering in codimension 1 of D_n is induced by the inclusion of lattices $\overline{N}'_0 \subseteq \overline{N}$. So, D_n is a weighted projective space if and only if $\overline{N}'_0 = \overline{N}$.

From the inclusions $\overline{N}' \subseteq \overline{N}'_0 \subseteq \overline{N}$ and the fact that $[\overline{N} : \overline{N}'] = p$, we see D_n is not a weighted projective space if and only if $\overline{N}'_0 = \overline{N}'$. Since $v_0 = -\sum_{i=1}^{n} v_i$, we see that $v_1, \ldots, v_n$ is a $\mathbb{Z}$-basis for N' and so $\overline{v}_1, \ldots, \overline{v}_{n-1}$ is a $\mathbb{Z}$-basis for $\overline{N}'$. Now, if $\overline{N}'_0 = \overline{N}'$, then $\overline{v}'_1 = \sum_{i=1}^{n-1} c_i\overline{v}_i$ for some $c_i \in \mathbb{Z}$. As a result, $\overline{v}_1 = \sum_{i=1}^{n-1} b_1c_i\overline{v}_i$, so $b_1 = 1$. Similarly, all $b_i = 1$, so $\sum_{i=0}^{n-1} \overline{v}'_i = 0$, i.e. $\Sigma_{\overline{N}'_0}$ is the fan for $\mathbb{P}^{n-1}$ so $\mathbb{P}^{n-1} \to D_n$ is the universal covering in codimension 1, identifying D_n with $\mathbb{P}^{n-1}/\mu_p$.

We may therefore assume that D_n is a weighted projective space. In order to show $D_n \simeq \mathbb{P}(1, \ldots, 1, p, \ldots, p)$, it is equivalent to show that every maximal cone of D_n has multiplicity dividing p. Given such a maximal cone σ, after reindexing we can assume $\sigma = \langle \overline{v}'_1, \ldots, \overline{v}'_{n-1}\rangle$. Since $\overline{v}_0 = -\sum_{i=1}^{n-1} \overline{v}_i$, we see $\overline{N}' = \mathbb{Z}\overline{v}_1 + \cdots + \mathbb{Z}\overline{v}_{n-1}$. From the inclusions

$$\overline{N}' \subseteq \mathbb{Z}\overline{v}'_1 + \cdots + \mathbb{Z}\overline{v}'_{n-1} \subseteq \overline{N}$$

and the fact that $[\overline{N} : \overline{N}'] = p$, we see

$$\operatorname{mult}(\sigma) = [\overline{N} : \mathbb{Z}\overline{v}'_1 + \cdots + \mathbb{Z}\overline{v}'_{n-1}] \in \{1, p\},$$

as desired.

Lastly, note that $b_i = \operatorname{mult}(\langle v_i, v_n\rangle)$ for $0 \leq i < n$. If all $b_i = 1$, then $\overline{N}' = \overline{N}'_0$, which, as we have observed above, is equivalent to the statement that D_n is not a weighted projective space. So, if D_n is a weighted projective space, then there must exist some $i < n$ for which the cone corresponding to $D_i \cap D_n$ has multiplicity $b_i > 1$. ☐

Corollary 6.9. *Proposition* 6.1 *holds for fake weighted projective spaces* W *of the form given in Lemma* 6.2(b). *In fact, the stronger conclusion* $-K_W \cdot C \leq \dim W$ *holds even if* W *does not have terminal singularities.*

Proof. We prove the statement by induction on $\dim W$. Let $W = \mathbb{P}^n/\mu_p$ as in Lemma 6.2(b), and let $P \in W(k)$. Let $v_0, \ldots, v_n \in N$ be the primitive generators for the rays of Σ_W, and denote by D_i the torus-invariant divisor corresponding to v_i. If $P \in T$, then the statement follows from Corollary 6.7. So, we may assume without loss of generality that $P \in D_0$. For $1 \leq i \leq n$, let D'_i denote the torus-invariant divisor on D_0 corresponding to v_i.

First suppose that D_0 is a weighted projective space. Then by Lemma 6.8, there exists $i \neq 0$ such that the multiplicity of the cone $\langle v_0, v_i\rangle$ is $m \geq 2$. Since D_0 is a weighted projective space, Lemma 6.3 yields a curve $C \subseteq D_0$ satisfying Proposition 6.1(a) and $C.D'_j \leq 1$ for all j. Then

$$-K_W \cdot C = (n+1)D_i \cdot C = \frac{n+1}{m}D'_i \cdot C \leq \frac{n+1}{m} \leq n.$$

Note, in particular, that this handles the base case of our induction. Indeed, there are no 1-dimensional fake weighted projective spaces of the form given in Lemma 6.2(b), so the base case is $n = 2$, in which case we necessarily have $D_0 \simeq \mathbb{P}^1$.

If D_0 is not a weighted projective space, then by Lemma 6.8, we know $D_0 \simeq \mathbb{P}^{n-1}/\mu_p$ as in Lemma 6.2(b). By induction on dimension, we can assume the existence of our desired C with $-K_{D_0} \cdot C \leq n - 1$. Since $\sum_{i=1}^{n} D'_i \cdot C = -K_{D_0} \cdot C$, by the Pigeonhole Principle, we may without loss of generality that $D'_1 \cdot C \leq \frac{n-1}{n}$. So, for all $0 \leq i \leq n$, we have $D_i \cdot C = D_1 \cdot C \leq D'_1 \cdot C \leq \frac{n-1}{n}$, which implies $-K_W \cdot C \leq \frac{1}{n}(n-1)(n+1) \leq n$. □

Putting these results together we have:

Proof of Proposition 6.1. Corollary 6.4 shows that Proposition 6.1(a) and (b) hold for all fake weighted projective spaces and that part (c) additionally holds for all weighted projective spaces. By Lemma 6.2, it remains to show Proposition 6.1 holds for fake weighted projective spaces of the form given in Lemma 6.2(b). This is handled in Corollary 6.9. □

7. Base case: P is in the exceptional locus

In this section we prove Proposition 4.10, thereby finishing the proof of Theorem 4.2, and hence also proving Theorem 1.7. We begin with a lemma that allows us to reduce to the case of fake weighted projective spaces.

Lemma 7.1. *Let X be a projective terminal $\mathbb{Q}$-factorial split toric variety and let $\pi\colon X \to Y$ be an elementary contraction corresponding to the extremal ray $\mathcal{R}$. Suppose $C \subseteq X$ is a curve contracted by π. If F is the reduction of the fiber of π containing C, then $-K_X \cdot C \leq -K_F \cdot C$.*

Proof. Let $v_1, \ldots, v_\ell$ be the rays of the fan Σ_X, and $D_i \subseteq X$ denote the torus-invariant divisor corresponding to v_i. Let $y = \pi(F)$. There is a unique torus-orbit closure $Z \subseteq Y$ such that y is contained in the torus T_Z of Z. Since the fibers of π are irreducible, $\pi^{-1}(Z)$ is also irreducible. So, the reduction of $\pi^{-1}(Z)$ is a torus-orbit closure $W \subseteq X$. Let $\tau \in \Sigma_X$ be the cone corresponding to W. Since F is positive-dimensional and it is a general fiber of $\pi|_W\colon W \to Z$, we see W is contained in the exceptional locus $\operatorname{Exc}(\pi)$.

By Lemma 14-1-7 and Corollary 14-2-2 of [Ma02], $\operatorname{Exc}(\pi)$ is the torus-orbit closure corresponding to the cone spanned by the rays v_i with $D_i \cdot C < 0$. Furthermore, the aforementioned results of [Ma02] show that the toric map from $\operatorname{Exc}(\pi)$ to its image corresponds to the quotient map $\eta\colon N/N_- \to N/N_{\neq 0}$, where N_- (resp. $N_{\neq 0}$) is the saturation of the sublattice generated by the v_i with $D_i \cdot C < 0$ (resp. $D_i \cdot C \neq 0$). So if $D_i \cdot C > 0$, then $\eta(\overline{v}_i) = 0$ and hence D_i does not contain a fiber of π. In particular, $D_i \cdot C \leq 0$ if v_i is a ray of τ.

Reordering the rays if necessary, we can assume $\tau = \langle v_1, \ldots, v_a \rangle$, and that $v_{a+1}, \ldots, v_b$ are the rays in $\operatorname{Star}(\tau)$ which are not in τ. Let m be the multiplicity of the cone $\langle v_1, \ldots, v_a \rangle$. For $a < i \leq b$, we let D'_i be the torus-invariant divisor on W corresponding to v_i, and let $m_i \geq 1$ be the multiplicity of the cone $\langle v_1, \ldots, v_a, v_i \rangle$. Since $D_i \cdot C = 0$ for $v_i \notin \operatorname{Star}(\tau)$, and since $D_i \cdot C \leq 0$ for $v_i \in \tau$,

$$-K_X \cdot C = \sum_{i=1}^{b} D_i \cdot C \leq \sum_{i=a+1}^{b} D_i \cdot C = \sum_{i=a+1}^{b} \frac{m}{m_i} D'_i \cdot C \leq \sum_{i=a+1}^{b} D'_i \cdot C = -K_W \cdot C.$$

Finally, since F is a general fiber of $\pi|_W$, we find $K_W|_F = K_F$, and so $-K_X \cdot C \leq -K_F \cdot C$. □

If $X \simeq \mathbb{P}^n$, then Theorem 4.2 follows from [McK07, Theorem 2.6]. Therefore, the following result finishes the proof of Proposition 4.10.

Proposition 7.2. *Let X be a projective terminal $\mathbb{Q}$-factorial split toric variety over a number field k and let $\pi\colon X \to Y$ be the elementary contraction corresponding to an extremal ray $\mathcal{R}$. Let $D \in \operatorname{Nef}(X) \cap \mathcal{R}^{\perp}$ be a $\mathbb{Q}$-divisor and $P \in X(k) \cap \operatorname{Exc}(\pi)$ a canonically bounded point. If $X \not\simeq \mathbb{P}^n$, then there exists an irreducible rational curve C through P such that C is unibranch at P, $-K_X \cdot C \leq \dim X$, and $\alpha_{P,C}(D) \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$ on X.*

Proof. Let F be the reduction of the fiber containing P. It follows from [Fu06, Remark 3.3] that F is a fake weighted projective space.

Suppose first that Y is a point. Then $F = X$. Since X has terminal singularities and is not isomorphic to projective space, Proposition 6.1(a) and (c) tell us there is an irreducible rational curve C through P which is unibranch at P and satisfies $-K_X \cdot C \leq \dim X$.

Next suppose Y is not a point. Then $\dim F \leq \dim X - 1$. By Proposition 6.1(a) and (b), there is an irreducible rational curve $C \subseteq F$ through P which is unibranch at P and satisfies $-K_F \cdot C \leq 1 + \dim F$. By Lemma 7.1, we have

$$-K_X \cdot C \leq -K_F \cdot C \leq 1 + \dim F \leq \dim X.$$

Lastly, regardless of whether or not Y is a point, by Theorem 2.8, if m denotes the multiplicity of C at P, we have $\alpha_{P,C}(D) = \frac{1}{m} C \cdot D = 0$, so $\alpha_{P,C}(D) = 0 \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$ on X. □

8. Finding the curve of best approximation

A curve $C \subseteq X$ is said to be a *curve of best approximation* with respect to D if $\alpha_{P,C}(D) = \alpha_P(D)$. The curve C constructed in Theorem 1.7 is not required to be a curve of best approximation, but only one that approximates P better than any Zariski dense sequence. In addition to the theoretical point raised in Remark 1.8 that there may be some Zariski-degenerate sequence with higher dimensional closure that approximates P better than C, there is the very practical point mentioned in Remark 1.9 that the curve C we find is *in fact* not always a curve of best approximation to P, as we discuss in this section.

For example, let k be a number field, and fix a place v of k. If X is the weighted projective space $\mathbb{P}(4,7,13)$, and D is the generator of the Picard group of X, then $D = -\frac{4\cdot 7\cdot 13}{4+7+13} K_X$. Assuming canonical boundedness of the point $P = [1:1:1]$, we find then that $\alpha_{P,\{x_i\}}(D) \geq \frac{91}{3}$. Our proof of Theorem 1.7 for this choice of X and P ultimately comes from Lemma 6.3. Specifically, the curve C we construct in this case is $x^7 = y^4$, which has D-degree 28. So, $\alpha_{P,C}(D) = 28 < \frac{91}{3} \leq \alpha_{P,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\}$. However, it is easy to see (for this particular X and P) that there are other curves which have smaller α-value, e.g. the curve $x^5 = yz$ has D-degree 20 and hence has α-value 20.

One may wonder whether $x^5 = yz$ is the curve of best approximation. The answer turns out to be interesting: a thorough search reveals that the curve C' given by the equation

$$x^8 y + x y^5 - 3x^3 y^2 z + z^3 = 0$$

satisfies $\alpha_{P,C'}(D) = 19.5$ provided that $\sqrt{-3} \in k_v \smallsetminus k$. This is because C', which has degree $C' \cdot D = 39$, is singular at P and the tangent directions to C' at P are distinct and split over the field $\mathbb{Q}(\sqrt{-3})$. So, if $\sqrt{-3} \in k_v \smallsetminus k$, then Theorem 2.7 implies $\alpha_{P,C'}(D) = \frac{39}{2} = 19.5$.

In fact, as we now explain, the curve C' is a curve of best approximation to P when $\sqrt{-3} \in k_v \smallsetminus k$. Let D' be the Weil divisor $x = 0$, so that $D = 91D'$. Letting $\mathfrak{m}_P$ be the maximal ideal at P, we see $\mathcal{O}_{X,P}/\mathfrak{m}_P^3$ has dimension $\binom{4}{2} = 6$. A straightforward computation shows that $H^0(14D')$ has dimension $7 > \binom{4}{2}$, and so there must be some non-zero section $g \in H^0(14D')$ that vanishes at P to order at least 3. In fact, one computes that, up to scalar, there is a unique such g, which defines the curve C''

$$x^{14} - 4x^9yz + x^7y^4 + 6x^4y^2z^2 - 4x^2y^5z + y^8 - xz^4 = 0.$$

Then the section $g^{13} \in H^0(13 \cdot 14D') = H^0(2D)$ vanishes at P to order at least 39. Thus, if $\pi\colon Y \to X$ denotes the blowup of X at P, with exceptional divisor E, then $2\pi^*D - 39E$ is effective. Let $B \subset X$ be the image of the asymptotic base locus of $2\pi^*D - 39E$. Then Theorem 3.3 of [MR16], shows that for any sequence $\{x_i\}$ not contained in B, we have $\alpha_{P,\{x_i\}}(D) \geq \frac{39}{2} = 19.5$. Thus, unconditionally (i.e. without even assuming P is canonically bounded), the curve C' must be a curve of best D-approximation to P, once we show that there is no curve in the locus B with a smaller α-value than C'.

To handle curves in the locus B, first note that the self-intersection $(2\pi^*D - 39E)^2 = -65$ is negative. Now, B is contained in the locus defined by the vanishing g^{13}, namely the divisor $13C''$. Thus, it suffices to show that $\alpha_{P,C''}(D) > \alpha_{P,C'}(D)$. This is the case since C'' has degree 56, so by Theorem 2.7, we see $\alpha_{P,C''}(D) \geq \frac{56}{2} = 28 > 19.5 = \alpha_{P,C'}(D)$. Therefore, C' is indeed a curve of best approximation to P, provided that $\sqrt{-3} \in k_v \smallsetminus k$.

In summary, the curve of best approximation depends in a subtle way on the number field k. In particular, if one wishes to show the existence of a curve of best D-approximation to P without assuming *a priori* that P is canonically bounded, one would need to provide an explanation for the non-trivial fact that C' is contained in the Zariski closed locus of exceptions to the canonical boundedness condition provided by Vojta's Conjecture, at least when $\sqrt{-3} \in k_v \smallsetminus k$.

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