J. Lesieutre and M. Satriano (2019) "Canonical Heights on Hyper-Kähler Varieties and the Kawaguchi–Silverman Conjecture,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–38
doi:10.1093/imrn/rnz067

Canonical Heights on Hyper-Kähler Varieties and the Kawaguchi–Silverman Conjecture

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The Kawaguchi–Silverman conjecture predicts that if $f : X \dashrightarrow X$ is a dominant rational-self map of a projective variety over $\overline{\mathbb{Q}}$, and P is a $\overline{\mathbb{Q}}$-point of X with a Zariski dense orbit, then the dynamical and arithmetic degrees of f coincide: $\lambda_1(f) = \alpha_f(P)$. We prove this conjecture in several higher-dimensional settings, including all endomorphisms of non-uniruled smooth projective threefolds with degree larger than 1, and all endomorphisms of hyper-Kähler manifolds in any dimension. In the latter case, we construct a canonical height function associated with any automorphism $f : X \to X$ of a hyper-Kähler manifold defined over $\overline{\mathbb{Q}}$. We additionally obtain results on the periodic subvarieties of automorphisms for which the dynamical degrees are as large as possible subject to log concavity.

1 Introduction

Let $f : X \dashrightarrow X$ be a dominant rational self-map of a smooth projective variety X defined over $\overline{\mathbb{Q}}$. There are two natural degree functions, one can associate with the dynamical system (X, f). The 1st measures the growth rate of the degrees of the iterates f^n. It is

Received July 13, 2018; Revised December 7, 2018; Accepted March 6, 2019
Communicated by Prof. Umberto Zannier

known as the *1st dynamical degree* and is defined as

$$\lambda_1(f) = \lim_{n\to\infty} \left((f^n)^*H \cdot H^{\dim X - 1}\right)^{1/n},$$

where H is a choice of ample divisor on X; a result of Dinh and Sibony [15] says that this limit exists and is independent of the choice of ample divisor H.

The 2nd notion is the *arithmetic degree*, which depends on a choice of $\overline{\mathbb{Q}}$-point P, and reflects the growth rate of the heights of the points $f^n(P)$. Given an ample divisor H on X, one can construct a corresponding *logarithmic Weil height* $h_H : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ that measures the arithmetic complexity of the point P. For example, if $X = \mathbb{P}^n$, $H = \mathcal{O}_{\mathbb{P}^n}(1)$, and $P = [a_0 : \ldots : a_n]$ is a $\mathbb{Q}$-point with the a_i coprime integers, then $h_H(P) = \log(\max\{|a_i|\})$. We refer to Proposition 2.25 for further properties of heights.

Letting h_H denote a logarithmic Weil height associated with an ample divisor H on X, and $h_H^+ = \max(h_H, 1)$, if the forward orbit of P is defined (i.e., if no $f^n(P)$ is contained in the indeterminacy locus), then we set

$$\underline{\alpha}_f(P) = \liminf_{n\to\infty} h_H^+(f^n(P))^{1/n}, \qquad \overline{\alpha}_f(P) = \limsup_{n\to\infty} h_H^+(f^n(P))^{1/n}.$$

Both of these quantities are again independent of the choice of ample divisor H [26, Proposition 12] and it is conjectured that they always coincide. When they do, $\alpha_f(P)$ is defined to be the common value. Whether or not $\underline{\alpha}_f(P)$ and $\overline{\alpha}_f(P)$ are equal remains open in general, but it is known when f is a morphism [25, Theorem 3], which will always be the case in this paper. The Kawaguchi–Silverman conjecture is then as follows.

Conjecture 1.1 (Kawaguchi–Silverman [26]). Let X be a smooth projective variety and let $f : X \dashrightarrow X$ be a dominant rational map defined over $\overline{\mathbb{Q}}$. Suppose that P is a $\overline{\mathbb{Q}}$-point of X. If the forward orbit of P under f is Zariski dense, then $\alpha_f(P)$ exists and is equal to $\lambda_1(f)$.

The conjecture is known in many cases; see [37, Remark 1.8] for a comprehensive list including abelian varieties [25, 48], automorphisms of smooth projective surfaces [23, 24], as well as certain product varieties [45]. Recently the conjecture was proved for all regular endomorphisms of smooth projective surfaces [37]; the proof in the case of surfaces relies heavily on the birational classification.

Our aim in this paper is to prove the conjecture in several higher-dimensional settings. A basic difficulty is that the classification of n-folds (for $n \geq 3$) is much more difficult and less complete than the classification of surfaces: there is no neat analog of the Enriques–Kodaira classification, and one must instead attempt to understand the interplay between the geometry of endomorphisms and the classification theory of higher-dimensional varieties.

We break up our analysis according to Kodaira dimension $\kappa(X)$. We note that by [40, Theorem A], if $\kappa(X) > 0$, then an iterate of f preserves the Iitaka fibration and so there is no $\overline{\mathbb{Q}}$-point P on X with a Zariski dense orbit; as a result Conjecture 1.1 vacuously holds. Thus, the only remaining cases to consider are those of Kodaira dimension 0 and $-\infty$.

Let us now discuss in detail our main results as well as several consequences. We say a smooth projective variety X is *Calabi–Yau* if $\dim X \geq 3$, $\mathcal{O}_X(K_X)$ is trivial and $h^0(\Omega^p_X) = 0$ for $0 < p < n$. We say X is *hyper-Kähler* if its complex analytification is simply connected and $H^0(X, \Omega^2_X)$ is spanned by a symplectic form.

Theorem 1.2. Conjecture 1.1 is true for automorphisms of hyper-Kähler manifolds.

Remark 1.3. In fact, we prove the following more general result in Theorem 2.28: Conjecture 1.1 holds for any automorphism of a normal projective variety X for which $h^1(X, \mathcal{O}_X) = 0$ and $\nu_+ + \nu_-$ is big for some good eigenvector pair (ν_+, ν_-), as defined in Definition 2.8. We also note that Theorem 1.2 applies to surjective endomorphisms since every surjective endomorphism of a hyper-Kähler manifold is an automorphism, see Lemma 2.6.

The key to proving Theorem 1.2 is to construct a canonical height function associated with f, following a strategy developed by Silverman [47] and Kawaguchi [23] in dimension 2. Along the way, we obtain various results on the periodic subvarieties of automorphisms for which the dynamical degrees are as large as possible (subject to log concavity). For instance,

Corollary 1.4.

1. Suppose $f : X \to X$ is an automorphism satisfying Condition (B) of Definition 2.10 and for which $\lambda_1(f) = \lambda_1(f^{-1})$, for example, f is any automorphism of a hyper-Kähler manifold. Then the union of the odd-dimensional f-periodic subvarieties of X is not Zariski dense.

2. Suppose that $f : X \to X$ is an automorphism of a threefold satisfying either $\lambda_1(f) = \lambda_2(f)^2$ or $\lambda_2(f) = \lambda_1(f)^2$. Then f admits only finitely many positive-dimensional periodic subvarieties. This situation can arise, for example, for automorphisms of abelian threefolds [33, Example 4.8.6].

As a consequence of Theorem 1.2, combined with the work of Sano [45], we are able to show that the conjecture holds for automorphisms of all varieties with $K_X \equiv 0$ as long as it holds for automorphisms of Calabi–Yau varieties.

Corollary 1.5. Let n be a positive integer. Then Conjecture 1.1 is true for all automorphisms of smooth projective varieties X with dimension at most n and K_X numerically trivial if and only if Conjecture 1.1 is true for all automorphisms of smooth Calabi–Yau varieties with dimension at most n.

Remark 1.6. The abundance conjecture implies that every smooth projective minimal variety X of Kodaira dimension 0 has K_X numerically trivial. Therefore, assuming the abundance conjecture in dimension at most n, Corollary 1.5 reduces Conjecture 1.1 for automorphisms of smooth projective minimal varieties of Kodaira dimension 0 to the special case of smooth Calabi–Yau varieties.

In the case of dimension 3, we obtain more detailed results for *endomorphisms* as well as automorphisms. Using results of Fujimoto [19], we show the following.

Proposition 1.7. Conjecture 1.1 holds for all surjective endomorphisms $f\colon X \to X$ of degree $\deg(f) > 1$ on smooth projective threefolds X of Kodaira dimension 0.

Since the abundance conjecture is known in dimension 3 [28], by Corollary 1.5 and Remark 1.6, to prove the conjecture for automorphisms of smooth minimal threefolds of Kodaira dimension 0, it is enough to handle the case of automorphisms of smooth Calabi–Yau threefolds. As such, we turn to the case of Calabi–Yau threefolds and prove the following technical result.

Theorem 1.8. Let f be an automorphism of a smooth Calabi–Yau threefold X. Suppose that either

1. $c_2(X)$ is strictly positive on $\mathrm{Nef}(X)$, or
2. there is a nonzero semi-ample class $D \in \mathrm{Nef}(X) \cap N^1(X)$ such that $c_2(X) \cdot D = 0$.

Then Conjecture 1.1 holds for (X, f).

Remark 1.9 (Understanding the hypotheses of Theorem 1.8). The essential point here is that Theorem 1.8 applies to all Calabi–Yau threefolds with sufficiently large Picard number, assuming [42, Question-Conjecture 2.6] and the semi-ampleness conjecture [32, Conjecture 2.1]. In particular, by Remark 1.6, this would resolve Conjecture 1.1 for all automorphisms of smooth minimal threefolds of Kodaira dimension 0 and sufficiently large Picard number.

Let us explain why this is the case. A theorem of Miyaoka [38] shows that either hypothesis (1) of Theorem 1.8 holds or $F := c_2(X)^{\perp} \cap \mathrm{Nef}(X)$ is a nonzero face of the nef cone of X. In [42, Question-Conjecture 2.6], Oguiso asks if F must always be rational when the Picard number $\rho(X)$ is sufficiently large. Provided this is true, there would be a nonzero rational class $D \in F$, and then the semi-ampleness conjecture [32, Conjecture 2.1] would tell us that after scaling D by a positive integer, we can assume it is semi-ample, that is, hypothesis (2) holds.

Finally, we turn to the case of Kodaira dimension $-\infty$. Here the closest analog of a minimal variety is one which has the structure of a Mori fiber space; this includes, for example, all rational normal scrolls. We prove Conjecture 1.1 in two special cases.

Theorem 1.10. Conjecture 1.1 holds for the following cases:

1. all automorphisms of threefolds that have the structure of a Mori fiber space.
2. all surjective endomorphisms of n-fold rational normal scrolls.

Remark 1.11. In the process of showing Theorem 1.10(2), we in fact prove a stronger result: if C is a smooth curve, then Conjecture 1.1 holds for all surjective endomorphisms of all projective bundles $\mathbb{P}_C(\mathcal{E})$ if and only if it holds in the case, where $\mathcal{E}$ is semistable of degree 0. See Corollary 6.8.

It is worth mentioning that in Section 3, we prove general results concerning the following set-up: $\pi : X \to Y$ is a surjective morphism of normal projective varieties over $\overline{\mathbb{Q}}$, f is a surjective endomorphism of X, g is a surjective endomorphism of Y, and $\pi \circ f = g \circ \pi$. We give several criteria by which one can reduce the conjecture for (X, f) to that of (Y, g), see Theorem 3.4.

2 Canonical Heights for Hyper-Kähler Automorphisms: Theorem 1.2

In this section, we treat Conjecture 1.1 for hyper-Kähler manifolds. There are many remarkable automorphisms and birational automorphisms of such varieties, see for example, [1] and [43]. In fact, we prove the conjecture for a wider class of varieties, namely those with trivial Albanese satisfying Condition (B) of Definition 2.10.

We begin by recalling the main definitions. We work throughout over $\overline{\mathbb{Q}}$, and where not otherwise stipulated, a variety is assumed to be defined over $\overline{\mathbb{Q}}$. Given a projective variety X, we write $N^j(X)$ for the numerical group of codimension-j cycles and $N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$ for the corresponding finite-dimensional $\mathbb{R}$-vector space. We use $\sim$ for the relation of linear equivalence of Cartier divisors, $\sim_{\mathbb{R}}$ for $\mathbb{R}$-linear equivalence, and $\equiv$ for numerical equivalence. Rational maps are denoted by "$\dashrightarrow$" and morphisms by "$\to$".

Suppose that $f : X \dashrightarrow X$ is a dominant rational map of a smooth projective variety, and fix an ample divisor H on X. The jth dynamical degree of f is the limit

$$\lambda_j(f) = \lim_{n\to\infty} \left(((f^n)^*H)^j \cdot H^{\dim X - j} \right)^{1/n}.$$

As noted in the introduction, the *1st dynamical degree* is obtained when $j = 1$, and this case occupies a place of particular importance in this note. In general, these limits are difficult to compute, since $(f^n)^*$ does not necessarily coincide with $(f^*)^n$ for rational maps. However, if $f : X \to X$ is a morphism, then $(f^n)^* = (f^*)^n$ and

$$\lambda_j(f) = \operatorname{SpecRad}\left(f^* : N^j(X)_{\mathbb{R}} \to N^j(X)_{\mathbb{R}} \right)$$

is simply the spectral radius of f^*, on the numerical group of codimension-j cycles, that is, the absolute value of the largest eigenvalue of f^* acting on $N^j(X)_{\mathbb{R}}$. When f is a morphism, we may also drop the smoothness hypothesis on X, and it suffices to assume that X is normal: there is no difficulty in pulling back Cartier divisors.

It will also be convenient to write $J(f,j)$ for the dimension of the largest $\lambda_j(f)$-Jordan block of $f^* : N^j(X) \to N^j(X)$ and $\tilde{J}(f,j) = J(f,j) - 1$. Then $((f^n)^*H)^j \cdot H^{\dim X - j}$ is bounded above and below by positive multiples of $n^{\tilde{J}(f,j)}\lambda_j(f)^n$.

Invariant fibrations play an important role in the study of rational maps in higher dimension, and the product formula of Dinh *et al.* [13] is useful in dealing with their dynamical degrees. Suppose that there exists a surjective morphism $\pi : X \to Y$ and a dominant rational map $g : Y \dashrightarrow Y$ with $g \circ \pi = \pi \circ f$. Let H' be an ample divisor on Y.

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \underset{g}{\dashrightarrow} & Y \end{array}$$

Definition 2.1. The *1st dynamical degree of f relative to π* is the limit

$$\lambda_1(\pi|_f) = \lim_{n\to\infty} \left((f^n)^*H \cdot \pi^*(H'^{\dim Y}) \cdot H^{\dim X - \dim Y - 1}\right)^{1/n}.$$

The definition of relative dynamical degrees can be extended to higher codimension and to the setting in which π itself is only a dominant rational map; we require only this simple case. The basic properties of dynamical degrees and their relative counterparts are worked out in [12–15]; a more algebro-geometric perspective (which, importantly, works on normal varieties) can be found in [10, 49]. The next theorem singles out some properties of the dynamical degrees that we will require throughout the paper.

Theorem 2.2.

1. Suppose that $f : X \dashrightarrow X$ is birational. Then $\lambda_1(f^{-1}) = \lambda_{\dim X-1}(f)$. Furthermore, if $\lambda_1(f) > 1$, then $\lambda_1(f^{-1}) > 1$.
2. If $f : X \dashrightarrow X$ admits an invariant fibration $\pi : X \to Y$ as above, then $\lambda_0(f|_\pi) = 1$ and $\lambda_1(f) = \max\{\lambda_1(g), \lambda_1(f|_\pi)\}$.
3. If $\dim Y = \dim X - 1$ and f is birational, then $g : Y \dashrightarrow Y$ is birational and $\lambda_1(f|_\pi) = 1$.
4. Let f (resp. g) be a surjective endomorphism of X (resp. Y) and assume that X and Y are normal projective varieties. If $\pi : X \to Y$ is a birational morphism such that $\pi \circ f = g \circ \pi$, then $\lambda_1(f) = \lambda_1(g)$.

Proof. To prove these claims requires using the properties of higher dynamical degrees $\lambda_p(f)$; since we do not otherwise make use of these degrees, we refer to the above references for the definitions.

The 1st fact follows from the log-concavity of dynamical degrees, which states that $\lambda_{p-1}(f)\lambda_{p+1}(f) \le \lambda_p(f)^2$ for each $1 \le p \le \dim X - 1$. Since $\lambda_p(f) \ge 1$ for any p, the hypothesis that $\lambda_1(f) > 1$ implies that $\lambda_p(f) > 1$ for each $1 \le p < \dim X$. If f is birational, then $\lambda_1(f^{-1}) = \lambda_{\dim X-1}(f) > 1$.

The claim in (2) that $\lambda_0(f|_\pi) = 1$ follows directly from the definition, while $\lambda_1(f) = \max\{\lambda_1(g)\lambda_0(f|_\pi), \lambda_0(g)\lambda_1(f|_\pi)\} = \max\{\lambda_1(g), \lambda_1(f|_\pi)\}$ is a case of the product formula of Dinh–Nguyên–Truong.

For (3), another application of the product formula yields $\lambda_{\dim X}(f) = \lambda_{\dim X-1}(g)\lambda_1(f|_\pi)$. Since f is birational, $\lambda_{\dim X}(f) = 1$, and so both terms on the right must be 1 as well.

Finally, (4) follows from [10, Theorem 1.(2)] and the discussion that follows. ■

In contrast to the dynamical degrees, the properties of the arithmetic degrees $\overline{\alpha}_f(P)$, $\underline{\alpha}_f(P)$, and $\alpha_f(P)$ are at present largely conjectural in general. There is nevertheless a close relationship between the arithmetic and dynamical degrees: it was shown in [37, Corollary 9.3] that if $f : X \to X$ is a surjective endomorphism with $\lambda_1(f) > 1$, then there exist points P with $\alpha_f(P) = \lambda_1(f)$; however, it remains open whether this equality holds for *every* point P with a dense orbit.

Remark 2.3. It was proved by Kawaguchi–Silverman [26, Theorem 4] and Matsuzawa [36, Theorem 1.4] that $\overline{\alpha}_f(P) \le \lambda_1(f)$ in general. As a result, the limit defining $\alpha_f(P)$ exists and is equal to $\lambda_1(f)$ if and only if $\lambda_1(f) \le \underline{\alpha}_f(P)$; indeed, if this inequality holds, then $\lambda_1(f) \le \underline{\alpha}_f(P) \le \overline{\alpha}_f(P) \le \lambda_1(f)$.

Furthermore, since we always have $1 \le \underline{\alpha}_f(P)$, if $\lambda_1(f) = 1$, then $\lambda_1(f) \le \underline{\alpha}_f(P)$, and so the conjecture holds. Hence, we can always restrict our attention to maps with $\lambda_1(f) > 1$.

Having recalled the main definitions, we collect some facts that will be useful in our study of hyper-Kähler manifolds.

Proposition 2.4 (The Beauville–Bogomolov–Fujiki form, see, e.g., [21, §23], [21, Prop. 25.14]). Suppose that X is a hyper-Kähler manifold of dimension $2m$. There exists a quadratic form $q(X)$ on $H^2(X, \mathbb{R})$ and a constant c_X such that for any divisor D, we have the following:

$$D^{2m} = c_X\, q_X(D)^m.$$

The form $q_X(-)$ has signature $(1, \rho(X) - 1)$ on $N^1(X)_{\mathbb{R}}$. If $\phi : X \to X$ is an automorphism, then the pullback ϕ^* preserves the form $q(-)$.

Example 2.5 ([3, §6], [43]). A basic example of a hyper-Kähler manifold is the Hilbert scheme of configurations of n points on a K3 surface S, which we denote by $\mathrm{Hilb}^n(S)$. If $f : S \to S$ is an automorphism, then the induced automorphism $f^{[n]} : \mathrm{Hilb}^n(S) \to \mathrm{Hilb}^n(S)$ of the Hilbert scheme satisfies $\lambda_1(f^{[n]}) = \lambda_1(f)$.

To begin, we first note that any surjective endomorphism of a hyper-Kähler manifold is actually an automorphism.

Lemma 2.6. Every surjective endomorphism of a hyper-Kähler manifold over $\overline{\mathbb{Q}}$ is an automorphism.

Proof. By faithfully flat and quasi-compact descent, it is enough to check this after base change to $\mathbb{C}$. Now if X is hyper-Kähler manifold over $\mathbb{C}$, we have $\chi(\mathcal{O}_X) = 1 + \frac{1}{2}\dim X$ [5, Lemma 14.21]. Next, let f be a surjective endomorphism of X. By [19, Lemma 2.3], f is a finite étale cover and so $\chi(\mathcal{O}_X) = \deg(f)\chi(\mathcal{O}_X)$. Since $\chi(\mathcal{O}_X) \neq 0$, we must have $\deg(f) = 1$, that is, f is an automorphism. ■

Having now reduced to the case of automorphisms, we roughly follow the strategy of Kawaguchi's proof of the conjecture in the case of surfaces. The following version of the Perron–Frobenius theorem plays an important role.

Lemma 2.7 ([7]). Suppose that V is a finite-dimensional real vector space and that $K \subset V$ is a closed, pointed, and convex cone. If $T : V \to V$ is a linear map for which $T(K) \subseteq K$, and the spectral radius of T is $\lambda > 1$, then there exists a λ-eigenvector for T, which is contained in K.

In fact, such an eigenvector can be found by choosing H a general element of the interior of K and taking a limit $\lim_{n\to\infty} \frac{1}{n^{\tilde{J}(f,1)}\lambda^n}(f^*)^n(H)$; the normalizing factor ensures that this limit converges to a nonzero eigenvector.

Definition 2.8. We will say that a class ν_+ in $N^1(X)_{\mathbb{R}}$ is a *leading eigenvector* for f if it is a nef class that is a $\lambda_1(f)$-eigenvector for f^*. We say that ν_+ is a *good eigenvector* if there exists an ample class H so that $\lim_{n\to\infty} \frac{1}{n^{\tilde{J}(f,1)}\lambda_1(f)^n}(f^*)^n(H) = \nu_+$ (recall that we write $\tilde{J}(f,1)$ for one less than the dimension of the $\lambda_1(f)$-Jordan block of f^*). In particular, if ν_+ lies in the relative interior of the intersection of $\mathrm{Nef}(X)$ with the $\lambda_1(f)$-eigenspace of f^*, then ν_+ is a good eigenvector [2].

We say that (ν_+, ν_-) is an *eigenvector pair* for f if ν_+ is a leading eigenvector for f and ν_- is a leading eigenvector for f^{-1}, and that it is a *good eigenvector pair* if there exists an ample divisor H for which $\lim_{n\to\infty} \frac{1}{n^{\tilde{J}(f,1)}\lambda_1(f)^n}(f^*)^n(H) = \nu_+$ and $\lim_{n\to\infty} \frac{1}{n^{\tilde{J}(f^{-1},1)}\lambda_1(f^{-1})^n}(f^*)^{-n}(H) = \nu_-$.

We say that (D_+, D_-) is an *eigendivisor pair* for f if D_+ and D_- are $\mathbb{R}$-divisors for which $f^*(D_+) \sim_{\mathbb{R}} \lambda_1(f)D_+$ and $(f^{-1})^*(D_-) \sim_{\mathbb{R}} \lambda_1(f^{-1})D_-$. If (D_+, D_-) is an eigendivisor pair, then the corresponding pair of numerical classes (ν_+, ν_-) is an eigenvector pair, and we say that (D_+, D_-) is good if (ν_+, ν_-) is. When $h^1(X, \mathcal{O}_X) = 0$, each numerical class has a unique lift to a linear equivalence class and these two notions coincide.

Corollary 2.9. If f is an automorphism of a normal projective variety X that satisfies $\lambda_1(f) > 1$, then a good eigenvector pair (ν_+, ν_-) exists for f. If moreover $h^1(X, \mathcal{O}_X) = 0$, then a good eigendivisor pair (D_+, D_-) exists for f.

Proof. Recall that $\lambda_1(f^{-1}) = \lambda_{\dim X-1}(f) > 1$ by Theorem 2.2(1). We obtain the result by applying Lemma 2.7 to the case, where $V = N^1(X)_{\mathbb{R}}$, $K = \mathrm{Nef}(X)$, and T is the pullback $f^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ or the pullback $(f^{-1})^*$. In fact, the invariant class is constructed as the limit $\lim_{n\to\infty} \frac{1}{n^{\tilde{J}(f,1)}\lambda_1(f)^n}(f^n)^*(H)$ for a general ample H, and so is a good eigenvector. Note in particular that ν_+ and ν_- belong to the cone $\mathrm{Nef}(X)$.

If $h^1(X, \mathcal{O}_X) = 0$, then the map $\mathrm{Pic}(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ is an isomorphism, and we may take $D_\pm$ to be the unique lift of $\nu_\pm$ to a linear equivalence class. ■

We next single out two special properties of hyper-Kähler manifolds and their automorphisms. We will prove Conjecture 1.1 for any automorphisms satisfying these properties, which includes some non-hyper-Kähler examples as well.

Definition 2.10. Suppose that $f : X \to X$ is an automorphism of a normal projective variety X. We say that f has property

(A) if $h^1(X, \mathcal{O}_X) = 0$;

(B) if $\nu = \nu_+ + \nu_-$ is big for some good eigenvector pair (ν_+, ν_-).

Recall that by Remark 2.3, the conjecture is known whenever $\lambda_1(f) = 1$. So there is never any harm in assuming $\lambda_1(f) > 1$. We next observe some easy cases in which Condition (B) holds; a more general criterion is presented in Theorem 2.21.

Lemma 2.11. Let $f : X \to X$ be an automorphism of a normal projective variety X and assume that $\lambda_1(f) > 1$. Then Condition (B) holds in all of the following cases:

1. the dimension of X is equal to 2;
2. the Picard rank of X is equal to 2;
3. X is a hyper-Kähler manifold.

Proof. In dimension 2, Condition (B) is a well-known consequence of the Hodge index theorem (see, e.g., [23, Proposition 2.5]).

Suppose instead that $\rho(X) = 2$. Then ν_+ and ν_- lie on the two boundary rays of $\mathrm{Nef}(X)$, and their sum is ample, which yields Condition (B).

We come at last to the hyper-Kähler case; the argument is the same as that in the 2D setting, but with the Beauville–Bogomolov form standing in for the usual intersection product. Let $\dim X = 2m$ and (ν_+, ν_-) be an eigenvector pair for f, whose existence is guaranteed by Corollary 2.9; define $\nu = \nu_+ + \nu_-$. Since ν is nef, its volume can be computed as the top self-intersection ν^{2k}, and ν is big if and only if this number is positive. We have

$$q(\nu_+) = q(f^*\nu_+) = \lambda_1(f)^2 q(\nu_+),$$

and so $q(\nu_+) = 0$ since $\lambda_1(f) > 1$. The same argument shows that $q(\nu_-) = 0$. Since the form $q_X(-)$ has signature $(1, \rho(X) - 1)$ on $\mathrm{Pic}(X)$, the maximal dimension of an isotropic subspace is 1, and so $q_X(\nu) \neq 0$. Since ν is nef, $\mathrm{Vol}(\nu) = \nu^{2m} = c_X q_X(\nu)^m > 0$, and we conclude that ν is big. ■

Remark 2.12. Notice that if a variety X has Picard rank 2 and an automorphism f with $\lambda_1(f) > 1$, then necessarily $K_X \equiv 0$. Otherwise, K_X would provide a nonzero 1-eigenvector, and since f^* is invertible and preserves the integral lattice $N^1(X) \subset N^1(X)_{\mathbb{R}}$, this would imply that both eigenvalues of f^* are equal to 1. However, there are many interesting examples in this case [44, 54].

We now turn to the proof of the Kawaguchi–Silverman conjecture in this setting. We do by constructing a canonical height function for the automorphism f.

Definition 2.13. Suppose that D is a $\mathbb{Q}$-divisor on a normal projective variety X. The *stable base locus* of D is the Zariski-closed subset of X defined by

$$\mathbf{B}(D) = \bigcap_{\substack{m \geq 1 \\ mD \text{ Cartier}}} \mathrm{Bs}(mD).$$

It is not hard to show that there exists an integer m_0 such that $\mathrm{Bs}(dm_0 D) = \mathbf{B}(D)$ for all sufficiently large integers d [31, Proposition 2.1.21]. It follows that $\mathbf{B}(D + D') \subset \mathbf{B}(D) \cup \mathbf{B}(D')$.

Suppose that D is an $\mathbb{R}$-divisor on a normal projective variety X. The *augmented base locus* $\mathbf{B}_+(D)$ is the Zariski-closed subset

$$\mathbf{B}_+(D) = \bigcap_{\substack{A \text{ ample} \\ D - A\ \mathbb{Q}\text{-divisor}}} \mathbf{B}(D - A).$$

We refer to [16] for a detailed treatment of the properties of the invariant $\mathbf{B}_+(D)$, but single out the following.

Lemma 2.14 ([16, Prop. 1.4, Examples 1.7–1.9, Prop. 1.5]).

1. $\mathbf{B}_+(D)$ depends only on the numerical class of D.
2. $\mathbf{B}_+(D)$ is a proper subset of X if and only if D is big.
3. For any $\mathbb{R}$-divisor D and any real $\lambda > 0$, we have $\mathbf{B}_+(D) = \mathbf{B}_+(\lambda D)$.
4. For any $\mathbb{R}$-divisors D_1 and D_2, we have $\mathbf{B}_+(D_1 + D_2) \subseteq \mathbf{B}_+(D_1) \cup \mathbf{B}_+(D_2)$.
5. Fix a norm $\|\cdot\|$ on $N^1(X)_{\mathbb{R}}$. For any $\mathbb{R}$-divisor D, there exists a constant ϵ such that for any ample $\mathbb{R}$-divisor A for which $\|A\| < \epsilon$ and $D - A$ is a $\mathbb{Q}$-divisor, we have $\mathbf{B}_+(D) = \mathbf{B}(D - A)$.

In view of (1), we sometimes write $\mathbf{B}_+(\nu)$, where ν is any class in $N^1(X)_{\mathbb{R}}$; this denotes $\mathbf{B}_+(D)$ for any D with numerical class ν.

Lemma 2.15. Suppose that D_1 is an $\mathbb{R}$-divisor and D_2 is a nef $\mathbb{R}$-divisor. Then

$$\mathbf{B}_+(D_1 + D_2) \subseteq \mathbf{B}_+(D_1).$$

Proof. First, choose an ample $\mathbb{R}$-divisor A_1 so that $D_1 - A_1$ is a $\mathbb{Q}$-divisor and

$$\mathbf{B}_+(D_1) = \mathbf{B}(D_1 - A_1).$$

Now, choose another ample $\mathbb{R}$-divisor A_2 for which $D_1 + D_2 - A_2$ is a $\mathbb{Q}$-divisor, $A_1 - A_2$ is ample, and

$$\mathbf{B}_+(D_1 + D_2) = \mathbf{B}(D_1 + D_2 - A_2).$$

It again follows from Lemma 2.14(5) that A_2 may be taken to be any sufficiently small ample divisor for which $D_1 + D_2 - A_2$ is a $\mathbb{Q}$-divisor. Note that the divisor $D_2 + (A_1 - A_2) = (D_1 + D_2 - A_2) - (D_1 - A_1)$ is again a $\mathbb{Q}$-divisor, and we may then compute the following:

$$\begin{aligned}\mathbf{B}_+(D_1 + D_2) = \mathbf{B}(D_1 + D_2 - A_2) &= \mathbf{B}\left((D_1 - A_1) + (D_2 + (A_1 - A_2))\right)\\ &\subseteq \mathbf{B}(D_1 - A_1) \cup \mathbf{B}(D_2 + (A_1 - A_2))\\ &= \mathbf{B}_+(D_1) \cup \mathbf{B}(D_2 + (A_1 - A_2)) = \mathbf{B}_+(D_1),\end{aligned}$$

where $\mathbf{B}(D_2 + (A_1 - A_2))$ is empty since D_2 is nef and $A_1 - A_2$ is ample. ■

Lemma 2.16. Suppose that D_1 and D_2 are nef $\mathbb{R}$-divisors. Then for any $a_1, a_2 > 0$, the locus $\mathbf{B}_+(a_1D_1 + a_2D_2)$ is independent of a_1 and a_2.

Proof. We show that for any $a_1, a_2 > 0$, we have $\mathbf{B}_+(a_1D_1 + a_2D_2) = \mathbf{B}_+(D_1 + D_2)$. Suppose first that $a_1 \geq a_2$. Recalling that $\mathbf{B}_+(D) = \mathbf{B}_+(\lambda D)$ according to Lemma 2.14(3), it follows from Lemma 2.15 that

$$\begin{aligned}
\mathbf{B}_+(a_1D_1 + a_2D_2) &= \mathbf{B}_+(a_2(D_1 + D_2) + (a_1 - a_2)D_1) \\
&\subseteq \mathbf{B}_+(a_2(D_1 + D_2)) = \mathbf{B}_+(D_1 + D_2), \\
\mathbf{B}_+(D_1 + D_2) &= \mathbf{B}_+(a_1(D_1 + D_2)) = \mathbf{B}_+((a_1D_1 + a_2D_2) + (a_1 - a_2)D_2) \\
&\subseteq \mathbf{B}_+(a_1D_1 + a_2D_2).
\end{aligned}$$

The case when $a_1 < a_2$ follows from the same argument, reversing the roles of D_1 and D_2. ■

Corollary 2.17. Suppose that $f : X \to X$ is an automorphism of a normal projective variety with $\lambda_1(f) > 1$, and let (ν_+, ν_-) be an eigenvector pair. Then $\mathbf{B}_+(\nu_+ + \nu_-)$ is invariant under f. Furthermore, if f satisfies Condition (B) and P is a $\overline{\mathbb{Q}}$-point of X with a Zariski dense orbit under f, then P is not contained in $\mathbf{B}_+(\nu_+ + \nu_-)$.

Proof. We have

$$\begin{aligned}
f\Big(\mathbf{B}_+(\nu_+ + \nu_-)\Big) &= \mathbf{B}_+\Big((f^{-1})^*(\nu_+ + \nu_-)\Big) \\
&= \mathbf{B}_+\Big(\lambda_1(f)^{-1}\nu_+ + \lambda_1(f^{-1})\nu_-\Big) = \mathbf{B}_+(\nu_+ + \nu_-),
\end{aligned}$$

where the final equality follows from Lemma 2.16.

If f satisfies Condition (B), then $\nu = \nu_+ + \nu_-$ is big, and so $\mathbf{B}_+(\nu)$ is a proper Zariski-closed subset of X, invariant under f. It follows that a point with a dense orbit cannot lie in $\mathbf{B}_+(\nu)$. ■

In fact, it is possible to give a more explicit description of the locus $\mathbf{B}_+(\nu)$ in terms of the dynamics of the map f. This relies on the following difficult result.

Theorem 2.18 (Nakamaye's theorem [17, Corollary 5.6]). Suppose that D is a nef $\mathbb{R}$-divisor. Then

$$\mathbf{B}_+(D) = \mathrm{Null}(D) = \bigcup_{\substack{V \subset X \\ D|_V \text{ is not big}}} V,$$

where the union runs over all positive dimensional subvarieties $V \subset X$.

Lemma 2.19. Suppose that $f : X \to X$ is an automorphism satisfying $\lambda_1(f) > 1$ and let (ν_+, ν_-) be a good eigenvector pair with $\nu = \nu_+ + \nu_-$. If $V \subseteq X$ is an f-periodic subvariety of dimension at least 1, then V is not contained in $\mathbf{B}_+(\nu)$ if and only if there exists $0 < a < \dim V$ such that we simultaneously have

$$\lambda_a(f|_V)^2 = \lambda_1(f)^a \, \lambda_1(f^{-1})^{\dim V - a},$$

and

$$\tilde{J}(f|_V, a) = a\tilde{J}(f, 1) + (\dim V - a)\tilde{J}(f^{-1}, 1).$$

Proof. Let $i\colon V \to X$ be the inclusion map. According to Nakamaye's theorem, V is not contained in $\mathbf{B}_+(\nu)$ if and only if $i^*\nu$ is big, which is equivalent to $(\nu_+ + \nu_-)^{\dim V} \cdot V > 0$. We may assume that V is f-invariant, since the statement is unaffected when replacing f by an iterate. Since (ν_+, ν_-) is a good eigenvector pair, we pick a suitable ample class H and compute the following:

$$\begin{aligned}
&(\nu_+ + \nu_-)^{\dim V} \cdot V \\
&= \lim_{n\to\infty} \left(\frac{1}{n^{\tilde{J}(f,1)}\lambda_1(f)^n}(f^*)^n H + \frac{1}{n^{\tilde{J}(f^{-1},1)}\lambda_1(f^{-1})^n}(f^*)^{-n} H \right)^{\dim V} \cdot V \\
&= \lim_{n\to\infty} \sum_{a=0}^{\dim V} \binom{\dim V}{a} \frac{1}{n^{a\tilde{J}(f,1)+(\dim V - a)\tilde{J}(f^{-1},1)}\lambda_1(f)^{na}\lambda_1(f^{-1})^{n(\dim V - a)}} \\
&\quad \left(((f^*)^n H)^a \cdot ((f^*)^{-n} H)^{\dim V - a} \right) \cdot V \\
&= \lim_{n\to\infty} \sum_{a=0}^{\dim V} \binom{\dim V}{a} \frac{1}{n^{a\tilde{J}(f,1)+(\dim V - a)\tilde{J}(f^{-1},1)}\lambda_1(f)^{na}\lambda_1(f^{-1})^{n(\dim V - a)}} \\
&\quad \left(((f^*)^{2n} H)^a \cdot H^{n(\dim V - a)} \cdot V \right).
\end{aligned}$$

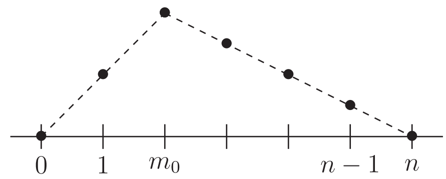


Fig. 1. The function h.

The quantity $((f^*)^{2n}H)^a \cdot H^{\dim V - a} \cdot V$ is bounded both below and above by positive multiples of $n^{\tilde{J}(f^2|_V, a)}\lambda_a(f^2|_V)^n = n^{\tilde{J}(f|_V,a)}\left(\lambda_a(f|_V)^2\right)^n$, and so the final expression is greater than 0 if and only if there is a value of a for which

$$\lambda_a(f|_V)^2 = \lambda_1(f)^a\,\lambda_1(f^{-1})^{\dim V - a}$$
$$\tilde{J}(f|_V, a) = a\tilde{J}(f,1) + (\dim V - a)\tilde{J}(f^{-1},1).$$

■

Used in combination with the log concavity of dynamical degrees, this provides a useful criterion for computing $\mathbf{B}_+(\nu)$, based on the following easy observation.

Lemma 2.20. Suppose that $h : \{0,\ldots,n\} \to \mathbb{R}_{\geq 0}$ is a concave function with $h(0) = 0$ and $h(n) = 0$. Let $h(1) = a > 0$ and $h(n-1) = b > 0$. Then for any $m \in \{0,\ldots,n\}$, we have $h(m) \leq am$ and $h(m) \leq b(n-m)$, so that $2h(m) \leq am + b(n-m)$.

Equality is possible only if $m_0 = \left(\frac{b}{a+b}\right)n$ is an integer and $h(m) = am$ for $0 \leq m \leq m_0$ and $h(m) = b(n-m)$ for $m_0 \leq m \leq n$. In this case, $2h(m_0) = am_0 + b(n-m_0)$.

Theorem 2.21. Suppose that $f : X \to X$ is an automorphism with $\lambda_1(f) > 1$ and good eigenvector pair (ν_+, ν_-), and let $\nu = \nu_+ + \nu_-$. Then we have the following:

1. ν is big if and only if

$$a = \left(\frac{\log\lambda_1(f^{-1})}{\log\lambda_1(f) + \log\lambda_1(f^{-1})}\right)(\dim X)$$

   is an integer, $\lambda_j(f) = \lambda_1(f)^j$ for $1 \leq j \leq a$, and $\lambda_j(f) = \lambda_1(f^{-1})^{\dim X - j}$ for $a \leq j \leq \dim X$.
2. Suppose that $\lambda_1(f) = \lambda_1(f^{-1})$. Then ν is big if and only if $\dim X = 2m$ is even, $\lambda_j(f) = \lambda_1(f)^j$ for $0 \leq j \leq m$, and $\lambda_j(f) = \lambda_1(f)^{2m-j}$ for $m \leq j \leq 2m$.

3. Suppose that ν is big and $V \subset X$ is f-periodic with $\dim V \geq 1$. Then V is not contained in $\mathbf{B}_+(\nu)$ if and only if there exists an integer a such that $0 < a < \dim V$ and all of the following hold:
   1. $\lambda_1(f|_V) > 1$ and $a = \left(\frac{\log \lambda_1(f|_V^{-1})}{\log \lambda_1(f|_V) + \log \lambda_1(f|_V^{-1})}\right)(\dim V)$;
   2. $\lambda_j(f|_V) = \lambda_1(f|_V)^j$ for $1 \leq j \leq a$, and $\lambda_j(f|_V) = \lambda_1(f|_V^{-1})^{\dim X - j}$ for $a \leq j \leq \dim X$;
   3. $\lambda_1(f|_V) = \lambda_1(f)$ and $\lambda_1(f^{-1}|_V) = \lambda_1(f^{-1})$;
   4. $\tilde{J}(f|_V, a) = a\tilde{J}(f, 1) + (\dim V - a)\tilde{J}(f^{-1}, 1)$.

Proof. Lemma 2.20 with $h(m) = \log \lambda_m(f)$ shows that $\lambda_a(f)^2 \leq \lambda_1(f)^a \lambda_1(f^{-1})^{\dim X - a}$ for any a. Since ν is not big if and only if $X = \mathbf{B}_+(\nu)$, parts (1) and (2) follow immediately from Lemma 2.19 and the condition for equality in Lemma 2.20.

We now turn to the proof of (3). We know from Lemma 2.19 that V is not contained in $\mathbf{B}_+(\nu)$ if and only if there exists $0 < a < \dim V$ such that (d) and $\lambda_a(f|_V)^2 = \lambda_1(f)^a \lambda_1(f^{-1})^{\dim V - a}$ both hold. Observe that

$$\lambda_a(f|_V)^2 \leq \lambda_1(f|_V)^a \lambda_1(f|_V^{-1})^{\dim V - a} \leq \lambda_1(f)^a \lambda_1(f^{-1})^{\dim V - a},$$

where the 1st inequality is an equality if and only if (a) and (b) hold, while the 2nd is an equality if and only if (c) holds. ■

Remark 2.22. In the case $\dim X = 2$, it follows from [23, Proposition 3.1(2)] that ν is big and the locus $\mathbf{B}_+(\nu)$ is precisely the union of the f-invariant curves. If $\dim X = 3$, then ν is big if and only if either $\lambda_1(f) = \lambda_2(f)^2$ or $\lambda_1(f)^2 = \lambda_2(f)$. If $\rho(X) = 2$, then ν is necessarily ample and so $\mathbf{B}_+(\nu)$ is empty.

Corollary 2.23.

1. Suppose that $f : X \to X$ is an automorphism satisfying Condition (B) and for which $\lambda_1(f) = \lambda_1(f^{-1})$ (e.g., f is an automorphism of a hyper-Kähler manifold). Then the union of the odd-dimensional f-periodic subvarieties of X is not Zariski dense.
2. Suppose that $f : X \to X$ is an automorphism of a threefold satisfying either $\lambda_1(f) = \lambda_2(f)^2$ or $\lambda_2(f) = \lambda_1(f)^2$. Then f admits only finitely many positive-dimensional periodic subvarieties.

Proof. The 1st claim is immediate from Theorem 2.21(3), since Condition (3a) cannot hold for an odd-dimensional subvariety. In particular any such subvariety is contained in the Zariski closed subset $\mathbf{B}_+(\nu) \subset X$.

For the 2nd, note that the condition on the dynamical degrees implies that f satisfies Condition (B). The ratio in part (3a) of Theorem 2.21 is $a = \frac{1}{3}$ or $a = \frac{2}{3}$ depending on which of the hypotheses holds. Either way, we conclude that any f-periodic subvariety is contained in $\mathbf{B}_+(\nu)$. In particular, there are only finitely many f-periodic surfaces in X. An f-periodic curve is either a component of $\mathbf{B}_+(\nu)$ or contained in one of these surfaces. Since a surface automorphism with $\lambda_1(f) > 1$ has only finitely many periodic curves, we conclude that the number of periodic curves of either type is finite, completing the proof. ■

Example 2.24 ([34, Example 5.2]). Suppose that $f : S \to S$ is an automorphism of a K3 surface with $\lambda_1(f) > 1$. Let $X = \mathrm{Hilb}^n(S)$ be the corresponding Hilbert scheme of n points on S. There is an induced automorphism $f^{[n]} : X \to X$, and $\lambda_1(f^{[n]}) = \lambda_1(f)$.

The f-periodic points p on S are Zariski dense [8], giving rise to f-periodic subvarieties V on X of any even codimension, as the images of $p \times \cdots \times p \times S \times \cdots \times S$ in X. These $f^{[n]}$-periodic subvarieties are Zariski dense, but they satisfy all four conditions of (3) of Theorem 2.21 and so are not contained in $\mathbf{B}_+(\nu)$. In particular, these subvarieties have even dimension.

If $f : S \to S$ has an invariant curve C, then the image $E \subset \mathrm{Hilb}^n(S)$ of the divisor $S \times S \times \cdots \times S \times C$ is contained in $\mathbf{B}_+(\nu)$: Conditions (a) and (b) of Theorem 2.21(3) both fail. Note that $\lambda_1(f|_E) = \lambda_1(f)$ here, but there is a smaller subvariety $V \subset E$ given as the image of $C \times \cdots \times C$ for which $\lambda_1(f^{[n]}|_V) = 1$.

Proposition 2.25 (The Weil height machine, e.g., [22, Theorem B.3.6]). Let X be a projective variety defined over $\overline{\mathbb{Q}}$. There exists a unique map

$$\mathrm{Pic}(X)_{\mathbb{R}} \to \frac{\left\{\text{functions } X(\overline{\mathbb{Q}}) \to \mathbb{R}\right\}}{\left\{\text{bounded functions } X(\overline{\mathbb{Q}}) \to \mathbb{R}\right\}}$$

with the following properties:

1. Normalization: if D is very ample, $\phi_D : X \to \mathbb{P}^n$ is the associated embedding, and h is the absolute logarithmic height [22, §B.2], then $h_D(P) = h(\phi_D(P)) + O(1)$.
2. Functoriality: if $\pi : X \to Y$ is a morphism, then $h_{X,\pi^*D}(P) = h_{Y,D}(\pi(P)) + O(1)$.

3. Additivity: $h_{X,D_1+D_2}(P) = h_{X,D_1}(P) + h_{X,D_2}(P) + O(1)$
4. Positivity: If D is effective, then $h_{X,D}(P) \geq O(1)$ for P outside the base locus of D.

By *a height function* for an $\mathbb{R}$-divisor class D, we mean a function $h_D : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ belonging to the class of height functions for D.

The augmented base locus is well suited to working with height functions associated with big $\mathbb{R}$-divisors. The next two lemmas give extensions of the positivity property and Northcott's lemma to this setting.

Lemma 2.26. Let X be a normal, projective variety over $\overline{\mathbb{Q}}$.

1. Suppose that D is a $\mathbb{Q}$-divisor. Then $h_{X,D}(P) \geq O(1)$ for P outside $\mathbf{B}(D)$.
2. Suppose that D is a $\mathbb{R}$-divisor. Then $h_{X,D}(P) \geq O(1)$ for P outside $\mathbf{B}_+(D)$.
3. Suppose that D is a big $\mathbb{R}$-divisor on X. Then for any M and N, there are only finitely many points P of $X(\overline{\mathbb{Q}}) \smallsetminus \mathbf{B}_+(D)$ with $[\mathbb{Q}(P) : \mathbb{Q}] < M$ and $h_D(P) < N$.

Proof. Fix an integer m with $\mathrm{Bs}(mD) = \mathbf{B}(D)$; then $h_{X,mD} = m\,h_{X,D} + O(1)$ according to the additivity property, and (1) follows.

For (2), according to Lemma 2.14(5) there exists an ample $\mathbb{R}$-divisor A so that $D - A$ is a $\mathbb{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - A)$. According to (1), we have $h_{D-A}(P) \geq O(1)$ for P outside $\mathbf{B}(D - A) = \mathbf{B}_+(D)$. Since $h_D = h_{D-A} + h_A + O(1)$, and since $h_A \geq O(1)$, this proves (2).

At last we prove (3); let D and A be as before. There is a constant C_1 such that $h_A(P) \leq h_D(P) - h_{D-A}(P) + C_1$ for all points P of $X(\overline{\mathbb{Q}})$. By (1), there is a constant C_2 so that $h_{D-A}(P) \geq C_2$ for any P in $X(\overline{\mathbb{Q}}) \smallsetminus \mathbf{B}(D - A) = X(\overline{\mathbb{Q}}) \smallsetminus \mathbf{B}_+(D)$. Now, if P is a point of $X(\overline{\mathbb{Q}}) \smallsetminus \mathbf{B}_+(D)$ with $h_D(P) < N$, we have

$$h_A(P) \leq h_D(P) - h_{D-A}(P) + C_1 \leq h_D(P) + C_1 - C_2 \leq N + C_1 - C_2.$$

It then follows from the Northcott theorem for the ample divisor A that there are only finitely many such P with $[\mathbb{Q}(P) : \mathbb{Q}] < M$ and $h_D(P) < N$, see Theorem B.3.2(g) and Remark B.3.2.1(i) of [22]. ■

With these results in place, we now construct a canonical height function for an automorphism satisfying Condition (B) and that admits an eigendivisor pair. Suppose that $f : X \to X$ is an automorphism of a normal projective variety satisfying these conditions, with (D_+, D_-) an eigendivisor pair for f. Define functions $\widehat{h}_{D_+} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ and $\widehat{h}_{D_-} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ by

$$\widehat{h}_{D_+}(P) = \lim_{n\to\infty} \frac{1}{\lambda_1(f)^n} h_{D_+}(f^n(P))$$
$$\widehat{h}_{D_-}(P) = \lim_{n\to\infty} \frac{1}{\lambda_1(f^{-1})^n} h_{D_-}(f^{-n}(P)).$$

The functoriality of the height function yields $h_{D_\pm}(P) - \lambda_1(f^{\pm 1})^{-1} h_{D_\pm}(f(P)) = O(1)$; it follows from an argument of Tate (cf. [47, §3]) that both of these limits exist and that $\widehat{h}_{D_\pm}$ is a height function for $D_\pm$. These functions furthermore satisfy the relations

$$\widehat{h}_{D_+}(f(P)) = \lambda_1(f)\widehat{h}_{D_+}(P), \quad \widehat{h}_{D_-}(f(P)) = \lambda_1(f^{-1})^{-1}\widehat{h}_{D_-}(P),$$

with no $O(1)$ term. Consider the function $\widehat{h}(P) : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ given by

$$\widehat{h}(P) = \widehat{h}_{D_+}(P) + \widehat{h}_{D_-}(P).$$

We next develop the properties of $\widehat{h}(P)$, closely following arguments of Kawaguchi [23, Theorem 5.2 and Proposition 5.5].

Theorem 2.27. Let X be a normal projective variety over $\overline{\mathbb{Q}}$. Let f be an automorphism of X with $\lambda_1(f) > 1$ and satisfying Condition (B), and suppose that f admits an eigendivisor pair (D_+, D_-). If $P \in X(\overline{\mathbb{Q}})$, then the function $\widehat{h}$ has the following properties:

1. $\widehat{h}$ is a height function for the big and nef divisor $D = D_+ + D_-$;
2. if $P \in X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$, then $\widehat{h}(P) \geq 0$, $\widehat{h}_{D_+}(P) \geq 0$, and $\widehat{h}_{D_-}(P) \geq 0$;
3. $\widehat{h}$ satisfies the Northcott property on $X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$;
4. If $P \in X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$, then $\widehat{h}_{D_+}(P) = 0$ if and only if $\widehat{h}_{D_-}(P) = 0$ if and only if $\widehat{h}(P) = 0$ if and only if P is f-periodic.

Proof. Part (1) is immediate from the fact that $\widehat{h}_{D_+}$ and $\widehat{h}_{D_-}$ are height functions for D_+ and D_-. Since D is big, it follows from Lemma 2.26(2) that there is a constant C such that $\widehat{h}(P) > C$ for any P not contained in $\mathbf{B}_+(D)$. Since $\mathbf{B}_+(D)$ is f-invariant by Corollary 2.17, if P is not contained in $\mathbf{B}_+(D)$ then neither is $f^n(P)$ for any integer n. Then

$$\begin{aligned}\widehat{h}(f^n(P)) + \widehat{h}(f^{-n}(P)) &= \widehat{h}_{D_+}(f^n(P)) + \widehat{h}_{D_-}(f^n(P)) + \widehat{h}_{D_+}(f^{-n}(P)) + \widehat{h}_{D_-}(f^{-n}(P)) \\ &= \lambda_1(f)^n\widehat{h}_{D_+}(P) + \lambda_1(f^{-1})^{-n}\widehat{h}_{D_-}(P) + \lambda_1(f)^{-n}\widehat{h}_{D_+}(P) + \lambda_1(f^{-1})^n\widehat{h}_{D_-}(P) \\ &= \left(\lambda_1(f)^n + \lambda_1(f)^{-n}\right)\widehat{h}_{D_+}(P) + \left(\lambda_1(f^{-1})^n + \lambda_1(f^{-1})^{-n}\right)\widehat{h}_{D_-}(P).\end{aligned}$$

The left side is bounded below by $2C$, and so we obtain

$$2C \leq \left(\lambda_1(f)^n + \lambda_1(f)^{-n}\right)\widehat{h}_{D_+}(P) + \left(\lambda_1(f^{-1})^n + \lambda_1(f^{-1})^{-n}\right)\widehat{h}_{D_-}(P)$$

and dividing by $\lambda_1(f)^n + \lambda_1(f)^{-n}$ yields

$$\frac{2C}{\lambda_1(f)^n + \lambda_1(f)^{-n}} \leq \widehat{h}_{D_+}(P) + \left(\frac{\lambda_1(f^{-1})^n + \lambda_1(f^{-1})^{-n}}{\lambda_1(f)^n + \lambda_1(f)^{-n}}\right)\widehat{h}_{D_-}(P) \qquad *$$

The analysis now depends on the relative sizes of $\lambda_1(f)$ and $\lambda_1(f^{-1})$. Suppose first that $\lambda_1(f) > \lambda_1(f^{-1})$; since the lemma is symmetric in f and f^{-1}, the case with the inequality reversed follows from the same argument. Taking the limit of equation $(*)$ as n tends to infinity, we obtain $\widehat{h}_{D_+}(P) \geq 0$. We have

$$\widehat{h}(f^{-n}(P)) = \lambda_1(f)^{-n}\widehat{h}_{D_+}(P) + \lambda_1(f^{-1})^n\widehat{h}_{D_-}(P).$$

The left side is again bounded below by C, and so

$$\left(\lambda_1(f)^{-n}\lambda_1(f^{-1})^{-n}\right)\widehat{h}_{D_+}(P) + \widehat{h}_{D_-}(P) \geq C\lambda_1(f^{-1})^{-n}.$$

Again taking the limit as n goes to infinity, we obtain $\widehat{h}_{D_-}(P) \geq 0$, and so $\widehat{h}(P) = \widehat{h}_{D_+}(P) + \widehat{h}_{D_-}(P) \geq 0$.

Suppose instead that $\lambda = \lambda_1(f) = \lambda_1(f^{-1})$. Taking the limit in equation $(*)$ then yields $0 \leq \widehat{h}_{D_+}(P) + \widehat{h}_{D_-}(P) = \widehat{h}(P)$. It follows that $\widehat{h}_{D_+}(P) \geq -\widehat{h}_{D_-}(P)$, and so

$$\widehat{h}_{D_+}(P) = \lambda^{-n}\widehat{h}_{D_+}(f^n(P)) \geq -\lambda^{-n}\widehat{h}_{D_-}(f^n(P)) = -\lambda^{-2n}\widehat{h}_{D_-}(P).$$

The non-negativity of $\widehat{h}_{D_+}(P)$ follows by taking the limit as n tends to infinity; non-negativity of $\widehat{h}_{D_-}(P)$ follows from a similar argument. This handles (2).

Lemma 2.26(3) combined with property (1) immediately implies (3).

We now turn to (4). First, if P is f-periodic, then $f^n(P) = P$ for some n, which implies directly from the definitions that $\widehat{h}_{D_+}(P)$ and $\widehat{h}_{D_-}(P)$ both vanish, and hence $\widehat{h}(P) = 0$. On the other hand, suppose that $\widehat{h}(P) = 0$ for some P in $X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$. By (2), it must be that $\widehat{h}_{D_+}(P) = 0$ and $\widehat{h}_{D_-}(P) = 0$. Then $\widehat{h}_{D_+}(f^n(P)) = 0$ and $\widehat{h}_{D_-}(f^n(P)) = 0$ for any integer n as well, so that $\widehat{h}(f^n(P)) = 0$. By Corollary 2.17, the locus $\mathbf{B}_+(D)$ is f-invariant, so $\{f^n(P) \mid n \in \mathbb{Z}\}$ is contained in $X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$. Since the $f^n(P)$ are of bounded degree over $\mathbb{Q}$ and all have height 0, the Northcott property (3) tells us the set of $f^n(P)$ is finite, and so P is f-periodic.

To finish the proof of (4), it remains to show that if $P \in X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$ and $\widehat{h}_{D_+}(P) = 0$, then P is f-periodic; the assertion that $\widehat{h}_{D_-}(P) = 0$ implies P is f-periodic will follow similarly. If $\widehat{h}_{D_+}(P) = 0$ and $n > 0$ then, letting $\lambda = \lambda_1(f)$, we have $\widehat{h}(f^n(P)) = \widehat{h}_{D_+}(f^n(P)) + \widehat{h}_{D_-}(f^n(P)) = \lambda^n \widehat{h}_{D_+}(P) + \lambda^{-n}\widehat{h}_{D_-}(P) = \lambda^{-n}\widehat{h}_{D_-}(P) \leq \widehat{h}_{D_-}(P)$. Since we have fixed our point P, we can view $\widehat{h}_{D_-}(P)$ as a constant and we have bounded $\widehat{h}(f^n(P))$ for all n. Again, since the set $\{f^n(P)\}$ has a bounded degree and is contained in $X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$, it follows from (3) that $\{f^n(P)\}$ is finite, and so P is f-periodic. ■

Theorem 2.28. Suppose that $f : X \to X$ is an automorphism of a normal projective variety satisfying Conditions (A) and (B). Then the Kawaguchi–Silverman conjecture holds for f.

Proof. Recall that we can always assume $\lambda_1(f) > 1$. Then by Corollary 2.9, there exists an eigendivisor pair (D_+, D_-), with $D = D_+ + D_-$ big. Let $P \in X(\overline{\mathbb{Q}})$ have a dense orbit under f. By Corollary 2.17, we know P does not lie in $\mathbf{B}_+(D)$. Since P is not f-periodic, we know from Theorem 2.27(2) and (4) that $\widehat{h}_{D_+}(P)$ and $\widehat{h}_{D_-}(P)$ are both strictly positive. Then

$$\alpha_f(P) \geq \liminf_{n\to\infty} h_D^+(f^n(P))^{1/n} = \liminf_{n\to\infty} (\lambda_1(f)^n \widehat{h}_{D_+}(P) + \lambda_1(f)^{-n}\widehat{h}_{D_-}(P))^{1/n} = \lambda_1(f),$$

where the inequality follows from [37, Remark 2.2] and the next equality from Theorem 2.27(1), which tells us that $h_D = \widehat{h} + O(1)$. ■

Remark 2.29. In fact, the proof of Theorem 2.28 shows that $\alpha_f(P) = \lambda_1(f)$ for any P in $X(\overline{\mathbb{Q}}) \setminus \mathbf{B}_+(D)$; it is not necessary to assume that P has a dense orbit.

It follows from Lemma 2.11 and Theorem 2.28 that the Kawaguchi–Silverman conjecture holds for automorphisms of hyper-Kähler manifolds; note that Condition (A) holds since a hyper-Kähler manifold is geometrically simply connected. The next lemma shows that it also holds for automorphisms of smooth varieties of Picard rank 2, slightly extending [46, Theorem 4.2(ii)].

Theorem 2.30. Suppose that X is a smooth projective variety, $\rho(X) = 2$, and $f : X \to X$ is an automorphism. Then the Kawaguchi–Silverman conjecture holds for f.

Proof. The map ϕ induces an automorphism $g : \mathrm{Alb}(X) \to \mathrm{Alb}(X)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ \big\downarrow a & & \big\downarrow a \\ \mathrm{Alb}(X) & \xrightarrow{\ g\ } & \mathrm{Alb}(X) \end{array}$$

commutes. The proof of Lemma 2.11 shows that neither eigenvalue of $f^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ is equal to 1, and so it must be that $K_X \equiv 0$. A form of abundance due to Nakayama [39] implies that K_X is torsion in $\mathrm{Pic}(X)$, so that $\kappa(X) = 0$. Since $\kappa(X) = 0$, a result of Kawamata (independent of the conjectures of the minimal model program (MMP)) implies that a is surjective with connected fibers [27].

If $\dim \mathrm{Alb}(X) = 0$, then $h^1(X, \mathcal{O}_X) = 0$, so that Condition (A) is satisfied. In this case, Conjecture 1.1 follows from Theorem 2.28. If $\dim \mathrm{Alb}(X) = \dim X$, then a is generically finite, and it must be birational since a has connected fibers. Then X is birational to an abelian variety, f descends to an automorphism of the abelian variety, and the conjecture holds by [48].

Suppose at last that a is not finite and that $\dim \mathrm{Alb}(X) > 0$. It must be that $\rho(X) \geq \rho(\mathrm{Alb}(X)) + 1$, since for any divisor D on Y, $\pi^* D$ has intersection 0 with a curve in the fiber of a. Since $\rho(X) = 2$, we have $\rho(\mathrm{Alb}(X)) = 1$. Taking H to be a generator of $\mathrm{Pic}(\mathrm{Alb}(X))$, it must be that $a^* H$ is a 1-eigenvector for ϕ^*, but neither eigenvalue of ϕ^* is equal to 1, so this case is impossible. ■

Notice that if $\rho(X) = 2$ and $h^1(X, \mathcal{O}_X) = 0$, we have proved something even stronger; since $D = D_+ + D_-$ is ample, $\mathbf{B}_+(D) = \varnothing$, and so $\alpha_f(P) = \lambda_1(f)$ for every non-periodic $\overline{\mathbb{Q}}$-point P, without assuming the orbit is Zariski-dense.

Lemma 2.31. Let $f : X \to X$ be an automorphism of a normal projective variety satisfying Condition (B), and let (ν_+, ν_-) be an eigenvector pair. If $i : V \to X$ is the inclusion of an f-periodic subvariety with $\dim V \geq 1$ and $\lambda_1(f|_V) < \lambda_1(f)$, then $i^* \nu_+ = 0$.

Proof. The pair (ν_+, ν_-) is also an eigenvector pair for f^n, so without loss of generality we can assume that $n = 1$, that is, V is fixed by f. Let $i : V \to X$ be the inclusion. Then $(f|_V)^*(i^*\nu_+) = i^*f^*\nu_+ = i^*(\lambda_1(f)\nu_+) = \lambda_1(f)(i^*\nu_+)$, so that $i^*\nu_+$ is a $\lambda_1(f)$-eigenvector for $(f|_V)^*$. Since the spectral radius of $(f|_V)^*$ is $\lambda_1(f|_V) < \lambda_1(f)$, this is impossible unless $i^*\nu_+ = 0$. ■

Definition 2.32. Suppose that $f : X \to X$ is an automorphism of a normal projective variety. Then $E(f)$ is the subset of X defined by

$$E(f) = \bigcup \left\{ V : \dim V \geq 1,\ V \text{ is } f\text{-periodic, } \lambda_1(f|_V) < \lambda_1(f), \text{ and } \lambda_1(f^{-1}|_V) < \lambda_1(f^{-1}) \right\}.$$

Theorem 2.33. Suppose that X is a normal projective variety and that $f : X \to X$ is an automorphism satisfying Condition (B). Then $E(f)$ is not Zariski dense in X.

Proof. It follows from Theorem 2.21(3c) that $E(f) \subset \mathbf{B}_+(\nu)$, where $\nu = \nu_+ + \nu_-$ for any good eigenvector pair. Since ν is big, $\mathbf{B}_+(\nu)$ is a proper Zariski-closed subset of X, and the claim follows. ■

Example 2.34. Let $g : S \to S$ be an automorphism of a K3 surface satisfying $\lambda_1(g) > 1$, and let $f = g \times \mathrm{id} : S \times \mathbb{P}^1 \to S \times \mathbb{P}^1$, which satisfies $\lambda_1(f) = \lambda_1(g)$. If p is any periodic point of g, then $V = p \times \mathbb{P}^1$ is f-periodic, and satisfies $\lambda_1(f|_V) = 1$, so that $V \subset E(f)$. Since the g-periodic points are dense in S (e.g., by [52, Theorem 1.2]), the set $E(f)$ is Zariski dense. However, f does not satisfy Condition (B), so Theorem 2.33 is not applicable.

The next result (which applies in particular in the hyper-Kähler case) is a higher-dimensional analog of a result of Cantat and Kawaguchi [9, Proposition 4.1], the latter having been shown for surfaces.

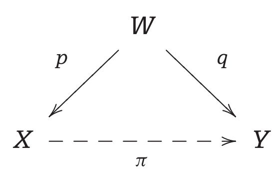
Proposition 2.35. Suppose that X has Kawamata log terminal (klt) singularities (e.g., that X is smooth), that $K_X \equiv 0$, and that $f : X \to X$ is an automorphism satisfying $\lambda_1(f) > 1$ and Condition (B). Then there exists a birational morphism $\pi : X \to Y$ such that f descends to an automorphism $g : Y \to Y$, and π contracts every connected component of $E(f)$ to a point.

Proof. Since f satisfies Condition (B), there is an eigenvector pair (ν_+, ν_-) with $\nu = \nu_+ + \nu_-$ big. When ν is represented by a $\mathbb{Q}$-divisor D, the claim follows quickly from Kawamata's basepoint-free theorem; D is semi-ample, and we take π to be the corresponding contraction. However, since ν does not typically have a $\mathbb{Q}$-divisor representative, we must resort to other methods, and we realize Y as the log canonical model of a klt pair (X, Δ) with $\Delta \equiv \epsilon\nu$.

Since ν is big, we may find $\epsilon > 0$ and an effective $\mathbb{R}$-divisor $\Delta \equiv \epsilon\nu$ such that (X, Δ) is klt [30, Corollary 2.35]. Note that $K_X + \Delta = \Delta$ is nef. It follows from [6] that there exists a log canonical model $\pi : X \dashrightarrow Y$ for the pair (X, Δ), which means that

1. π is a birational contraction (i.e., π is birational and π^{-1} does not contract any divisors);
2. π is $(K_X + \Delta)$-negative (in the sense of [6]);
3. taking $\Gamma = \pi_*\Delta$, we have $K_Y + \Gamma$ ample.

We argue now that if $K_X + \Delta$ is big and nef, the map π is in fact a morphism (a standard fact, for which we do not know a convenient reference). Take a resolution of the rational map π:



Since π is $(K_X + \Delta)$-negative, we have $p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E$, with $E \geq 0$. It follows from [39] that

$$E = N_\sigma(q^*(K_Y + \Gamma) + E) = N_\sigma(p^*(K_X + \Delta)) = 0,$$

and so $p^*(K_X + \Delta) = q^*(K_Y + \Gamma)$. It then follows from [6, 3.6.6(2)] that π is a morphism, and that $K_X + \Delta = \pi^*A$, where A is ample. Since $K_X \equiv 0$ by assumption, this means that $\epsilon\nu \equiv \pi^*A$.

Suppose that V is an irreducible component of $E(f)$. Letting $i\colon V \to X$ be the inclusion map, Lemma 2.31 applied to f and f^{-1} shows that $i^*\nu_+ = i^*\nu_- = 0$, and so $i^*\nu = 0$. Since $D = \pi^*A$, it follows that all such subvarieties V are contracted to points by π.

It remains to check that f induces an automorphism $g : Y \to Y$. We claim first that every subvariety contracted to a point by π is also contracted by $\pi \circ f$. The varieties contracted by π are precisely those V for which $i^*\nu = 0$, where $i\colon V \to X$ is the inclusion map. Since ν_+ and ν_- are nef, this is possible only if $i^*\nu_+ = i^*\nu_- = 0$. The varieties contracted by $f \circ \pi$ are those on which $(f^{-1})^*(\nu) = \lambda_1(f)^{-1}\nu_+ + \lambda_1(f^{-1})^{-1}\nu_-$ restricts to 0, which is the same set of varieties.

The map $\pi : X \to Y$ is birational with Y normal and so satisfies $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ by Zariski's main theorem, and since $f \circ \pi$ contracts every fiber of π, it follows from the rigidity lemma [11, Lemma 1.15(b)] that it factors through π. This yields a map $g : S \to S$ with $f \circ \pi = \pi \circ g$. An inverse to g is obtained by applying the same argument to f^{-1}. ■

As a consequence of Theorem 1.2, we reduce Conjecture 1.1 for automorphisms of smooth varieties X with $K_X \equiv 0$ to the case of Calabi–Yau varieties. This is done in Corollary 1.5.

Proof of Corollary 1.5 Let X be a smooth projective $\overline{\mathbb{Q}}$-variety with numerically trivial canonical class, and $f\colon X \to X$ an automorphism. By [4, Proposition 3.1], there is an abelian variety A, Calabi–Yau varieties Y_i, and hyper-Kähler manifolds Z_j all defined over $\overline{\mathbb{Q}}$, and there is a finite étale cover $\pi : \widetilde{X} \to X$, where $\widetilde{X} = A \times \prod_i Y_i \times \prod_j Z_j$. Applying Condition (3) of [4, Proposition 3.1] to $f \circ \pi$ yields a map $\widetilde{f}$ making the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array}$$

commute. Since π is finite étale, by degree considerations, we see $\widetilde{f}$ is an automorphism. By [37, Lemma 3.2], the conjecture for f follows from that of $\widetilde{f}$, so we may assume X itself is a product $A \times \prod_i Y_i \times \prod_j Z_j$ as above.

Recall that Conjecture 1.1 holds for f if and only if it holds for an iterate of f. Since the Y_i and Z_j are simply connected, their 1st Betti numbers are trivial, so after possibly replacing f by an iterate, we may assume by Theorem 4.6 and Lemma 5.1 of [45] that $f = f_0 \times \prod_i g_i \times \prod_j h_j$ with f_0 an endomorphism of A, g_i an endomorphism of Y_i, and h_j an endomorphism of Z_j. Applying the same argument to f^{-1}, we may assume $f^{-1} = f_0' \times \prod_i g_i' \times \prod_j h_j'$. Since $\mathrm{id} = ff^{-1} = f_0 f_0' \times \prod_i g_i g_i' \times \prod_j h_j h_j'$, it follows that $f_0^{-1} = f_0'$, $g_i^{-1} = g_i'$, and $h_j^{-1} = h_j'$; so, f_0, g_i, and h_j are all automorphisms.

By [45, Lemma 3.2], the conjecture for f then follows from the conjecture for f_0, g_i, and h_j. Conjecture 1.1 is known for abelian varieties by [48], and we proved in Theorem 1.2 that the conjecture holds for hyper-Kähler manifolds. Thus, Conjecture 1.1 for f is reduced to that of each g_i, that is, automorphisms of Calabi–Yau varieties of dimension at most n. ■

3 Interplay Between Conjecture 1.1, Fibrations, and Birational Maps

In this section, we collect some results that will be used throughout the rest of the paper. We consider the following general situation: suppose that $f : X \to X$ is surjective, and there are morphisms $\pi : X \to Y$ and $g : Y \to Y$ with $\pi \circ f = g \circ \pi$. Under these circumstances, we are in some cases able to reduce Conjecture 1.1 for f to the conjecture for g. There are a number of natural fibrations $\pi : X \dashrightarrow Y$ to which one might hope to apply these results on a given variety X, for example, the canonical model, the albanese map, Mori fiber spaces, and the maximal rationally connected (mrc) quotient. Such canonically defined fibrations play a fundamental role in the study of self-maps of higher-dimensional varieties [53]. Recall, as stated in the introduction, that for a regular morphism f and a point P with a dense orbit, the limit defining $\alpha_f(P)$ exists, that is, $\underline{\alpha}_f(P) = \overline{\alpha}_f(P)$.

Lemma 3.1. Assume that X and Y are normal projective varieties over $\overline{\mathbb{Q}}$ and let f (resp. g) be a surjective endomorphism of X (resp. Y). If $\pi : X \to Y$ is a surjection such that $\pi \circ f = g \circ \pi$ and $P \in X(\overline{\mathbb{Q}})$ has a dense orbit under f, then $\alpha_f(P) \geq \alpha_g(\pi(P))$. Moreover, if π is birational and X and Y are $\mathbb{Q}$-factorial, then $\alpha_f(P) = \alpha_g(\pi(P))$.

Proof. We first show $\alpha_f(P) \geq \alpha_g(\pi(P))$. Let H be an ample Cartier divisor on Y. Since P has a dense orbit under f, it follows that $\pi(P)$ has a dense orbit under g. So, the limit defining $\alpha_g(\pi(P))$ exists and we have

$$\alpha_g(\pi(P)) = \lim_{n\to\infty} h_H^+(g^n(\pi(P)))^{1/n} = \lim_{n\to\infty} h_{\pi^*H}^+(f^n(P))^{1/n}.$$

By [37, Remark 2.2] (cf. the proof of [26, Proposition 12]), we obtain

$$\alpha_f(P) = \overline{\alpha}_f(P) \geq \limsup_{n\to\infty} h_{\pi^*H}^+(f^n(P))^{1/n} = \alpha_g(\pi(P)).$$

The cited references are formulated under the hypothesis that X is smooth, so that a dominant rational map induces a pullback map ϕ^* on $N^1(X)$. However, since we assume that f and g are regular morphisms, there are no difficulties associated with repeatedly pulling back Cartier divisors, and the same arguments go through on any normal projective variety (see [26, Remarks 8 and 20]).

It remains to handle the case where π is birational. This follows from the proofs of Lemma 3.3 and Theorem 3.4(ii) in [37]. The statement is again formulated under a smoothness hypothesis, but it suffices to assume that X and Y are normal and $\mathbb{Q}$-factorial. The negativity lemma holds as long as X and Y are normal, and the Weil height machine (see [22, Theorem B.3.2] or Proposition 2.25) remains valid for Cartier divisors on singular varieties by [22, Remark B.3.2.1]. The $\mathbb{Q}$-factoriality assumption is needed so that the definition of the divisor E in [37, Proof of Lemma 3.3] makes sense; to form $p^*p_*q^*H_Y$, we must be able to pull back a Weil divisor. ■

Corollary 3.2. Assume that X and Y are normal, $\mathbb{Q}$-factorial projective varieties over $\overline{\mathbb{Q}}$, and let f (resp. g) be a surjective endomorphism of X (resp. Y). If $\pi : X \to Y$ is a birational morphism such that $\pi \circ f = g \circ \pi$, then Conjecture 1.1 holds for (X, f) if and only if it holds for (Y, g).

Proof. Let $P \in X(\overline{\mathbb{Q}})$. Then P has a dense orbit under f if and only if $\pi(P)$ has a dense orbit under g. Indeed, since π is surjective, it is clear that density of the f-orbit of P implies density of the g-orbit of $\pi(P)$. Conversely, suppose the g-orbit of $\pi(P)$ is dense and let $U \subset X$ be a dense open subset where $\pi|_U$ is an isomorphism. Given any open $V \subset X$, we see $V \cap U \neq \varnothing$ and so $\pi(V \cap U)$ contains some $g^n(\pi(P))$. Thus, $V \cap U$ contains $f^n(P)$, proving density of the f-orbit of P.

To finish the proof, note that Lemma 3.1 and Theorem 2.2(4) tell us $\alpha_f(P) = \alpha_g(\pi(P))$ and $\lambda_1(f) = \lambda_1(g)$. So, $\alpha_f(P) = \lambda_1(f)$ if and only if $\alpha_g(\pi(P)) = \lambda_1(g)$. ■

Combining Corollary 3.2 with [24, Theorem 10] yields the following result.

Corollary 3.3. Let X be a normal, $\mathbb{Q}$-factorial projective surface over $\overline{\mathbb{Q}}$. If f is an automorphism of X, then Conjecture 1.1 holds for (X, f).

Proof. Let $\pi : \widetilde{X} \to X$ be the minimal resolution. By [35, Theorem 4-6-2(i)], there exists an automorphism $\widetilde{f}$ of $\widetilde{X}$ such that $\eta \circ \widetilde{f} = f \circ \eta$. By [24, Theorem 2(c)], Conjecture 1.1 is known for $(\widetilde{X}, \widetilde{f})$ and hence also known for (X, f) by Corollary 3.2. ■

Theorem 3.4. Let $\pi : X \to Y$ be a surjective morphism of normal projective varieties over $\overline{\mathbb{Q}}$. Suppose f (resp. g) is a surjective endomorphism of X (resp. Y) such that $g \circ \pi = \pi \circ f$. If $\lambda_1(f|_\pi) \leq \lambda_1(g)$ and Conjecture 1.1 holds for (Y, g), then Conjecture 1.1 also holds for (X, f). The condition $\lambda_1(f|_\pi) \leq \lambda_1(g)$ holds in particular if f is birational and $\dim Y = \dim X - 1$.

Proof. We begin by showing that $\lambda_1(g) = \lambda_1(f)$. By Theorem 2.2(2), we have

$$\lambda_1(f) = \max\{\lambda_1(g), \lambda_1(f|_\pi)\} = \lambda_1(g).$$

Next, let $P \in X(\overline{\mathbb{Q}})$ have a dense orbit under f, so that $\pi(P)$ has a dense orbit under g. Then by Lemma 3.1, we obtain

$$\alpha_f(P) \geq \alpha_g(\pi(P)) = \lambda_1(g) = \lambda_1(f),$$

where $\alpha_g(\pi(P)) = \lambda_1(g)$ because the conjecture holds for (Y, g). By Remark 2.3, we then know that $\alpha_f(P) = \lambda_1(f)$. Therefore, the conjecture holds for (X, f).

Theorem 2.2(3) tells us that $\lambda_1(f|_\pi) = 1$ whenever f is birational and $\dim Y = \dim X - 1$. Since $\lambda_1(g) \geq 1$, the inequality follows. ■

The following consequence of Theorem 3.4 is applied in the proofs of Theorem 1.8(2) and Theorem 1.10(1).

Corollary 3.5. Let $\pi : X \to Y$ be a surjective morphism of normal projective varieties over $\overline{\mathbb{Q}}$ with X a threefold and Y a $\mathbb{Q}$-factorial surface. Suppose f (resp. g) is an automorphism of X (resp. Y) such that $g \circ \pi = \pi \circ f$. Then Conjecture 1.1 holds for (X, f).

Proof. By Corollary 3.3, we know Conjecture 1.1 holds for (Y, g). Since f is birational and $\dim Y = \dim X - 1$, Conjecture 1.1 for (X, f) follows from Theorem 3.4. ■

4 Endomorphisms of Kodaira Dimension 0 Threefolds: Proposition 1.7

The goal of this brief section is to prove Conjecture 1.1 for all smooth threefolds X of Kodaira dimension 0 and surjective endomorphisms f with $\deg(f) > 1$. The crux of the argument is a theorem of Fujimoto that it is possible to run the minimal model program on X while only contracting f-periodic rays.

Proof of Proposition 1.7 By [19, Lemma 2.3], f is a finite étale cover and so $\chi(\mathcal{O}_X) = \deg(f)\chi(\mathcal{O}_X)$. Then $\chi(\mathcal{O}_X) = 0$ since $\deg(f) > 1$. By [19, Corollary 4.4] and its proof, we know that all extremal contractions of X are of type (E1) (the inverse of the blowup along a smooth curve), so the minimal model of X is smooth, and f descends to a surjective endomorphism of a minimal model of X. The argument of [19] is based on a run of the MMP and holds over any algebraically closed field of characteristic 0, so the minimal model of X is defined over $\overline{\mathbb{Q}}$. By Theorem 2.2(4), the Kawaguchi–Silverman conjecture holds for f if and only if it holds for the induced endomorphism of the minimal model of X. We may therefore assume X itself is minimal.

The abundance conjecture is known in dimension 3 by [28], so $K_X \equiv 0$. By [4, Proposition 3.1], there is a finite étale cover $\pi : \widetilde{X} \to X$ with $\widetilde{X} = A \times \prod_i Y_i \times \prod_j Z_j$, where A is an abelian variety, Y_i are Calabi–Yau varieties, and Z_j are hyper-Kähler manifolds all defined over $\overline{\mathbb{Q}}$. Applying Condition (3) of [4, Proposition 3.1] to $f \circ \pi$ yields a map $\widetilde{f}$ making the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array}$$

commute. We see $\widetilde{f}$ is finite étale with $\deg(\widetilde{f}) = \deg(f)$. From [19, Main Theorem A] Case 3, we know that $\widetilde{X}$ is an abelian threefold or $E \times Z$ with E and elliptic curve and Z a K3 surface; the reason $\widetilde{X}$ cannot be a Calabi–Yau threefold is that $\pi_1(X)$ is infinite, see [19, Claim, pg. 66]. By [45, Theorem 1.3], Conjecture 1.1 is known for products of abelian varieties and K3 surfaces, so it is known for $\widetilde{f}$. By [37, Lemma 3.2], the conjecture for f follows. ∎

5 Automorphisms of Calabi–Yau Threefolds: Theorem 1.8

Proof of Theorem 1.8 We first handle case (1). By [4, Lemma 7.1] we know that $\{D \in \mathrm{Nef}(X) \mid c_2(X) \cdot D \leq M\}$ is compact for all $M \geq 0$. So the function $D \mapsto c_2(X) \cdot D$ achieves a minimum positive value on $N^1(X) \cap \mathrm{Amp}(X)$ and this value is achieved by only finitely many D_i. Taking the sum of these finitely many D_i, we obtain an ample class A that is fixed by f^*. It follows that some iterate f^n lies in the connected component of the identity $\mathrm{Aut}^0(X) \subseteq \mathrm{Aut}(X)$. Since X is a Calabi–Yau threefold, $\dim \mathrm{Aut}^0(X) = \dim H^0(X, TX) = 0$, and we conclude that f has finite order, so the conjecture holds vacuously.

We now turn to case (2). Let $\pi : X \to Y$ be the contraction map associated with D; since $D \cdot c_2(X) = 0$, this is referred to as a c_2-contraction. Oguiso shows in [42, Theorem 4.3] that there are only finitely many c_2-contractions, and so after replacing f by a further iterate, we can assume $f^*[D] = [D]$. By [4, Proposition 6.1(a)], we know that f descends to an automorphism g of Y. Since $D \neq 0$, we see $\dim Y > 0$.

Let us first suppose that $\dim Y = 1$. By hypothesis, there is a rational point $P \in X(\overline{\mathbb{Q}})$ with a Zariski dense orbit under f, so $\pi(P) \in Y$ has a Zariski dense orbit under g. As a result, Y must be rational or an elliptic curve; since X has trivial Albanese, we see $Y \simeq \mathbb{P}^1$. Let $Z \subseteq \mathbb{P}^1$ be the locus of points t, where the fiber X_t is singular. Then $g(Z) = Z$. Since Z is a finite set, after replacing f by a further iterate, we can assume g

fixes Z point-wise. By [50, Theorem 0.2], we know that Z contains at least three points. It follows that g is the identity since it fixes at least three points of $\mathbb{P}^1$. In other words, there exists a rational function on X that is invariant under some iterate of f, which contradicts the fact that X has a point with a dense orbit.

The case in which $\dim Y = 2$ is an immediate consequence of Corollary 3.5; that Y is normal and $\mathbb{Q}$-factorial is proved in [41, pg. 18].

Finally, we handle the case, where $\dim Y = 3$, that is, D is big. Since contractions have connected fibers, π is birational. Then $D = \pi^*H$ for some ample divisor H on Y. Then $\pi^*(g^*H) = f^*(\pi^*H) = f^*D = D = \pi^*H$, which shows that $g^*H = H$, and so $\lambda_1(g) = 1$. Theorem 2.2 (4) shows that $\lambda_1(f) = \lambda_1(g) = 1$, and the conjecture holds for f by Remark 2.3.

6 Mori Fiber Spaces

6.1 Automorphisms of threefold Mori fiber spaces: Theorem 1.10(1)

We prove Theorem 1.10(1) after a preliminary lemma.

Definition 6.1. A *Mori fiber space* is a projective morphism $\pi : X \to S$ such that X is terminal and $\mathbb{Q}$-factorial, $-K_X$ is π-ample, and $\rho(X/S) = 1$.

Lemma 6.2. Let $\pi : X \to S$ be a Mori fiber space. If f is a surjective endomorphism of X, then after replacing f by a suitable iterate f^m, we may assume that there is an endomorphism $g : S \to S$ such that $g \circ \pi = \pi \circ f$. If f is an automorphism then g is also an automorphism.

Proof. We claim first that some iterate of f maps fibers to fibers. This is a consequence of an observation of Wiśniewski [51, Theorem 2.2] (see also [29, Exercise III.1.19]); on a given variety, there are only finitely many K_X-negative extremal rays on the closed cone of curves $\overline{\mathrm{NE}}(X)$ yielding Mori fiber space structures.

The existence of the map g is a consequence of the rigidity lemma [11, Lemma 1.15(b)], as in the proof of Proposition 2.35, since a Mori fiber space necessarily satisfies $\pi_*\mathcal{O}_X = \mathcal{O}_S$. ■

Theorem 6.3. Suppose that $\pi : X \to Y$ is a Mori fiber space. Suppose that $f : X \to X$ and $g : Y \to Y$ are automorphisms with $\pi \circ f = g \circ \pi$. If Conjecture 1.1 holds for g, then it holds for f.

Proof. Recall that the 1st relative dynamical degree is defined by

$$\lambda_1(\pi|_f) = \lim_{n\to\infty} \left((f^n)^* H \cdot \pi^*(H'^{\dim Y}) \cdot H^{\dim X - \dim Y - 1}\right)^{1/n}.$$

Here $\pi^*(H'^{\dim Y})$ is the class of a fiber of π, and $\pi^*(H'^{\dim Y}) \cdot H^{\dim X - \dim Y - 1}$ is the class of some curve in the fiber. Since π is a Mori fiber space, all curves contained in fibers are proportional in $N_1(X)$, and since f is an automorphism defined over π, this class must be invariant under f. It follows that $\lambda_1(\pi|_f) = 1$. The claim is then a consequence of Theorem 3.4. ■

Proof of Theorem 1.10 (1) Let X be a threefold, f an automorphism of X, and $\pi : X \to S$ a Mori fiber space structure. After replacing f by an iterate, by Lemma 6.2 we may assume that there is an automorphism $g : S \to S$ such that $\pi \circ f = g \circ \pi$. Since $\dim S \leq 2$ and g is an automorphism, Conjecture 1.1 is known for (S, g), and the conjecture for (X, f) follows from Theorem 6.3. ■

6.2 Endomorphisms of rational normal scrolls: Theorem 1.10 (2)

Let C be a smooth projective curve over $\overline{\mathbb{Q}}$, $\mathcal{E}$ a vector bundle on C of rank n, and $X = \mathbb{P}_C(\mathcal{E})$. By [18, Theorem 9.6], the Chow group of X is given by

$$\begin{aligned} A^*(X) &= A^*(C)[D]/(D^n + c_1(\mathcal{E})D^{n-1} + c_2(\mathcal{E})D^{n-2} + \cdots + c_n(\mathcal{E})) \\ &= A^*(C)[D]/(D^n + c_1(\mathcal{E})D^{n-1}F), \end{aligned}$$

where F is the class of a fiber. So $A^*(X)$ is generated by the divisor classes F and D and we have the relations $F^2 = 0$, $FD^{n-1} = 1$, and $D^n = -c_1(\mathcal{E})D^{n-1}F = -c_1(\mathcal{E})$; the 2nd relation holds because $DF = D|_F$ is the class of a hyperplane on $F = \mathbb{P}^{n-1}$ and so $FD^{n-1} = (D|_F)^{n-1} = 1$.

The nef cone of X is given by the following, which generalizes a result of Miyaoka [38, Theorem 3.1]. Recall that the slope $\mu(\mathcal{E})$ is defined to be $c_1(\mathcal{E})/\mathrm{rank}(\mathcal{E})$. We let $\mu_{\min}(\mathcal{E})$ and $\mu_{\max}(\mathcal{E})$ denote the minimum, resp. maximum, slope of the graded pieces appearing in the Harder–Narasimhan filtration of $\mathcal{E}$.

Lemma 6.4. $\mathrm{Nef}(X)$ is the cone generated by F and $D - \mu_{\min}(\mathcal{E})F$.

Proof. See, for example, [39, Lemma 4.4.1] or [20, Lemma 2.1]. ■

Given a surjective endomorphism f of $X = \mathbb{P}_C(\mathcal{E})$, in order to verify Conjecture 1.1 we may replace f by an iterate. Since the structure map $\pi : X \to C$ is a Mori fiber space, by Lemma 6.2 we can replace f by an iterate and assume that there is an endomorphism g of C such that $\pi \circ f = g \circ \pi$. We assume we are in this situation throughout this section. Let

$$\delta := \frac{\deg(f)}{\deg(g)}.$$

Lemma 6.5. The action of f^* on $N^1(X)$ is given by

$$f^*(F) = \deg(g)\, F, \qquad f^*(D) = (\deg(g) - \delta^{1/(n-1)})\mu_{\min}(\mathcal{E})\, F + \delta^{1/(n-1)}\, D$$

and has eigenvalues $\lambda_1(g) = \deg(g)$ and $\delta^{1/(n-1)}$. Moreover,

$$\lambda_1(f) = \max(\lambda_1(g), \delta^{1/(n-1)}).$$

Proof. It is clear that F is an eigenvector with eigenvalue $\deg(g) = \lambda_1(g)$: since F is a fiber it is of the form $\pi^{-1}(P_0)$ for a point $P_0 \in C$ and we have

$$f^*F = f^*\pi^*P_0 = \pi^*g^*P_0 = \pi^*(\deg(g)P_0) = \deg(g)F.$$

Next, let $f^*D = cF + dD$. Notice that with respect to the basis F, D for $N^1(X)$, the matrix for f^* is upper triangular with diagonal entries $\deg(g)$ and d. So, the eigenvalues for $(f^p)^*$ are given by $\deg(g)^p$ and d^p. Since $\lambda_1(f) = \lim_{p\to\infty} \mathrm{SpecRad}((f^p)^*)^{1/p}$, we see $\lambda_1(f) = \mathrm{SpecRad}(f^*) = \max(\deg(g), d)$. So, we need only show $d = \delta^{1/(n-1)}$, that is, that $\deg(f) = d^{n-1}\deg(g)$. Notice that

$$\begin{aligned}\deg(f) = \deg(f)D^{n-1}F = f_*f^*(FD^{n-1}) &= f_*(f^*F \cdot (f^*D)^{n-1})\\ &= \deg(g)f_*(F \cdot (cF + dD)^{n-1}) = \deg(g)f_*(d^{n-1}FD^{n-1}) = d^{n-1}\deg(g).\end{aligned}$$

So, we have now shown that the eigenvalues of f^* are $\lambda_1(g) = \deg(g)$ and $\delta^{1/(n-1)}$, and that $\lambda_1(f) = \max(\lambda_1(g), \delta^{1/(n-1)})$.

Lastly, we must calculate c. To do so, we use Lemma 6.4. Notice that the determinant of the action of f^* on $N^1(X)$ is $\deg(f) > 0$ so the action is orientation-preserving. Since f is finite, for all D' we know D' is ample if and only if f^*D' is ample. As a result, the boundary rays of $\mathrm{Nef}(X)$ are each sent to themselves. Thus, the eigenvectors for f^* are given by F and $D-\mu_{\min}(\mathcal{E})F$. In particular, $d(D-\mu_{\min}(\mathcal{E})F) = f^*(D-\mu_{\min}(\mathcal{E})F) = cF + dD - \deg(g)\mu_{\min}(\mathcal{E})F$, and so

$$c = (\deg(g) - d)\mu_{\min}(\mathcal{E}),$$

proving the lemma. ■

Proposition 6.6. One of the following holds: $\lambda_1(f) = \lambda_1(g)$ or $\mu_{\min}(\mathcal{E}) = -\mu(\mathcal{E})$.

Proof. From Lemma 6.5, we know $f^*D = cF + dD$, where $c = (\deg(g) - d)\mu_{\min}(\mathcal{E})$ and $d = \delta^{1/(n-1)}$. Recalling that $D^n = -c_1(\mathcal{E})$, we have

$$-\deg(f)c_1(\mathcal{E}) = f_*f^*(D^n) = f_*(cF + dD)^n = ncd^{n-1} - d^n c_1(\mathcal{E}).$$

Substituting for c, we have

$$-\deg(f)c_1(\mathcal{E}) = n(\deg(g) - d)\mu_{\min}(\mathcal{E})d^{n-1} - d^n c_1(\mathcal{E}) = n(\deg(f) - d^n)\mu_{\min}(\mathcal{E}) - d^n c_1(\mathcal{E})$$

and so

$$d^n(c_1(\mathcal{E}) + n\mu_{\min}(\mathcal{E})) = \deg(f)(c_1(\mathcal{E}) + n\mu_{\min}(\mathcal{E})).$$

Thus, $\mu_{\min}(\mathcal{E}) = -c_1(\mathcal{E})/n =: -\mu(\mathcal{E})$ or $d^n = \deg(f)$. This latter equality is equivalent to $d = \deg(g) = \lambda_1(g)$, which by Lemma 6.5, implies $\lambda_1(f) = \lambda_1(g)$. ■

We next need the following basic result concerning the Harder–Narasimhan filtration.

Lemma 6.7. If $\mathcal{E}$ is a vector bundle that is not semistable, then $\mu_{\max}(\mathcal{E}) > \mu(\mathcal{E}) > \mu_{\min}(\mathcal{E})$.

Proof. Let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_{\ell-1} \subsetneq \mathcal{E}_\ell = \mathcal{E}$$

be the Harder–Narasimhan filtration of $\mathcal{E}$, so that $\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E}_1)$ and $\mu_{\min}(\mathcal{E}) = \mu(\mathcal{E}/\mathcal{E}_{\ell-1})$. By construction, $\mathcal{E}_1$ is the maximal destabilizing subbundle of $\mathcal{E}$, that is, for all subbundles $0 \neq \mathcal{F} \subseteq \mathcal{E}$ we have (1) $\mu(\mathcal{E}_1) \geq \mu(\mathcal{F})$ and (2) if $\mu(\mathcal{E}_1) = \mu(\mathcal{F})$, then $\mathcal{F} \subseteq \mathcal{E}_1$. So, we see $\mu(\mathcal{E}_1) \geq \mu(\mathcal{E})$ and we cannot have equality since then we would have $\mathcal{E} = \mathcal{E}_1$ that is not possible as $\mathcal{E}_1$ is semistable and $\mathcal{E}$ is not. We have therefore shown $\mu_{\max}(\mathcal{E}) > \mu(\mathcal{E})$.

To show $\mu(\mathcal{E}) > \mu_{\min}(\mathcal{E})$, we induct on ℓ. We first recall the general result that follows immediately from the definition of slope: if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of nontrivial vector bundles, then $\mu(\mathcal{F}') > \mu(\mathcal{F})$ if and only if $\mu(\mathcal{F}) > \mu(\mathcal{F}'')$.

Since $\mathcal{E}$ is not semistable, we have $\ell \geq 2$. When $\ell = 2$ we have a short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2/\mathcal{E}_1 \to 0$$

and since we have already shown $\mu(\mathcal{E}_1) > \mu(\mathcal{E})$, we know $\mu(\mathcal{E}) > \mu(\mathcal{E}_2/\mathcal{E}_1) = \mu_{\min}(\mathcal{E})$.

Next suppose $\ell \geq 3$. Then

$$0 \neq \mathcal{E}_2/\mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_{\ell-1}/\mathcal{E}_1 \subsetneq \mathcal{E}/\mathcal{E}_1$$

is the Harder–Narasimhan filtration of $\mathcal{E}/\mathcal{E}_1$; it has length $\ell - 1 \geq 2$ and so $\mathcal{E}/\mathcal{E}_1$ is not semistable. Then by induction, $\mu(\mathcal{E}/\mathcal{E}_1) > \mu_{\min}(\mathcal{E}/\mathcal{E}_1) = \mu(\mathcal{E}/\mathcal{E}_{\ell-1}) = \mu_{\min}(\mathcal{E})$. Since we have shown $\mu(\mathcal{E}_1) > \mu(\mathcal{E})$, we know $\mu(\mathcal{E}) > \mu(\mathcal{E}/\mathcal{E}_1)$ and so $\mu(\mathcal{E}) > \mu_{\min}(\mathcal{E})$. ■

Corollary 6.8. Let C be a smooth curve. Then the following are equivalent:

1. Conjecture 1.1 holds for all surjective endomorphisms of varieties of the form $\mathbb{P}_C(\mathcal{E})$
2. Conjecture 1.1 holds for all surjective endomorphisms of varieties of the form $\mathbb{P}_C(\mathcal{E})$ with $\mathcal{E}$ semistable of degree 0.

Proof. By Proposition 6.6, we know $\lambda_1(f) = \lambda_1(g)$ or $\mu_{\min}(\mathcal{E}) = -\mu(\mathcal{E})$. Suppose $\lambda_1(f) = \lambda_1(g)$ and $P \in X(\overline{\mathbb{Q}})$ has a dense orbit under f. Then $\pi(P)$ has a dense orbit under g, so $\alpha_g(\pi(P)) = \lambda_1(g)$ since the conjecture is known for curves. Then Lemma 3.1 shows $\alpha_f(P) \geq \alpha_g(\pi(P)) = \lambda_1(g) = \lambda_1(f)$, and hence $\alpha_f(P) = \lambda_1(f)$ by Remark 2.3.

We next turn to the case where $\mu_{\min}(\mathcal{E}) = -\mu(\mathcal{E})$. Since $X = \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ for any line bundle $\mathcal{L}$, choosing $\mathcal{L}$ with sufficiently negative degree, we can assume $\mu(\mathcal{E}) < 0$. If $\mathcal{E}$ is not semistable, then by Lemma 6.7 we have $\mu(\mathcal{E}) > \mu_{\min}(\mathcal{E}) = -\mu(\mathcal{E})$ that is a contradiction. So, $\mathcal{E}$ must be semistable, in which case $\mu(\mathcal{E}) = \mu_{\min}(\mathcal{E}) = -\mu(\mathcal{E})$, so $\mu(\mathcal{E}) = 0$, that is, $\deg \mathcal{E} = 0$. ■

We are now ready to prove Conjecture 1.1 in the case where $C = \mathbb{P}^1$, that is, the case of rational normal scrolls.

Proof of Theorem 1.10 (2) By Corollary 6.8, we need only prove the conjecture for semistable degree 0 vector bundles on $\mathbb{P}^1$. Such vector bundles are all trivial, so $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ in which case the conjecture holds by [45, Theorem 1.3]. ■

Funding

This work was supported by the National Science Foundation [DMS-1700898, DMS-1246844 to J.L.]; and the Natural Sciences and Engineering Research Council of Canada [Discovery grant RGPIN-2015-05631 to M.S.].

Acknowledgments

We thank Fei Hu for a number of helpful comments, including a method to weaken the hypotheses of Theorem 2.27. We are also grateful to Lawrence Ein and Mihai Fulger for useful remarks. We also thank the anonymous referees for their thorough reading of the paper and many helpful suggestions.

References

[1] Amerik, E. and M. Verbitsky. "Construction of automorphisms of hyperkähler manifolds." *Compos. Math.* 153, no. 8 (2017): 1610–21.

[2] Bayraktar, T. "Green currents for meromorphic maps of compact Kähler manifolds." *J. Geom. Anal.* 23, no. 2 (2013): 970–98.

[3] Beauville, A. "Variétés Kähleriennes dont la première classe de Chern est nulle." *J. Differ. Geom.* 18, no. 4 (1983): 755–82.

[4] Bell, J., D. Ghioca, Z. Reichstein, and M. Satriano. "On the Medvedev–Scanlon conjecture for minimal threefolds of non-negative Kodaira dimension." *New York J. Math.* 23 (2017): 1185–203.

[5] Besse, A. L. "Einstein manifolds." *Classics in Mathematics*. Berlin: Springer, 2008. Reprint of the 1987 edition. *MR 2371700*.

[6] Birkar, C., P. Cascini, C. D. Hacon, and J. McKernan. "Existence of minimal models for varieties of log general type." *J. Amer. Math. Soc.* 23, no. 2 (2010): 405–68.

[7] Birkhoff, G. "Linear transformations with invariant cones." *Amer. Math. Monthly* 74 (1967): 274–6.

[8] Cantat, S. "Dynamique des automorphismes des surfaces $K3$." *Acta Math.* 187, no. 1 (2001): 1–57.

[9] Cantat, S. "Dynamics of automorphisms of compact complex surfaces." *Frontiers in Complex Dynamics* , 463–514. Princeton Math. Ser. 51. Princeton, NJ: Princeton University Press, 2014.

[10] Dang, N.-B. "*Degrees of iterates of rational maps on normal projective varieties*." (2017): preprinthttps://arxiv.org/abs/1701.07760, 2017.

[11] Debarre, O. *Higher-dimensional Algebraic Geometry*. New York: Universitext, Springer, 2001.

[12] Dinh, T.-C. and V.-A. Nguyên. "Comparison of dynamical degrees for semi-conjugate meromorphic maps." *Comment. Math. Helv.* 86, no. 4 (2011): 817–40.

[13] Dinh, T.-C., V.-A. Nguyên, and T. T. Truong. "On the dynamical degrees of meromorphic maps preserving a fibration." *Commun. Contemp. Math.* 14, no. 6 (2012): 1250042.

[14] Dinh, T.-C. and N. Sibony. "Green currents for holomorphic automorphisms of compact Kähler manifolds." *J. Amer. Math. Soc.* 18, no. 2 (2005): 291–312.

[15] Dinh, T.-C. and N. Sibony. "Une borne supérieure pour l'entropie topologique d'une application rationnelle." *Ann. of Math. (2)* 161, no. 3 (2005): 1637–44.

[16] Ein, L., R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa. "Asymptotic invariants of base loci." *Ann. Inst. Fourier (Grenoble)* 56, no. 6 (2006): 1701–34.

[17] Ein, L., R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa. "Restricted volumes and base loci of linear series." Amer. *J. Math.* 131, no. 3 (2009): 607–51.

[18] Eisenbud, D. and J. Harris. *3264 and All That—A Second Course in Algebraic Geometry*. Cambridge: Cambridge University Press, 2016. MR 3617981.

[19] Fujimoto, Y. "Endomorphisms of smooth projective 3-folds with non-negative Kodaira dimension." *Publ. Res. Inst. Math. Sci.* 38, no. 1 (2002): 33–92.

[20] Fulger, M. "The cones of effective cycles on projective bundles over curves." *Math. Z.* 269, nos. 1–2 (2011): 449–59.

[21] Gross, M., D. Huybrechts, and D. Joyce. *Calabi–Yau Manifolds and Related Geometries*. Berlin: Universitext, Springer, 2003. Lectures from the Summer School held in Nordfjordeid, *June* 2001.

[22] Hindry, M. and J. H. Silverman. "Diophantine geometry." *Graduate Texts in Mathematics* 201. New York: Springer,2000. MR 1745599.

[23] Kawaguchi, S. "Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint." *Amer. J. Math.* 130, no. 1 (2008): 159–86.

[24] Kawaguchi, S. and J. H. Silverman. "Examples of dynamical degree equals arithmetic degree." *Michigan Math. J.* 63, no. 1 (2014): 41–63.

[25] Kawaguchi, S. and J. H. Silverman. "Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties." *Trans. Amer. Math. Soc.* 368, no. 7 (2016): 5009–35.

[26] Kawaguchi, S. and J. H. Silverman. "On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties." *J. Reine Angew. Math.* 713 (2016): 21–48.

[27] Kawamata, Y. "Characterization of abelian varieties." *Compos. Math.* 43, no. 2 (1981): 253–76.

[28] Kawamata, Y. "Abundance theorem for minimal threefolds." *Invent. Math.* 108, no. 2 (1992): 229–46.

[29] Kollár, J. "Rational curves on algebraic varieties." Ergeb. Math. Grenzgeb. (3) Folge. *A Series of Modern Surveys in Mathematics Results in Mathematics and Related Areas. 3rd Series*. A Series of Modern Surveys in Mathematics 32. Berlin: Springer, 1996.

[30] Kollár, J. and S. Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Math. 134. Cambridge: Cambridge Univ. Press, 1998. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original.

[31] Lazarsfeld, R. *Positivity in algebraic geometry. I*. Berlin: Springer, 2004.

[32] Lazic, V., K. Oguiso, and T. Peternell. "Nef line bundles on Calabi–Yau threefolds, I." *Int. Math. Res. Not. IMRN* (2018). https://doi.org/10.1093/imrn/rny191.

[33] Lo Bianco, F. "*On the cohomological action of automorphisms of compact Kähler three-folds*." (2017): preprint https://arxiv.org/abs/1712.08453.

[34] Lo Bianco, F. "On the primitivity of birational transformations of irreducible holomorphic symplectic manifolds." *Int. Math. Res. Not. IMRN* 2019, no. 1 (2019): 1–32.

[35] Matsuki, K. *Introduction to the Mori Program*. New York: Universitext, Springer, 2002.

[36] Matsuzawa, Y. "*On upper bounds of arithmetic degrees*." (2017): preprint https://arxiv.org/abs/1606.00598.

[37] Matsuzawa, Y., K. Sano, and T. Shibata. "Arithmetic degrees and dynamical degrees of endomorphisms on surfaces." *Algebra Number Theory* 12, no. 7 (2018): 1635–57. MR 3871505.

[38] Miyaoka, Y. "The Chern classes and Kodaira dimension of a minimal variety." *Algebraic Geometry, Sendai, 1985, pp. 449–476*. Adv. Stud. Pure Math. 10. North-Holland, Amsterdam, 1987. MR 946247.

[39] Nakayama, N. *Zariski-decomposition and Abundance*. *MSJ Mem.* 14. Tokyo: Mathematical Society of Japan, 2004.

[40] Nakayama, N. and D.-Q. Zhang. "Building blocks of étale endomorphisms of complex projective manifolds." *Proc. Lond. Math. Soc. (3)* 99, no. 3 (2009): 725–56.

[41] Oguiso, K. "On algebraic fiber space structures on a Calabi–Yau 3-fold." *Int. J. Math.* 4, no. 3 (1993): 439–65. With an appendix by Noboru Nakayama.

[42] Oguiso, K. "On the finiteness of fiber-space structures on a Calabi–Yau 3-fold." *J. Math. Sci.* (N.Y.) 106, no. 5 (2001): 3320–3335. *Algebraic geometry*, 11.

[43] Oguiso, K. "A remark on dynamical degrees of automorphisms of hyperkähler manifolds." *Manuscripta Math.* 130, no. 1 (2009): 101–11.

[44] Oguiso, K. "Automorphism groups of Calabi–Yau manifolds of Picard number 2." *J. Algebraic Geom.* 23, no. 4 (2014): 775–95.

[45] Sano, K. "*Dynamical degree and arithmetic degree of endomorphisms on product varieties*." (2016): preprint https://arxiv.org/pdf/1604.04174.pdf.

[46] Shibata, T. "Ample canonical heights for endomorphisms on projective varieties." (2017): preprint https://arxiv.org/abs/1710.05278.

[47] Silverman, J. H. "Rational points on $K3$ surfaces: a new canonical height." *Invent. Math.* 105, no. 2 (1991): 347–73.

[48] .Silverman, J. H. "Arithmetic and dynamical degrees on abelian varieties." *J. Théor. Nombres Bordeaux* 29, no. 1 (2017): 151–67.

[49] Truong, T. "*(Relative) dynamical degrees of rational maps over an algebraic closed field*." (2015): preprint https://arxiv.org/abs/1501.01523.

[50] Viehweg, E. and K. Zuo. "On the isotriviality of families of projective manifolds over curves." *J. Algebraic Geom.* 10, no. 4 (2001): 781–99.

[51] Wiśniewski, J. "On contractions of extremal rays of Fano manifolds." *J. Reine Angew. Math.* no. 417 (1991): 141–58.

[52] Junyi, X. "Periodic points of birational transformations on projective surfaces." *Duke Math. J.* 164, no. 5 (2015): 903–32.

[53] Zhang, D.-Q. "Dynamics of automorphisms on projective complex manifolds." *J. Differ. Geom.* 82, no. 3 (2009): 691–722.

[54] Zhang, D.-Q. "Birational automorphism groups of projective varieties of Picard number two." *Automorphisms in Birational and Affine Geometry*, 231–8. Springer Proc. Math. Stat. 79. Cham: Springer, 2014.