Evidence for a Spectral Interpretation of the Zeros of L-Functions

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS

JUNE, 1998

Abstract

By looking at the average behavior (n-level density) of the low lying zeros of certain families of L-functions, we find evidence, as predicted by function field analogs, in favor of a spectral interpretation of the non-trivial zeros in terms of the classical compact groups. This is further supported by numerical experiments for which an efficient algorithm to compute L-functions was developed and implemented.

Acknowledgements

When Mike Rubinstein woke up one morning he was shocked to discover that he was writing the acknowledgements to his thesis. After two screenplays, a 40000 word manifesto, and many fruitless attempts at making sushi, something resembling a detailed academic work has emerged for which he has people to thank.

Peter Sarnak- from Chebyshev's Bias to $\mathrm{USp}(\infty)$. For being a terrific advisor and teacher. For choosing problems suited to my talents and involving me in this great project to understand the zeros of L-functions.

Zeev Rudnick and Andrew Oldyzko for many discussions and comments. Odlyzko, also for providing me with a program that was used to generate some of the solid curves in the introduction. Rudnick further for inviting me to warm and sunny Israel where part of this thesis work was done.

Brian Conrad, and Henryk Iwaniec for some references. Jeff Viaclovsky for suggesting the subject heading: 'l.h.s. = r.h.s.' (but not for insisting on an acknowledgement).

Little E. (for sitting on my back, and more).

April, 1998

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Dedicated to my sister Ronit

Chapter 1

Introduction

In this thesis, I obtain evidence for a spectral interpretation of the zeros of L-functions. The Langland's program [12] [24] [19] predicts that all L-functions can be written as products of $\zeta(s)$ and L-functions attached to automorphic cuspidal representations of GL_M over $\mathbb{Q}$. Such an L-function is given intially (for $\Re s$ sufficiently large) as an Euler product of the form

$$L(s,\pi) = \prod_p L(s,\pi_p) = \prod_p \prod_{j=1}^{M} (1-\alpha_\pi(p,j)p^{-s})^{-1}. \tag{1.0.1}$$

Basic properties of such L-functions are described in [33]. The L-functions that arise in the $m=1$ case are the Riemann zeta function $\zeta(s)$, and Dirichlet L-functions $L(s,\chi)$, χ a primitive character. For $m=2$, the L-functions in question are associated to cusp forms or Maass forms of congruence subgroups of $SL_2(\mathbb{Z})$.

The Riemann Hypothesis (RH) for $L(s,\pi)$ asserts that the non-trivial zeros of $L(s,\pi)$, $\{1/2+i\gamma_\pi\}$, all have $\gamma_\pi \in \mathbb{R}$. **(our L-functions are always normalized so that the critical line is through $\Re s = 1/2$).**

A vague suggestion of Polya and Hilbert suggests an approach that one might take in establishing RH. They hypothesized (for $\zeta(s)$) that one might be able to associate the non-trivial zeros of ζ to the eigenvalues of some operator acting on some Hilbert space, thus (depending on the properties of the operator) forcing the zeros to lie on a line.

The first evidence in favor of this approach was obtained by Montgomery [23] who derived (under certain restrictions) the pair correlation ((1.2.1) with $n=2$) of the zeros of $\zeta(s)$. Together with an observation of Freeman Dyson, who pointed out that the Gaussian Unitary Ensemble (GUE), consisting of $N \times N$ random Hermitian matrices (see [22] for a more precise definition), has the same pair correlation (as $N \to \infty$), seems to suggest that the relevant operator, at least for $\zeta(s)$, might be Hermitian. Extensive computations of Odlyzko [25] [26] further seem to bolster the Hermitian nature of the zeros of $\zeta(s)$, as might the work of Rudnick and Sarnak [33] where, under certain restrictions, the n-level correlations of $\zeta(s)$ and $L(s,\pi)$ are found to be the same as those of the GUE.

However, recent developments suggest that, rather than being Hermitian, the relevant operators for L-functions belong to the classical compact groups. (This is consistent with the above work of Montgomery, Odlyzko, and Rudnick-Sarnak since all the classical compact groups have the same n-level correlations as of the GUE (as $N \to \infty$). See Section 1.2). First, analogs with function field zeta functions, where there is a spectral interpretation of the zeros in terms of Frobenius on cohomology, point towards the classical compact groups [16]. Second, even though all the mentioned families of matrices have the same n-level correlations, there is another statistic, called n-level density, which is sensitive to the particular family. By looking at this statistic for zeros of L-functions, one finds the fingerprints of the classical compact groups. For $n = 1$ this was done, for quadratic twists of $\zeta(s)$, and with certain restrictions, by Ozluk and Snyder [29]. A stronger result (which takes into account certain non-diagonal contributions and allows one to choose test funtions whose fourier transform is supported in $(-2/M, 2/M)$) which applies for $\zeta(s)$ as well as all $L(s,\pi)$ was obtained by Katz and Sarnak [16]. The general case, $n \geq 1$, is worked out (again, with some restrictions) in this thesis.

Further evidence in favor of a spectral interpretation of the zeros comes from numerical experimentation. For this, an algorithm was developed and implemented to compute zeros of various L-functions. The data obtained supports the predictions of the function field and in particular the role played by the classical compact groups.

The next few sections describe the above in greater detail.

1.1 Matrices, Eigenvalues, Functional Equations

Let A be a diagonalizable $N \times N$ matrix, $p(z) = \det(zI - A)$ its characteristic polynomial, and $\{\lambda_i\}_{i=1}^N$ its sequence of eigenvalues. The following table summarizes some properties of several types of matrices:

A	$\det A$	$\{\lambda_i\}$	functional equation for $p(z)$
Hermitian: $A = A^*$	$\det A \in \mathbb{R}$	$\lambda_i \in \mathbb{R}$	$p(z) = \overline{p(\overline{z})}$, i.e. real coeff.
U(N): $AA^* = I$	$\|\det A\| = 1$	$\|\lambda_i\| = 1$	$p(z) = (-z)^N (\det A)\overline{p(1/\overline{z})}$
USp(N): $AA^* = I$, N even $A^t J A = J$, $J = \begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}$	$\det A = 1$	$\|\lambda_i\| = 1$, eigenvalues form conjugate pairs	$p(z) = z^N p(1/z)$, $p(z) = \overline{p(\overline{z})}$, i.e. real coeff.
O(N): $AA^t = I$ $a_{ij} \in \mathbb{R}$	$\det A = \pm 1$	$\|\lambda_i\| = 1$, eigenvalues form conjugate pairs	$p(z) = (-z)^N (\det A) p(1/z)$, real coeff. (since $a_{ij} \in \mathbb{R}$)

Note that the functional equation for U(N) mimics the functional equation of a typical L-function. Let

$$p(z) = \sum_{i=0}^{N} a_i z^i \tag{1.1.1}$$

$$p_2(z) = \sum_{i=0}^{N} \overline{a}_i z^i$$

.

Then, changing variables, $z = (1-s)/s$, (which maps the unit circle, $|z| = 1$, to the critical line, $\Re s = 1/2$, with the point $z = 1$ going to the point $s = 1/2$), we get, for U(N),

$$s^N p((1-s)/s) = \varepsilon (1-s)^N p_2(s/(1-s))$$

(with $|\varepsilon| = 1$). This is of the same form as the functional equation for L-functions

$$L(s, \pi) \longleftrightarrow L(1-s, \tilde{\pi})$$

where $\tilde{\pi}$ is the contragredient of π and has $\alpha_{\tilde{\pi}}(p, j) = \overline{\alpha}_\pi(p, j)$.

Furthermore, matrices in USp(N) or O(N) have characteristic polynomials with real coefficients and functional equations that mimic the functional equations and symmetries of zeros (which come in conjugate pairs) of self contragredient L-functions (i.e. $\alpha_\pi(p,j) \in \mathbb{R}$)

$$\begin{gathered} L(s,\pi) \longleftrightarrow L(1-s,\pi) \\ 1/2 + i\gamma \text{ is a zero} \longleftrightarrow 1/2 - i\gamma \text{ is a zero.} \end{gathered} \tag{1.1.2}$$

On the other hand, it is hard to see any relevance of the GUE in this picture.

1.2 n-level correlations

The n-level correlation of a set of N numbers $B = \{\lambda_j\}_{j=1}^N$ measures the correlations between the differences of the numbers. For a box $Q \in \mathbb{R}^{n-1}$, we define the n-level correlation of the set B to equal

$$\frac{1}{N}\#\left\{(\lambda_{j_1}-\lambda_{j_2},\dots,\lambda_{j_{n-1}}-\lambda_{j_n}) \in Q : 1 \le j_1,\dots,j_n \le N, j_{i_1} \neq j_{i_2} \text{ if } i_1 \neq i_2\right\} \tag{1.2.1}$$

One can also define, more generally, the n-level correlation for an arbitrary test function. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying

$$f(x + t(1,\dots,1)) = f(x) \qquad \text{for } t \in \mathbb{R}.$$

Then the n-level correlation of the set B with respect to the test function f is defined by

$$R_B^{(n)}(f) = \frac{1}{N} \sum_{\substack{1 \le j_1,\dots,j_n \le N \\ \text{distinct}}} f(\lambda_{j_1},\dots,\lambda_{j_n}).$$

Notice that (1.2.1) is equal to $R_B^{(n)}\left(\chi_Q(x_1-x_2,\dots,x_{n-1}-x_n)\right)$ where χ_Q is the characteristic function of Q.

Katz and Sarnak [17] prove that the n-level correlations of all of the classical compact groups (as $N \to \infty$) are, for typical elements of the group, the same as that of the GUE.

More precisely, assume that $f : \mathbb{R}^n \to \mathbb{R}$, is bounded, Borel measurable, symmetric in its variables, satisfies $f(x+t(1,\dots,1)) = f(x)$ for $t \in \mathbb{R}$, and is supported in some neighborhood of the diagonal $x_1 = \dots = x_n$ (i.e. there is an $\alpha \geq 0$ such that $f(x) = 0$ whenever $\max_{j,k} |x_j - x_k| > \alpha$).

Let $G(N)$ denote any of the following groups of matrices: U(N), SU(N), O(N), SO(N), USp(N).

For a matrix A in one of the classical compact groups, write its eigenvalues as $\lambda_j = e^{i\theta_j}$, with

$$0 \leq \theta_1 \leq \dots \leq \theta_N < 2\pi, \tag{1.2.2}$$

and let $B_A = \{\theta_1 N/(2\pi), \dots, \theta_N N/(2\pi)\}$. The normalization by $N/(2\pi)$ is such that the mean spacing between neighboring elements of B_A is equal to 1.

Then [17]

$$\int_{G(N)} R_{B_A}^{(n)}(f) dA \sim \frac{1}{N} \int_0^N \cdots \int_0^N W_U^{(n)}(x) f(x) dx \tag{1.2.3}$$

with dA the Haar measure on $G(N)$ (normalized so that $\int_{G(N)} dA = 1$) and where

$$W_U^{(n)}(x) = \det \left(\frac{\sin \pi (x_j - x_k)}{\pi (x_j - x_k)} \right)_{\substack{1 \leq j \leq n \\ 1 \leq k \leq n}} \tag{1.2.4}$$

($W_U^{(n)}(x)$ is also the correlation function of the GUE).

Returning to zeros of L-functions, we let

$$N_\pi(T) = \# \{\gamma_\pi : |\gamma_\pi| < T\}$$

denote the number of non-trivial zeros of $L(S, \pi)$ with $|\gamma_\pi| < T$. Then, as $T \to \infty$ [33]

$$N_\pi(T) \sim \frac{M}{\pi} T \log T. \tag{1.2.5}$$

Label the zeros as follows

$$\dots \Re\gamma_\pi^{(-2)} \leq \Re\gamma_\pi^{(-1)} < 0 \leq \Re\gamma_\pi^{(1)} \leq \Re\gamma_\pi^{(2)} \leq \dots .$$

and let

$$\tilde{\gamma}_\pi^{(j)} = \gamma_\pi^{(j)} \frac{M}{2\pi} \log |\gamma_j| \tag{1.2.6}$$

(we have normalized so that the average vertical spacing between neighboring zeros is equal to 1).

Rudnick and Sarnak [33] prove (under certain assumptions) that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\le |j_1|,\dots,|j_n|\le N \\ j_{i_1}\neq j_{i_2}}} f(\tilde{\gamma}_\pi^{(j_1)}, \dots, \tilde{\gamma}_\pi^{(j_n)}) \\ &= \lim_{N\to\infty} \frac{1}{N} \int_{-N}^{N} \cdots \int_{-N}^{N} W_U^{(n)}(x) f(x) dx. \end{aligned} \tag{1.2.7}$$

The assumptions used are:

- $f(x + t(1, \dots, 1)) = f(x)$ for $t \in \mathbb{R}$.
- f smooth and symmetric in $x_1, \dots, x_n$; $|f(x)| \to 0$ rapidly as $|x| \to \infty$ in the hyperplane $\sum_{j=1}^n x_j = 0$.
- $\hat{f}(u_1, \dots, u_n) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x\cdot u} dx$ is supported in $\sum_{j=1}^n |u_j| < 2/M$.
- RH for $L(s, \pi)$.

If one removes the latter assumption (RH), one can prove a smoothed version of (1.2.7). Presumably, (1.2.7) also holds for $f = \chi_Q(x_1 - x_2, \dots, x_{n-1} - x_n)$ (see Section 1.4.2 for numerical evidence in favor of this statement).

Thus, while (1.2.7) points towards a spectral interpretation of the zeros, it does not tell us whether the underlying operator is Hermitian or in one of the classical compact groups. Analogs with function field zeta functions [16] [17] point towards the classical compact groups being the basic symmetry, and the question arises whether there is another statistic that can be used to obtain further information.

1.3 n-level density

Again, let A be an $N \times N$ matrix in one of the classical compact groups, and denote its eigenvalues as in (1.2.2).

Assume that $f : \mathbb{R}^n \to \mathbb{R}$, is bounded, Borel measurable, and is compactly supported. Then, letting

$$H^{(n)}(A,f) = \sum_{\substack{1\le j_1,\dots,j_n\le N\\ \text{distinct}}} f\left(\theta_{j_1}N/(2\pi),\dots,\theta_{j_n}N/(2\pi)\right),$$

Katz and Sarnak [17, Lemma AD.4.2] obtain the following family dependent result:

$$\lim_{N\to\infty}\int_{G(N)} H^{(n)}(A,f)dA = \int_{\mathbb{R}^n_{\ge 0}} W_G^{(n)}(x)f(x)dx \qquad (1.3.1)$$

for the following families:

G	$W_G^{(n)}$
$\mathrm{U}(N)$,$\mathrm{U}_\kappa(N)$	$\det\left(K_0(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}}$
$\mathrm{USp}(N)$	$\det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}}$
$\mathrm{SO}(2N)$, $\mathrm{O}^-(2N+1)$	$\det\left(K_1(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}}$
$\mathrm{SO}(2N+1)$, $\mathrm{O}^-(2N)$	$\det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}} + \sum_{\nu=1}^n \delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\ne\nu\le n\\1\le k\ne\nu\le n}}$

with

$$K_\varepsilon(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \varepsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}.$$

Here

$$\begin{aligned}\mathrm{U}_\kappa(N) &= \{A\in\mathrm{U}(N) : \det{}^\kappa(A)=1\}\\ \mathrm{SO}(N) &= \{A\in\mathrm{O}(N) : \det A = 1\}\\ \mathrm{O}^-(N) &= \{A\in\mathrm{O}(N) : \det A = -1\}.\end{aligned}$$

The delta functions in the $\mathrm{SO}(2N+1)$, $\mathrm{O}^-(2N)$ case are accounted for by the eigenvalue $\lambda_1 = 1$ (notice, for $\mathrm{O}(N)$, that $\lambda = 1$ is an eigenvalue if N is even and $\det A = -1$ (i.e. $A \in \mathrm{O}^-(2N)$), or if N is odd and $\det A = 1$ (i.e. if $A\in\mathrm{SO}(2N+1)$). Removing this zero from (1.3.1) would yield the same $W_G^{(n)}$ as for USp. For ease of notation, we refer to the third $W_G^{(n)}$ above (i.e. $\det\left(K_1(x_j,x_k)\right)$) as the scaling density of O^+ and the fourth $W_G^{(n)}$ as the scaling density of O^- (this notation, because the former comes from orthogonal matrices with even functional equations, $p(z) = z^N p(1/z)$, while the latter comes from orthogonal matrices with odd functional equations $p(z) = -z^N p(1/z)$).

One could also form a similar statistic for the eigenvalues of the GUE (where we would normalize the eigenvalues according to the Wigner semi-circle law), and obtain the same answer (as $N \to \infty$) as for $\mathrm{U}(N)$.

The function $W_G^{(n)}(x)$ is called the n-level scaling density of the group $G(N)$, and its non-universality can be used to detect which group lies behind which family L-function.

Notice that the normalization by $N/(2\pi)$ is such that the mean spacing is 1 and that only the low lying eigenvalues (those with $\theta \leq c/N$ for some constant c) contribute to $H^{(n)}(A, f)$. So, (1.3.1) measures how the low lying eigenvalues of matrices in $G(N)$ fall near the point 1 (as $N \to \infty$).

1.4 Thesis results

1.4.1 n-level density for families of L-functions

In Chapter 2 of this thesis, I consider the analog of (1.3.1) for the zeros of families of L-functions. One looks at the average behaviour of the low lying non-trivial zeros (i.e. those close to the real axis) of a family of L-functions hoping to find evidence (as predicted by functional field analogs [16]) in favor of a spectral interpretation in terms of the classical compact groups.

Indeed, if we take quadratic twists of $\zeta(s)$, $\{L(s, \chi_d)\}$, as our family of L-functions, where $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's symbol and we restrict ourselves to primitive χ_d, we find evidence of a $\mathrm{USp}(\infty)$ symmetry. This is Theorem 2.1.

More generally, we take a self contragredient automorphic cuspidal representations of GL_M over $\mathbb{Q}$, $\pi = \tilde{\pi}$, i.e. one whose L-function has real coefficients, $\alpha_\pi(p, j) \in \mathbb{R}$, and look at the family of quadratic twists, $\{L(s, \pi \otimes \chi_d)\}$. The low lying zeros of this family behave as if they are coming from either $\mathrm{USp}(\infty)$ or from $\mathrm{O}_\pm(\infty)$ (here the $\pm$ is to indicate that we need to consider seperately the $L(s, \pi \otimes \chi_d)$'s with even (resp. odd) functional equations). We describe this result in Theorem 2.2. It confirms the connection to the classical compact groups, and gives an answer that cannot be confused with the corresponding statistic for the GUE.

One could also look at families of higher order twists, such as twists by cubic characters. In those cases we expect, from the form of the functional equation and from analogs with function field zeta functions, to see evidence of $\mathrm{U}_\kappa(\infty)$. We outline, in Section 2.7 the modifications required to the proof of Theorem 2.2 for such families.

1.4.2 Numerical experiments

Two kinds of experiments were performed. The first was designed to compare the n-level correlations of the zeros of a fixed L-function against that predicted by (1.2.7) (with f equal to a characteristic function). The second experiment aimed at verifying that the n-level densities of families of L-functions reflect those of the classical compact groups (as just described). We call these two experiments the T-experiment and the d-experiment since, in the former, we look at zeros of a fixed L-function with $|\gamma| < T$, and, in the latter, we look at zeros of families of L-functions $\{L(s, \pi \otimes \chi_d)\}$.

The T-experiment

Because Odlyzko [25] [26] and Rumely [34] have performed the T-experiment for $\zeta(s)$ and Dirichlet L-functions respectively, my computations for the T-experiment were restricted to $GL_2/\mathbb{Q}$ L-functions.

We collected the first several thousand zeros of four different L-functions: three elliptic curve L-functions, and the L-function associated to Ramanujan's τ function. To compute these L-functions we used the algorithm described in Chapter 3.

A description of elliptic curve L-functions is given on page 72. The three elliptic curves examined were

$$
\begin{aligned}
E_{11} &: \; y^2 + y = x^3 - x^2 - 10x - 20 \\
E_{37} &: \; y^2 + y = x^3 - x \\
E_{36} &: \; y^2 = x^3 + 1
\end{aligned}
$$

These have conductors $Q = 11, 37, 36$ respectively (computed using the `PARI` number theory package), hence the notation. The ranks are $r = 0, 1, 0$ respectively (the first two are computed in Fermigier [11], and the last in Birch and Swinnerton-Dyer [3]). Furthermore, E_{36} is a curve with complex multiplication.

The Ramanujan L-function, $L_\tau(s)$, has a Dirichlet series given by

$$L_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{11/2}} n^{-s}, \qquad \Re s > 1$$

where

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

The Dirichlet coefficients were obtained, in the case of $L_\tau(s)$, by taking the 8th power of a truncated form of the Jacobi identity

$$\prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}$$

and, in the case of elliptic curve L-functions, using `PARI`. Note that the above identity allows us to compute the first B $\tau(n)$'s with $O(B^{3/2})$ basic arithmetic operations.

Zeros were found by looking for sign changes of the L-function on the critical line (after rotating it suitably so as to be real on the critical line).

We used the formulae

$$
\begin{aligned}
N_\tau(T) &\sim \frac{2T}{\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{11}{2} \\
N_{E_Q}(T) &\sim \frac{2T}{\pi} \log\left(\frac{Q^{1/2} T}{2\pi e}\right) + \frac{1}{2}
\end{aligned}
$$

to verify that all zeros had been found, though not in any rigorous fashion. In principle, using Turing's method [9], this could be turned into a proof that none of the zeros to height slightly less than T were off the critical line. (note that, because E_{37} has rank 1, when comparing against the zeros with $\gamma > 0$ one should evaluate $\frac{1}{2}(N_{E_{37}}(T) - 1)$ to account for the simple zero at $s = 1/2$).

Normalizing the zeros

$$\tilde{\gamma} = \frac{\gamma}{\pi} \log\left(\frac{Q^{1/2}\gamma}{2\pi e}\right), \qquad \gamma > 0 \tag{1.4.1}$$

(for $L_\tau(s)$ take $Q = 1$) we compared

$$\frac{(\beta-\alpha)^{-1}}{N} \sum_{\substack{1\le j_1, j_2 \le N \\ \text{distinct}}} \chi_{[\alpha,\beta)}\left(\tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2}\right), \qquad [\alpha,\beta) = [0, 1/12), \ldots [47/12, 48/12) \tag{1.4.2}$$

with $1 - (\sin(\pi x)/(\pi x))^2$ (for the first N zeros with $\gamma > 0$). The results are displayed in Figure 1.1.

Remark : *For numerical comparisons, (1.4.1) provides a more accurate normalization than (1.2.6).*

From a graphical point of view, it is hard to display information concerning the higher order correlations ($n \ge 3$). Instead one can look at a statistic that involves knowing all the n-level correlations (for characteristic functions) [17], namely the nearest neighbor spacings distribution. In Figure 1.2 we display

$$\frac{(\beta-\alpha)^{-1}}{N-1} \sum_{1\le j \le N-1} \chi_{[\alpha,\beta)}\left(\tilde{\gamma}_{j+1} - \tilde{\gamma}_{j}\right), \qquad [\alpha,\beta) = [0, 1/12), \ldots [35/12, 36/12) \tag{1.4.3}$$

against the nearest neighbor spacings distribution of (any of) the classical compact groups (which coincides with that of the GUE). Numerical values for this curve were obtained from Andrew Odlyzko (see [25] for a description of how this was computed).

The d-experiment

Zeros were collected for quadratic twists of $\zeta(s)$ and $L_\tau(s)$. Let

$$L(s,\chi_d) = \sum_{n=1}^{\infty} \chi_d(n) n^{-s}, \qquad \Re s > 1$$
$$L_\tau(s,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)\tau(n)}{n^{11/2}} n^{-s}, \qquad \Re s > 1$$

and

$$\Lambda(s,\chi_d) = (q/\pi)^{s/2}\Gamma((s+\mathfrak{a})/2)L(s,\chi_d), \qquad \text{where } \mathfrak{a} = (1-\chi_d(-1))/2$$
$$\Lambda_\tau(s,\chi_d) = (q/(2\pi))^{s}\Gamma(s+11/2)L_\tau(s,\chi_d),$$

where $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's symbol. Then Λ and Λ_τ extend to entire functions and satisfy the functional equations

$$\Lambda(s,\chi_d) = \Lambda(1-s,\chi_d)$$
$$\Lambda_\tau(s,\chi_d) = \chi_d(-1)\Lambda_\tau(1-s,\chi_d).$$

(see [6], [4]). Note that $L_\tau(s,\chi_d)$ has a zero at $s = 1/2$ if $\chi_d(-1) = -1$.

We collected the first few zeros of various $L(s,\chi_d)$'s and $L_\tau(s,\chi_d)$'s by looking for sign changes of the L-function (rotated so as to be real) on the critical line. No attempt was made to verify RH for the L-functions in the d-experiment.

For $L(s,\chi_d)$ we collected the low lying zeros for the following sets of d's:

$$\begin{aligned} D = &\{2 < |d| < 90002 : |d| \text{ prime}\}, \\ &\left\{10^9 < |d| < 10^9 + 385414 : |d| \text{ prime}\right\}, \\ &\left\{10^{12} < |d| < 10^{12} + 200000 : |d| \text{ prime}\right\}, \end{aligned} \tag{1.4.4}$$

and for $L_\tau(s,\chi_d)$:

$$\begin{aligned} D_+ = &\{10000 < |d| < 166082 : \chi_d(-1) = 1, |d| \text{ prime}\}, \\ &\{350000 < |d| < 650000 : \chi_d(-1) = 1, |d| \text{ prime}\}, \\ D_- = &\{10000 < |d| < 166100 : \chi_d(-1) = -1, |d| \text{ prime}\}, \\ &\{350000 < |d| < 650000 : \chi_d(-1) = -1, |d| \text{ prime}\}, \end{aligned} \tag{1.4.5}$$

(i.e., for $L_\tau(s,\chi_d)$, we consider even twists seperately from odd twists, since the latter always have a zero at $s = 1/2$).

To compute $L(s,\chi_d)$ and $L_\tau(s,\chi_d)$ we used the algorithm described in Chapter 3. The $\tau(n)$'s were evaluated as in the T-experiment, and $\chi_d(n)$ was computed using quadratic reciprocity.

Because the number of operations required to compute $L(s,\chi_d)$ grows with $|d|^{1/2}$, but with $|d|$ for $L_\tau(s,\chi_d)$, the data obtained is much more extensive for $L(s,\chi_d)$.

According to Theorem 2.2, we expect, for $L(s,\chi_d)$, the low lying zeros to behave like the low lying eigenvalues of matrices in $\mathrm{USp}(N)$, N large. For $L_\tau(s,\chi_d)$ we expect to see evidence of either O^+ or O^- according to whether $\chi_d(-1) = 1$ or -1 respectively.

Normalizing the zeros

$$\begin{aligned}\tilde{\gamma} &= \frac{\gamma}{2\pi}\log(|d|/\pi), &\quad \text{for } L(s,\chi_d)\\ \tilde{\gamma} &= \frac{\gamma}{\pi}\log(|d|/\pi), &\quad \text{for } L_\tau(s,\chi_d)\end{aligned}$$

we compared, for $L(s,\chi_d)$,

$$\frac{(\beta-\alpha)^{-1}}{|D|}\sum_{d\in D}\sum_{\gamma_d}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d\right), \qquad [\alpha,\beta) = [0,1/12),\ldots[23/12,24/12)$$

to the 1-level density of $\mathrm{USp}(\infty)$, namely $1-\sin(2\pi x)/(2\pi x)$. We also looked at the distribution of the lowest lying zero, $\tilde{\gamma}_d^{(1)}$,

$$\frac{(\beta-\alpha)^{-1}}{|D|}\sum_{d\in D}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d^{(1)}\right), \qquad [\alpha,\beta) = [0,1/12),\ldots[35/12,36/12)$$

and of the 2nd lowest lying zero, $\tilde{\gamma}_d^{(2)}$,

$$\frac{(\beta-\alpha)^{-1}}{|D|}\sum_{d\in D}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d^{(2)}\right), \qquad [\alpha,\beta) = [0,1/12),\ldots[47/12,48/12).$$

We compared these to the density functions of the corresponding distributions for $\mathrm{USp}(\infty)$. These are given [17] by

$$\nu_1(\mathrm{USp})(t) = -\frac{d}{dt}E_{-,0}(t) \tag{1.4.6}$$

and

$$\nu_2(\mathrm{USp})(t) = -\frac{d}{dt}(E_{-,0}(t) + E_{-,1}(t)) \tag{1.4.7}$$

respectively, where

$$E_{-,0}(t) = \prod_{j=0}^{\infty}(1 - \lambda_{2j+1}(2t))$$
$$E_{-,1}(t) = \sum_{k=1}^{\infty} \lambda_{2k+1}(2t) \prod_{j=0, j\neq k}^{\infty} (1 - \lambda_{2j+1}(2t)).$$

Here, the $\lambda_j(t)$'s are the eigenvalues of the integral equation

$$\int_{-t/2}^{t/2} (\sin(\pi(x-y))/\pi(x-y)) f(y) dy = \lambda(t) f(x)$$

and

$$1 \geq \lambda_0(s) \geq \lambda_1(s) \geq \lambda_2(s) \geq \dots .$$

They were computed using the same program, obtained from Andrew Odlyzko, as in [25].

The above suggests that the means of the 1st (resp. 2nd) lowest lying zeros are

$$\frac{1}{|D|}\sum_{d\in D}\tilde{\gamma}_d^{(1)} \to \int_0^{\infty} t\nu_1(\mathrm{USp})(t)dt = .78\dots$$
$$\frac{1}{|D|}\sum_{d\in D}\tilde{\gamma}_d^{(2)} \to \int_0^{\infty} t\nu_2(\mathrm{USp})(t)dt = 1.76\dots$$

In Figures 1.3, 1.6, 1.7 we display the results of these comparisons for the sets of d's in (1.4.4). Because the means converge slowly to the predicted means, we also display, in the right column of these figures, the same statistics, but with the zeros $\tilde{\gamma}_d$ renormalized

$$.78\left(\frac{1}{|D|}\sum_{d\in D}\tilde{\gamma}_d^{(1)}\right)^{-1}\tilde{\gamma}_d$$

for Figures 1.3 and 1.6, and

$$1.76\left(\frac{1}{|D|}\sum_{d\in D}\tilde{\gamma}_d^{(2)}\right)^{-1}\tilde{\gamma}_d$$

for Figure 1.7.

For $L_\tau(s,\chi_d)$, $\chi_d(-1)=1$, we compare, in Figure 1.4,

$$\frac{(\beta-\alpha)^{-1}}{|D_+|}\sum_{d\in D_+}\sum_{\gamma_d}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d\right), \qquad [\alpha,\beta)=[0,1/12),\ldots[23/12,24/12)$$

to the 1-level density of O^+, namely $1+\sin(2\pi x)/(2\pi x)$. For $\chi_d(-1)=-1$, we compare, in Figure 1.5,

$$\frac{(\beta-\alpha)^{-1}}{|D_-|}\sum_{d\in D_-}\sum_{\gamma_d>0}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d\right), \qquad [\alpha,\beta)=[0,1/12),\ldots[23/12,24/12)$$

to the 1-level density of O^- (with the eigenvalue $\lambda=1$ removed), namely $1-\sin(2\pi x)/(2\pi x)$. Note that no δ function appears since we have removed the zero at $s=1/2$.

We also look at

$$\frac{(\beta-\alpha)^{-1}}{|D_+|}\sum_{d\in D_+}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d^{(1)}\right), \qquad [\alpha,\beta)=[0,1/12),\ldots[35/12,36/12)$$

and

$$\frac{(\beta-\alpha)^{-1}}{|D_-|}\sum_{d\in D_-}\chi_{[\alpha,\beta)}\left(\tilde{\gamma}_d^{(1)}\right), \qquad [\alpha,\beta)=[0,1/12),\ldots[35/12,36/12)$$

comparing these to the density functions of the corresponding statistic for O^+ and O^-. These are given by

$$\nu_1(\mathrm{O}^+)(t) = -\frac{d}{dt}\prod_{j=0}^{\infty}(1-\lambda_{2j}(2t))$$
$$\nu_2(\mathrm{O}^-)(t) = \nu_1(\mathrm{USp})(t)$$

whose means are $.78\ldots$ and $.32\ldots$ respectively.

Finally, for $\mathrm{U}(\infty)$, the analogous density, $\nu_1(\mathrm{U})(t)$ is given by

$$\nu_1(\mathrm{U})(t) = -\frac{d}{dt}\prod_{j=0}^{\infty}(1-\lambda_j(t))$$

and has mean $.59\ldots$. This was used in Figure 1.10.

The algorithm

In Chapter 3 an efficient algorithm for computing any imaginable L-function is given. We only focus in this introduction on a special case that covers all the L-functions mentioned above.

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

be absolutely convergent in a half plane, $\Re(s) > \sigma_1$. Assume that $L(s)$ extends to an entire function and satisifes a functional equation of the form

$$\Lambda(s) = Q^s \Gamma(\gamma s + \lambda) L(s) = \omega \overline{\Lambda(1-\overline{s})}$$

with $Q \in \mathbb{R}^+$, $\gamma \in \{1/2, 1\}$, $\Re\lambda \geq 0$, and $\omega \in \mathbb{C}$, $\omega \neq 0$. One also needs to assume a rate of growth on $L(s)$, but we ignore this detail here. Then, the formula used to compute $L(s)$ was a truncated version of

$$\begin{aligned} Q^s\Gamma(\gamma s+\lambda)L(s)\delta^{-s} &= (\delta/Q)^{\lambda/\gamma} \sum_{n=1}^{\infty} b(n) n^{\lambda/\gamma} G\left(\gamma s + \lambda, (n\delta/Q)^{1/\gamma}\right) \\ &+ \frac{\omega}{\delta}(Q\delta)^{-\overline{\lambda}/\gamma} \sum_{n=1}^{\infty} \overline{b}(n) n^{\overline{\lambda}/\gamma} G\left(\gamma(1-s)+\overline{\lambda}, (n/(\delta Q))^{1/\gamma}\right) \end{aligned} \quad (1.4.8)$$

where δ is a complex valued parameter with $|\delta| = 1$, and $\Re\delta^{1/\gamma} > 0$. As $\Im s$ grows, $\delta^{1/\gamma}$ is chosen to lie within $O(1/|\Im s|)$ of the imaginary axis (for the d-experiment, a value of $\delta = 1$ was used since, in that experiment, we only wanted to collect the low lying zeros). This paramater helps to balance the exponentially small size of $|\Gamma(\gamma s + \lambda)|$ (as $|\Im s| \to \infty$). Without it, one would face great difficulty in computing the zeros of $L(s)$ to large T. The idea of using such a parameter goes back to Hardy, who used it to derive his approximate functional equation.

In the above,

$$G(z, w) = w^{-z}\Gamma(z, w) = \int_1^{\infty} e^{-wx} x^{z-1} dx, \qquad \Re(w) > 0$$

where $\Gamma(z, w)$ is the incomplete gamma function. In order to compute $G(z, w)$, three different expressions were used. The details can be found in Section 3.3.2. After performing some precomputations we are led to a formula which superficially resembles the Riemann-Siegel formula in that there is a main sum and correction terms

$$L(s) = \sum_{n=1}^{\beta} \frac{b(n)}{n^s} + \omega \frac{\Gamma(\gamma(1-s)+\overline{\lambda})}{\Gamma(\gamma s + \lambda)} Q^{1-2s} \sum_{n=1}^{\beta} \frac{\overline{b}(n)}{n^{1-s}} + \text{ corrections}$$

(where β is roughly equal to $Q\,|\gamma s + \lambda|^{\gamma}$). The most time consuming part of the algorithm is in computing the main sum. But even here improvements are possible and we can evaluate many instances of the main sum for essentially the same computational cost of a single evaluation (see page 89).

Tests of correctness

The first test that an algorithm for computing an L-function must pass, is that, once suitably rotated, it be real on the critical line. If, in (1.4.8), δ is complex, only a miraculous combination of bugs would yield real values.

Besides this simple test, the following were also checked. Whenever there was overlap with other people's computations, I verified whether my program produced the same zeros. Fermigier [11] computed the first few zeros ($T = 15$) to four places after the decimal point for many elliptic curve L-functions, including the L-functions associated to E_{11} and E_{37}. My results agree with his. For $L_\tau(s)$, Yoshida [41] computed the zeros to height $T = 100$, and my data agrees to all his 12 decimal places, as well as that of Spira [35] (first 3 zeros to all places listed). Keiper [18] computed the first 5018 zeros of $L_\tau(s)$, and 2028 zeros near the 20001st zero, but his data (obtained in a file from Andrew Odlyzko) agrees with Yoshida's, Spira's, and mine to only the 5th (or more) decimal place. My data also agrees with the Dirichlet L-function computations of Rumely [34] and of Davies and Haselgrove [7].

In the T-experiment, the zeros were checked against the predictions of the asymptotic formula for $N(T)$. Finally, in both the T and d experiments, the data collected agrees with our theoretical predictions, as depicted in Figures 1.1 - 1.10 (however, a lack of agreement could either indicate a problem with our computations or with our theory! So the latter test, alone, would not be very convincing).

References to other computations

Riemann himself computed the first few zeros of $\zeta(s)$, and detailed numerical studies were initiated almost as soon as computers were invented. See Edwards [9] for a historical survey of these computations. To date, the most impressive computations for $\zeta(s)$ have been those of Odlyzko [26] and Van de Lune, te Riele, Winter [39]. The latter were interested in verifying the Riemann Hypothesis and their computations showed that the first $1.5 \cdot 10^9$ nontrivial zeros of $\zeta(s)$ fall on the critical line. Odlyzko's computations were more concerned with examining the distribution of the spacings between neighboring zeros, though the Riemann Hypothesis was also checked for the intervals examined. In [26], Odlyzko computed 79 million consecutive zeros of $\zeta(s)$ lying near the 10^{20}th zero (and more recently, about half a billion zeros in a slightly higher region). The Riemann-Siegel formula has been at the heart of these computations.

Dirichlet L-functions were not computed on machines until 1961 when Davies and Haselgrove [7] looked at several such L-functions with moduli ≤ 163. More recently, Rumely [34] using summation by parts, computed the first several thousand zeros for many Dirichlet L-functions with small moduli [34]. He both verified RH and looked at statistics of neighboring zeros.

Yoshida [42] [41] has also used summation by parts (though in a different manner) to compute the first few zeros of certain higher degree ($M = 2, 3, 4$) L-functions.

Lagarias and Odlyzko [21] have computed the low lying zeros of several Artin L-functions. More recently, Fermigier [11] computed the zeros to height $T = 15$, of several hundred L-functions attached to elliptic curves. Both use expansions involving the incomplete gamma function. Lagarias and Odlyzko also note that one could compute higher up in the critical strip by introducing the parameter δ (as explained above) but did not implement it since it led to difficulties (which are overcome in this thesis) concerning computing $G(z, w)$ with both z and w complex.

Other computations of L-functions included those of Berry and Keating [2] ($\zeta(s)$), Paris [30] ($\zeta(s)$), Tollis [38] (Dedekind zeta functions), Spira [35] ($L_\tau(s)$), and Keiper [18] ($L_\tau(s)$).

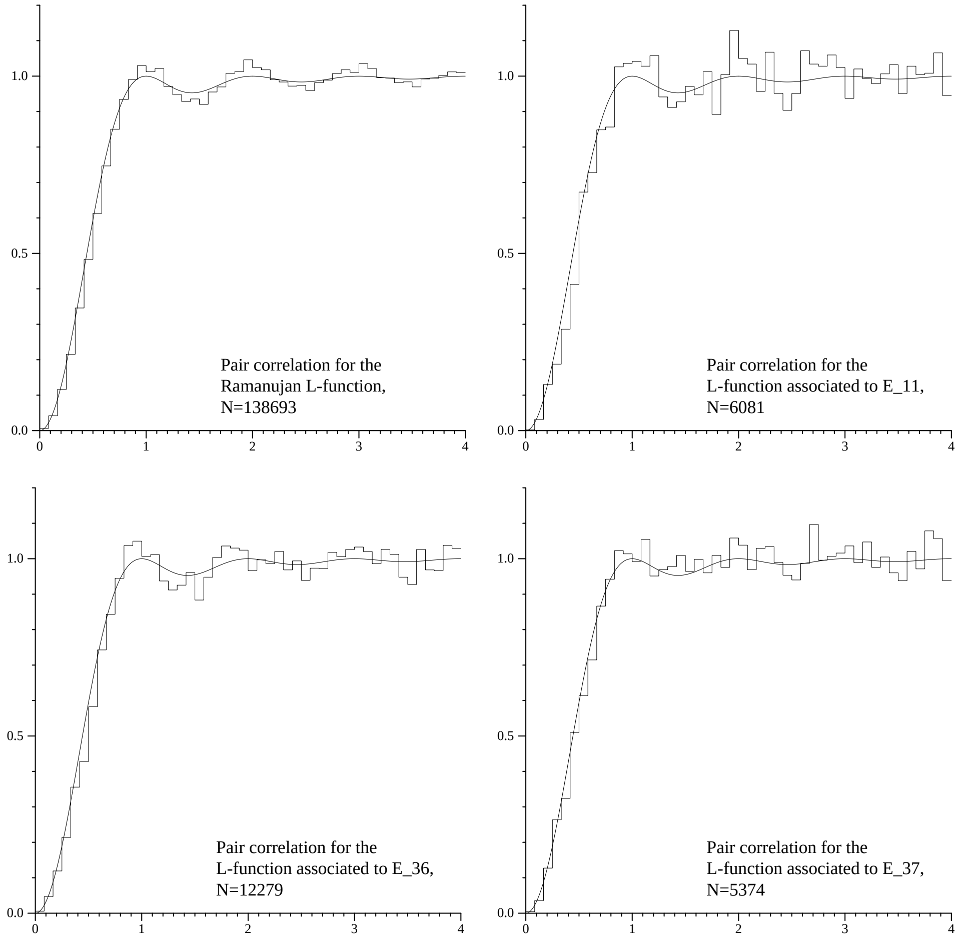

Figure 1.1: Pair correlations for the four L-functions, compared to $1-(\sin(\pi x)/(\pi x))^2$. The most extensive data is for $L_\tau(s)$ (138693 zeros). These fits are comparable to the pictures we get from $\zeta(s)$ using the same number of zeros.

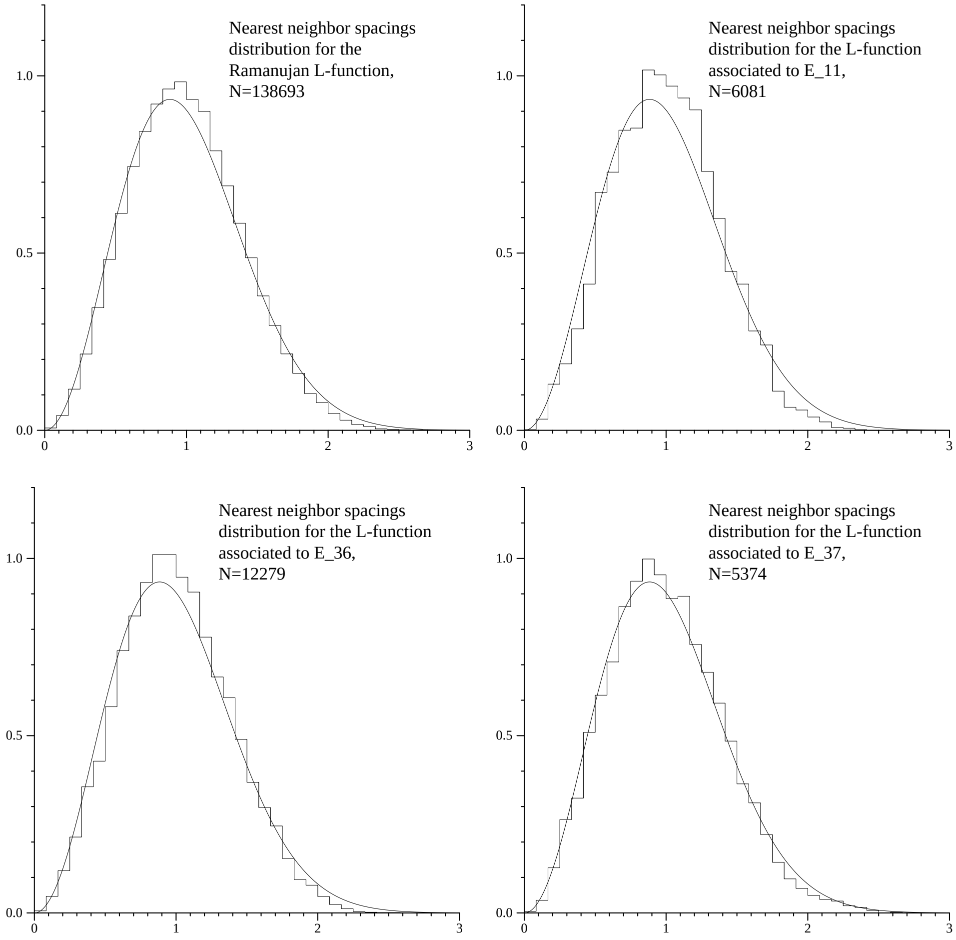

Figure 1.2: Nearest neighbor spacings distribution for the four L-functions. Again, the quality of the fits are comparable to that of $\zeta(s)$ for the same number of zeros.

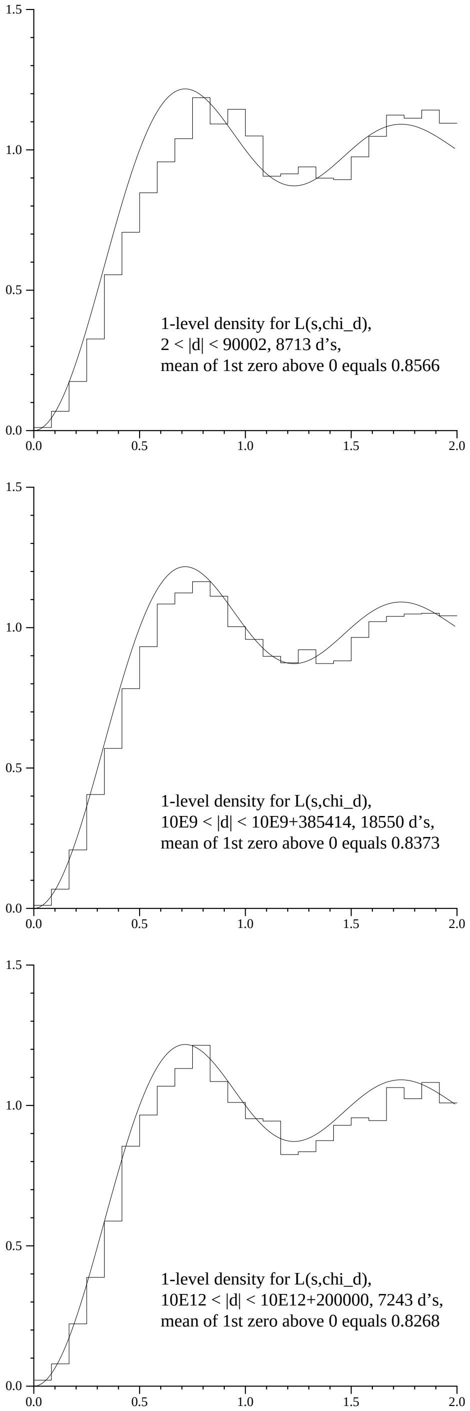

Figure 1.3: 1-level density for $L(s, \chi_d)$, compared to $1 - \sin(2\pi x)/(2\pi x)$.

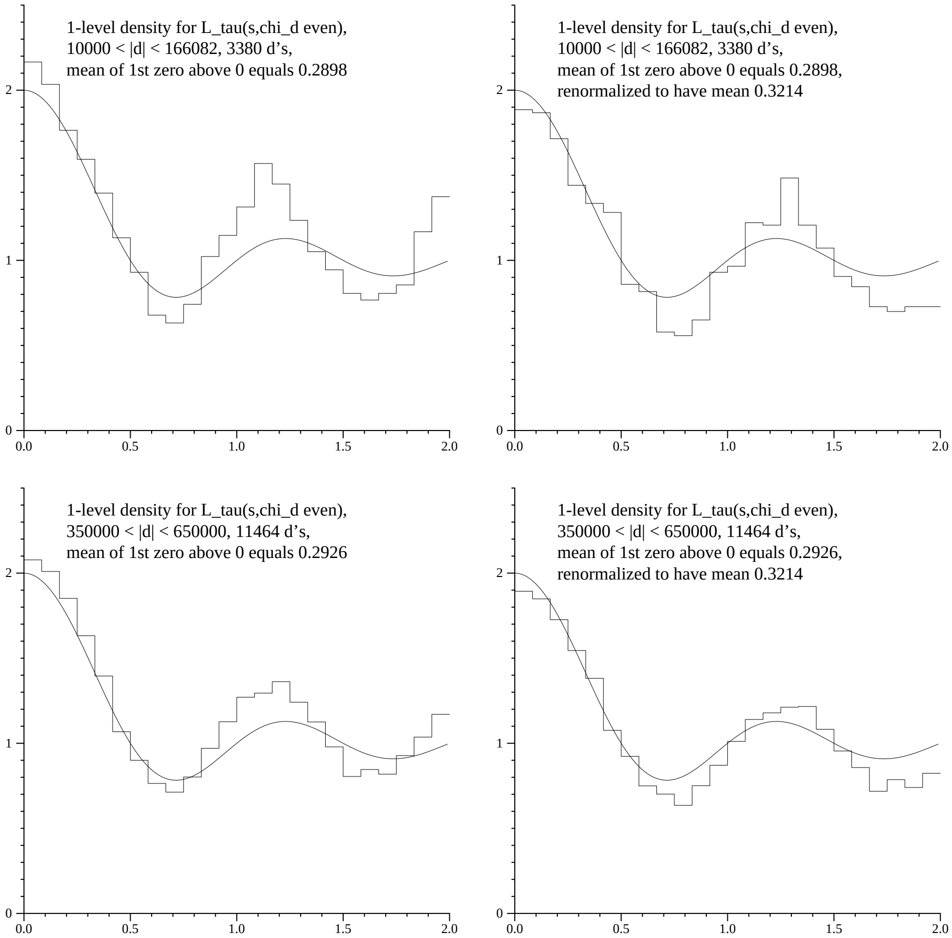

Figure 1.4: 1-level density for $L_\tau(s,\chi_d)$, even twists, compared to $1+\sin(2\pi x)/(2\pi x)$.

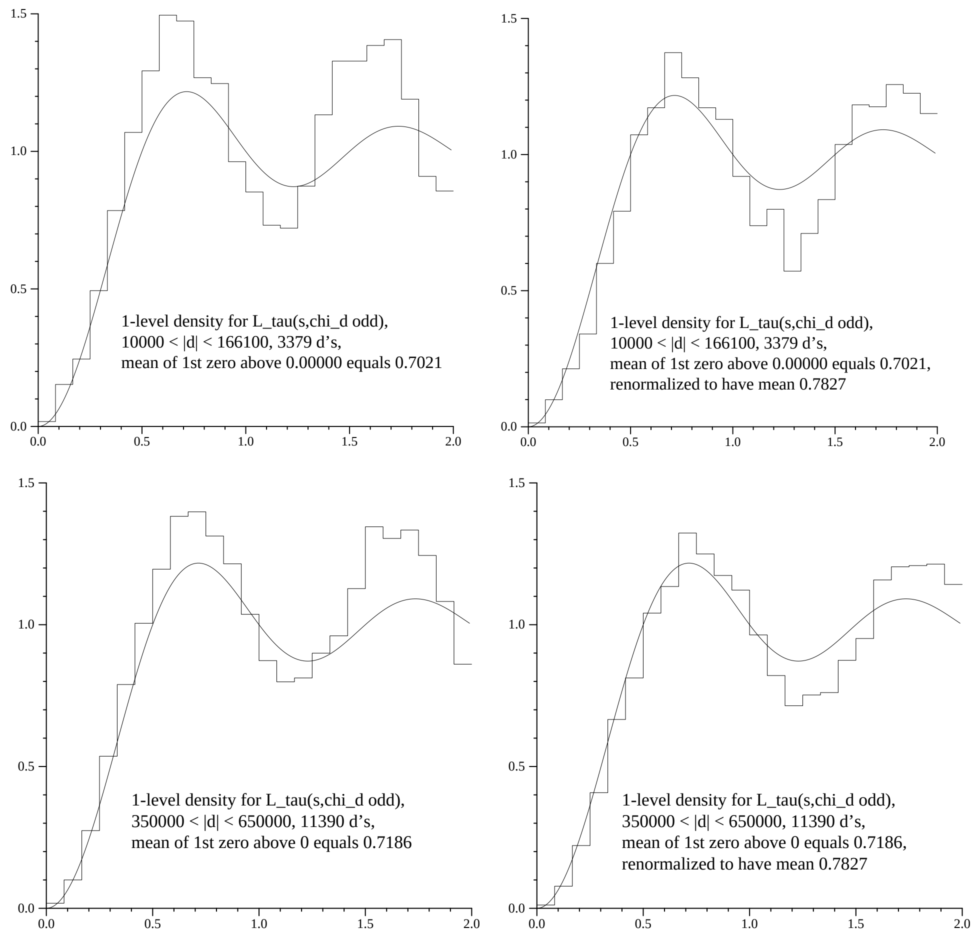

Figure 1.5: 1-level density for $L_\tau(s,\chi_d)$, odd twists, compared to $1-\sin(2\pi x)/(2\pi x)$. We have ignored the zero at $\gamma = 0$ so as not to get a δ function at the origin.

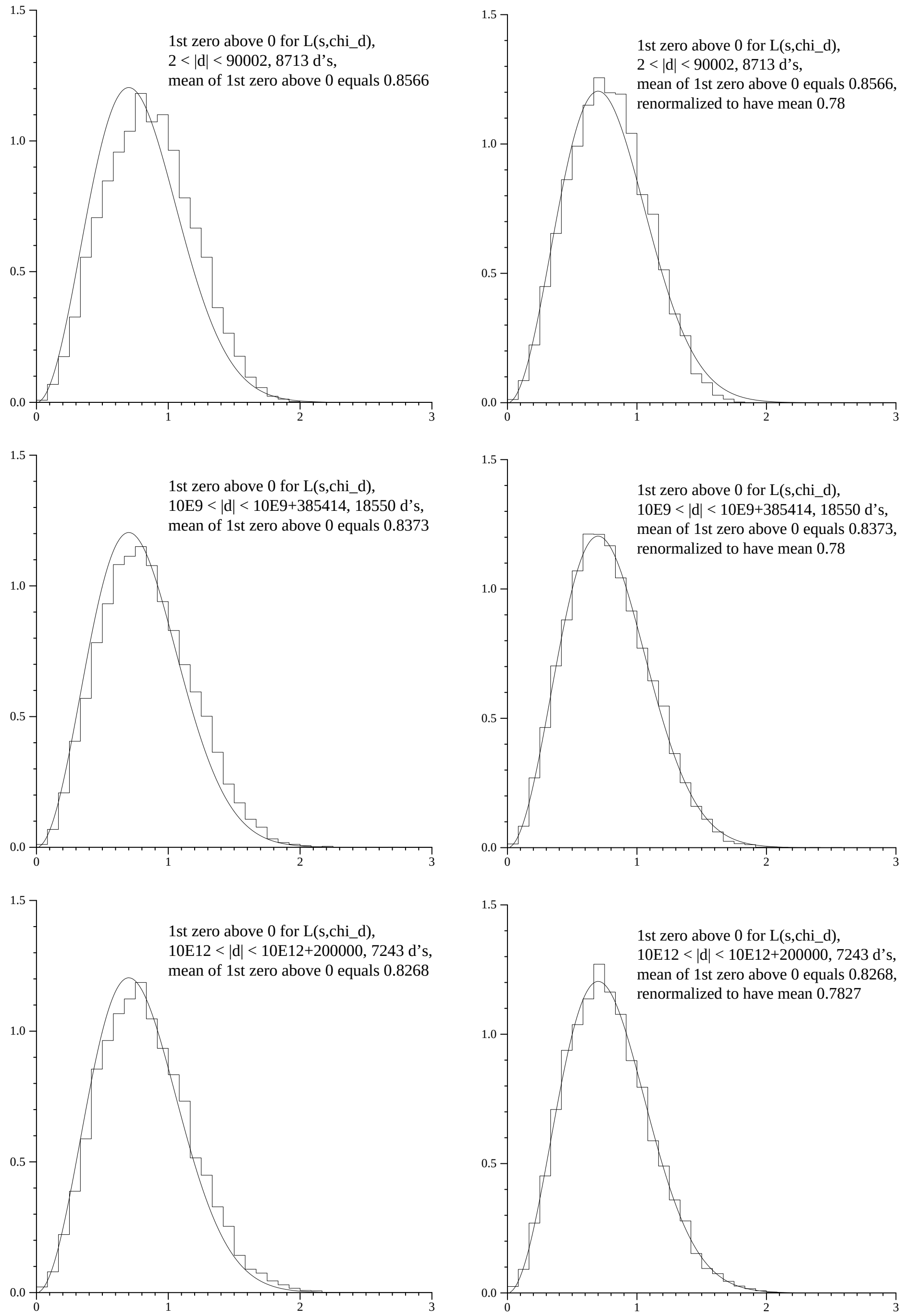

Figure 1.6: Distribution of the 1st zero above 0 for $L(s, \chi_d)$, as compared to $\nu_1(\mathrm{USp})(t)$.

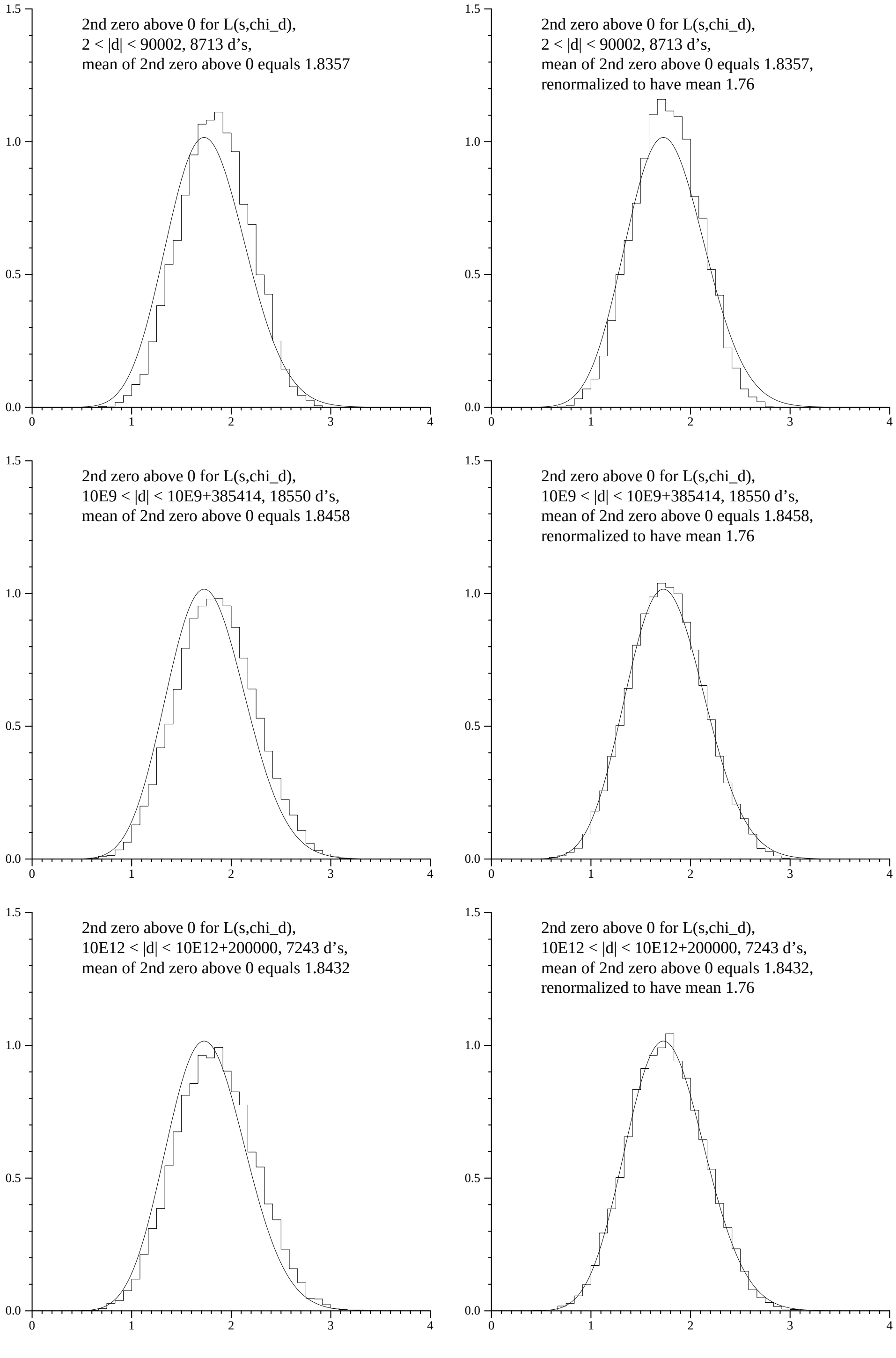

Figure 1.7: Distribution of the 2nd zero above 0 for $L(s, \chi_d)$, as compared to $\nu_2(\mathrm{USp})(t)$.

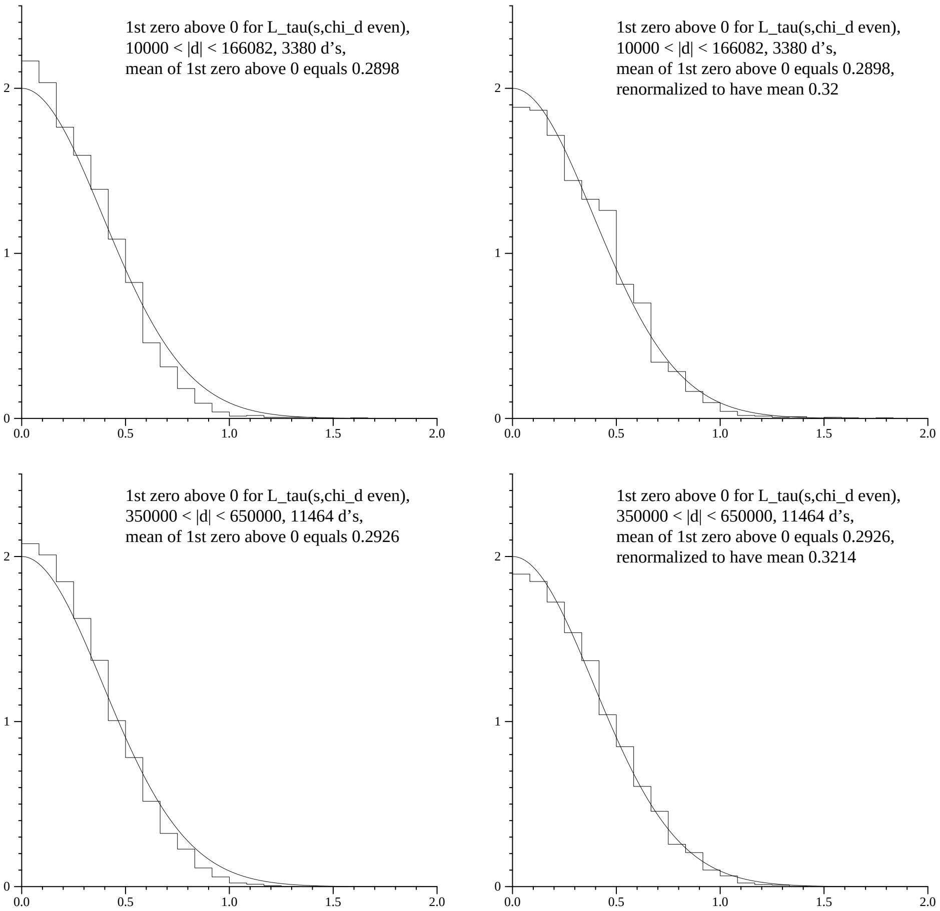

Figure 1.8: Distribution of the 1st zero above 0 for even quadratic twists of $L_\tau(s)$, as compared to $\nu_1(\mathrm{O}^+)(t)$

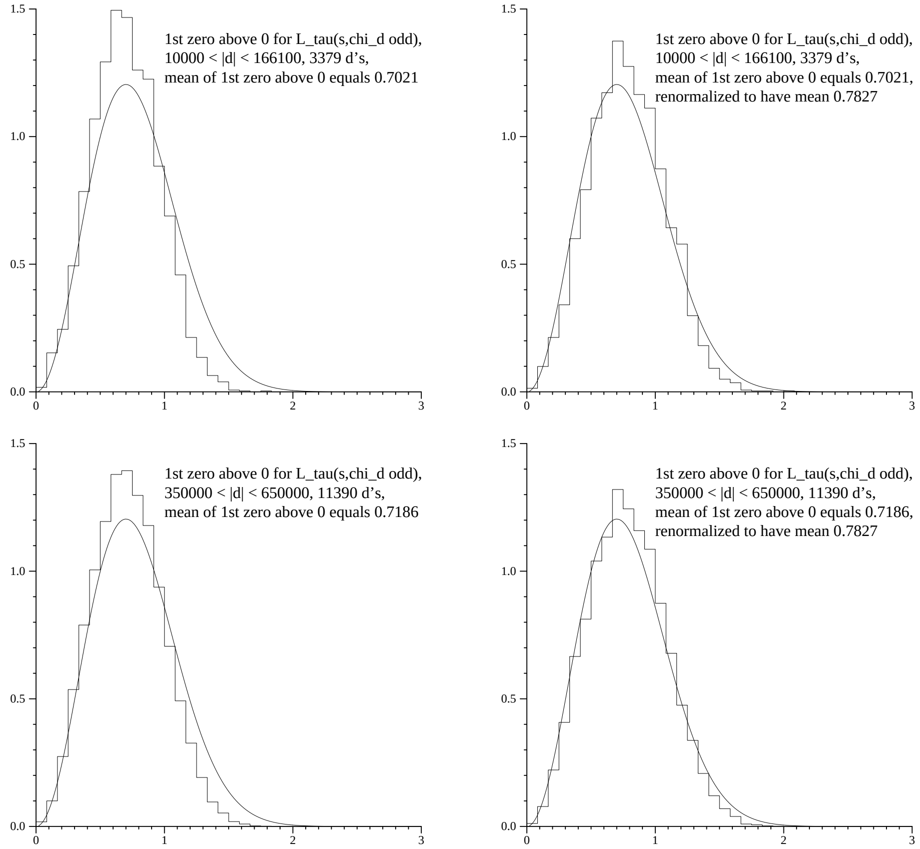

Figure 1.9: Distribution of the 1st zero above 0 for odd quadratic twists of $L_\tau(s)$, as compared to $\nu_2(\mathrm{O}^-)(t)$.

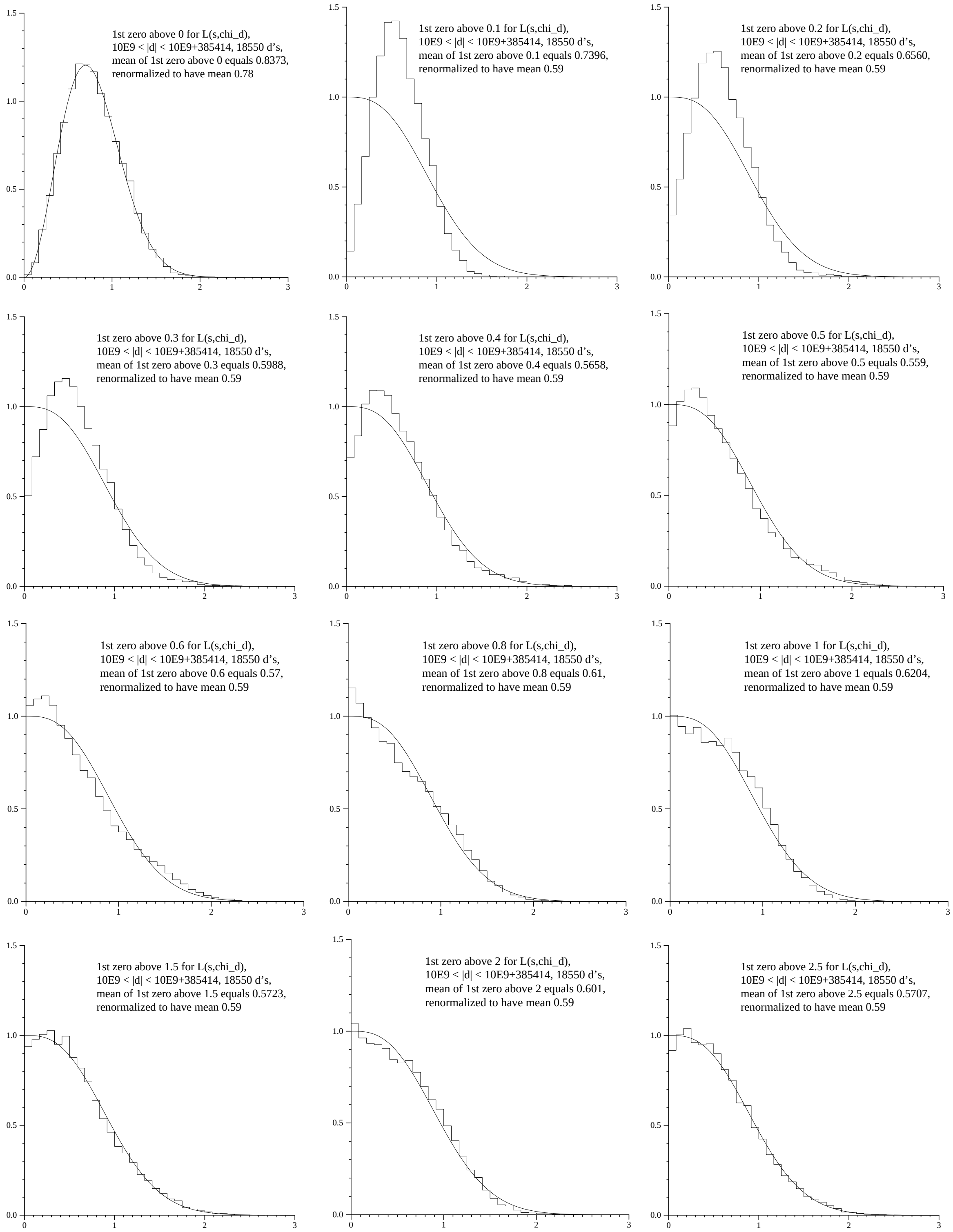

Figure 1.10: 1st zero above 'h' for $L(s, \chi_d)$. We see the picture changing from USp to U. This illustrates that the central point $s = 1/2$ is essentially the only place where the distinct behavior of $\{L(s, \chi_d)\}$ can be detected. Away from $s = 1/2$, we get a universal answer, i.e. that of $\mathrm{U}(\infty)$.

Chapter 2

n-level density

2.1 Main Theorem

Write the non-trivial zeros of $L(s, \chi_d)$ as

$$1/2 + i\gamma_d^{(j)}, \qquad j = \pm 1, \pm 2, \ldots$$

where

$$0 \leq \Re\gamma_d^{(1)} \leq \Re\gamma_d^{(2)} \leq \Re\gamma_d^{(3)} \ldots$$

and

$$\gamma_d^{(-j)} = -\gamma_d^{(j)}. \tag{2.1.1}$$

Here $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's symbol and we restrict ourselves to primitive χ_d. Let D denote the set of such d's, and let $D(X) = \{d \in D : X/2 \leq |d| < X\}$.

Notice that we are *not* assuming the Riemann Hypothesis for $L(s, \chi_d)$ since we allow that the $\gamma_d^{(j)}$'s be complex.

Theorem 2.1: *Let*

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i) \tag{2.1.2}$$

where each f_i is even and in $S(\mathbb{R})$ (i.e. smooth and rapidly decreasing). Assume further that $\hat{f}(u_1, \ldots, u_n) = \prod_{i=1}^{n} \hat{f}_i(u_i)$ is supported in $\sum_{i=1}^{n} |u_i| < 1$, where

$$\hat{f}(u) \stackrel{def}{=} \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot u} dx. \tag{2.1.3}$$

Then

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \sum_{j_1,\cdots,j_n}^{*} f\left(L\gamma_d^{(j_1)}, L\gamma_d^{(j_2)}, \dots, L\gamma_d^{(j_n)}\right)$$
$$= \int_{\mathbb{R}^n} f(x) W_{USp}^{(n)}(x)dx, \tag{2.1.4}$$

where

$$L = \frac{\log X}{2\pi}$$
$$W_{USp}^{(n)}(x_1, \dots, x_n) = \det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\le n \\ 1\le k\le n}}$$
$$K_{-1}(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} - \frac{\sin(\pi(x+y))}{\pi(x+y)}$$

and where $\sum^{*}_{j_1,\cdots,j_n}$ *is over* $j_k = \pm 1, \pm 2, \dots,$ *with* $j_{k_1} \neq \pm j_{k_2}$ *if* $k_1 \neq k_2$.

Plan: We first use the explicit formula to study the l.h.s. of (2.1.4), and end up expressing it in terms of the $\hat{f}_i$'s. Parseval is then applied to the r.h.s. of (2.1.4), and terms are matched with the l.h.s. .

Remark : *The condition f_i even is not essential to the proof, nor is the assumption that f be of the form $\prod f_i$. At the expense of more cumbersome writing, these can be removed.*

2.2 l.h.s.

By (2.1.1), (2.1.2), and since $f_i(-x) = f_i(x)$,

$$\frac{1}{|D(X)|} \sum_{d\in D(X)} \sum_{j_1,\cdots,j_n}^{*} f\left(L\gamma_d^{(j_1)}, L\gamma_d^{(j_2)}, \dots, L\gamma_d^{(j_n)}\right)$$
$$= \frac{2^n}{|D(X)|} \sum_{d\in D(X)} \sum_{\substack{j_1,\cdots,j_n \\ \text{positive} \\ \text{and} \\ \text{distinct}}} \tilde{f}_d(j_1, \dots, j_n) \tag{2.2.1}$$

where

$$\tilde{f}_d(j_1, \dots, j_n) = \prod_{i=1}^{n} f_i\left(L\gamma_d^{(j_i)}\right). \tag{2.2.2}$$

In order to apply the explicit formula to (2.2.1), we need to circumvent the fact that the j_i's are distinct. By combinatorial sieving, as in [33, pg 305], the r.h.s. of (2.2.1) is

$$\frac{2^n}{|D(X)|} \sum_{d \in D(X)} \sum_{\underline{F}} (-1)^{n-\nu(\underline{F})} \left(\prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell| - 1)! \right) w_{\underline{F}}$$

where $\underline{F}$ ranges over all ways of decomposing $\{1, 2, \dots, n\}$ into disjoint subsets $[F_1, \dots, F_\nu]$, and where

$$w_{\underline{F}} = \sum_{\substack{j_1, \dots, j_\nu \\ \text{positive}}} \tilde{f}_d(\ell_{\underline{F}}(j_1, \dots, j_\nu)).$$

Here $\ell_{\underline{F}} : \mathbb{R}^\nu \to \mathbb{R}^n$, $\ell_{\underline{F}}(x_1, \dots, x_\nu) = (y_1, \dots, y_n)$ with $y_i = x_j$ if $i \in \underline{F}_\ell$.

For example, for $n = 3$, the possible $\underline{F}$'s are: $[\{1,2,3\}]$, $[\{1,2\}, \{3\}]$, $[\{1,3\}, \{2\}]$, $[\{2,3\}, \{1\}]$, $[\{1\}, \{2\}, \{3\}]$, and $\ell_{[\{1,3\},\{2\}]}(x_1, x_2) = (x_1, x_2, x_1)$.

Thus, (2.2.1) is

$$\frac{2^n}{|D(X)|} \sum_{d \in D(X)} \sum_{\underline{F}} (-1)^{n-\nu(\underline{F})} \left(\prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell| - 1)! \right) \sum_{\substack{j_1, \dots, j_{\nu(\underline{F})} \\ \text{positive}}} \tilde{f}_d(\ell_{\underline{F}}(j_1, \dots, j_{\nu(\underline{F})}))$$

which, by (2.2.2), equals

$$\frac{2^n}{|D(X)|} \sum_{d \in D(X)} \sum_{\underline{F}} \frac{(-1)^{n-\nu(\underline{F})}}{2^{\nu(\underline{F})}} \prod_{\ell=1}^{\nu(\underline{F})} \left((|F_\ell| - 1)! \sum_{\gamma_d} \prod_{i \in F_\ell} f_i(L\gamma_d) \right). \tag{2.2.3}$$

In the innermost sum, we are going over all $\gamma_d^{(j)}$ (instead of $j > 0$) and hence the $1/2^{\nu(\underline{F})}$. This is justified by (2.1.1) and because we are assuming that the f_i's are even.

Let

$$F_\ell(x) = \prod_{i \in F_\ell} f_i(x). \tag{2.2.4}$$

By the explicit formula (see [33, 2.16], with, in the notation of that paper, $h(r) = F_\ell(Lr)$, $g(y) = (1/\log X)\hat{F}_\ell(-y/\log X)$)

$$\sum_{\gamma_d} F_\ell\left(L\gamma_d^{(j)}\right) = \int_{\mathbb{R}} F_\ell(x)dx + O(1/\log X)$$
$$- \frac{2}{\log X}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{\log X}\right) \tag{2.2.5}$$

(note that $\hat{F}_\ell(x)$ is even since each f_i is even. We have also used the facts that $F_\ell(x)$ is rapidly decreasing and $\Gamma'(s)/\Gamma(s) = O(\log|s|)$, to replace the Γ'/Γ terms in [33, 2.16] by $O(1/\log X)$. Note further that $\hat{F}_\ell$ is compactly supported (see Claim 1, page 34)).

Plugging (2.2.5) into (2.2.3) (without the $O(1/\log X)$ term, a step that is justified in Lemma 2, page 39), we see, on multiplying out the product over ℓ in (2.2.3), that (2.2.3) is

$$\frac{1}{|D(X)|}\sum_{d\in D(X)}\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\prod_{\ell=1}^{\nu(\underline{F})}(|F_\ell|-1)!(C_\ell + D_\ell)$$

where

$$C_\ell = \int_{\mathbb{R}} F_\ell(x)dx$$
$$D_\ell = -\frac{2}{\log X}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{\log X}\right).$$

When we expand the product over ℓ, we obtain $2^{\nu(\underline{F})}$ terms, each a product of C_ℓ's and D_ℓ's. A typical term can be written as

$$\prod_{\ell\in S^c} C_\ell \prod_{\ell\in S} D_\ell$$

for some subset S of $\{1,2,\ldots,\nu(\underline{F})\}$. (empty products are taken to be 1). The product of the C_ℓ's contributes to (2.2.3) a factor of

$$\prod_{\ell\in S^c}\int_{\mathbb{R}} F_\ell(x)dx.$$

The product of the D_ℓ's equals

$$\left(\frac{-2}{\log X}\right)^{|S|}\prod_{\ell\in S}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{\log X}\right)$$

which, by Lemma 1 (below), contributes, a factor of

$$\sum_{\substack{S_2\subseteq S\\|S_2|\text{even}}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)du\right)\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du\right)$$

from which we find that (2.2.3) (and hence (2.2.1)) tends, as $X\to\infty$, to

$$\boxed{\begin{aligned}&\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\left(\prod_{\ell=1}^{\nu(\underline{F})}(|F_\ell|-1)!\right)\sum_{S}\left(\prod_{\ell\in S^c}\int_{\mathbb{R}}F_\ell(x)dx\right)\\&\sum_{\substack{S_2\subseteq S\\|S_2|\text{even}}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)du\right)\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du\right)\end{aligned}} \tag{2.2.6}$$

Here S ranges over all $2^{\nu(\underline{F})}$ subsets of $\{1,2,\ldots,\nu(\underline{F})\}$, and S^c denotes the complement of S. The rest of the notation is as in Lemma 1.

Lemma 1:

$$\begin{aligned}&\lim_{X\to\infty}\frac{1}{|D(X)|}\sum_{d\in D(X)}\left(\frac{-2}{\log X}\right)^{k}\prod_{j=1}^{k}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_{\ell_j}\left(\frac{\log m}{\log X}\right)\\&=\sum_{\substack{S_2\subseteq S\\|S_2|\,even}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)du\right)\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du\right)\end{aligned} \tag{2.2.7}$$

where $S=\{l_1,\ldots,l_k\}$. $\sum_{\substack{S_2\subseteq S\\|S_2|\,even}}$ is over all subsets S_2 of S whose size is even. $\sum_{(A;B)}$ is over all ways of pairing up the elements of S_2. $F_\ell(x)$ is defined in (2.2.4).

For example, if $S=\{1,2,5,7\}$, the possible S_2's are $\emptyset$, $\{1,2\}$, $\{1,5\}$, $\{1,7\}$, $\{2,5\}$, $\{2,7\}$, $\{5,7\}$, $\{1,2,5,7\}$.

And if $S_2 = \{1,2,5,7\}$, then the possible $(A;B)$'s are $(1,2;5,7)$, $(1,2;7,5)$, $(1,5;2,7)$. These correspond, respectively, to matching 1 with 5 and 2 with 7, 1 with 7 and 2 with 5, 1 with 2 and 5 with 7. Note that our notation is not unique. For example, $(1,2;5,7) \equiv (7,1;2,5)$.

Lemma 1 is obtained in a sequence of Claims.

Claim 1: *Suppose that* $\prod_{i=1}^n \hat{f}_i(u_i)$ *is supported in* $\sum_{i=1}^n |u_i| \leq \alpha$. *Then* $\prod_{j=1}^k \hat{F}_{\ell_j}(u_j)$ *is supported in* $\sum_{j=1}^k |u_j| \leq \alpha$.

Proof. By (2.2.4)

$$\begin{aligned} \hat{F}_\ell(u) &= \int_{\mathbb{R}} \prod_{i \in F_\ell} f_i(x) e^{2\pi i u x} dx \\ &= \int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i \in F_\ell} dx_i f_i(x_i) \right) e^{2\pi i u \sum_{i \in F_\ell} x_i / |F_\ell|} \prod_{m=2}^{|F_\ell|} \delta(x_{i_m} - x_{i_1}) \\ &= \int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i \in F_\ell} dv_i \hat{f}_i(v_i) \right) \delta\left(u - \sum_{i \in F_\ell} v_i \right), \end{aligned} \tag{2.2.8}$$

the last step following from Parseval's formula. (note: if $|F_\ell| = 1$ then the product over m is taken to be 1). Hence,

$$\prod_{j=1}^k \hat{F}_{\ell_j}(u_j) = \int_{\mathbb{R}^{\Sigma_1^k |F_{\ell_j}|}} \left(\prod_{i \in \bigcup F_{\ell_j}} dv_i \hat{f}_i(v_i) \right) \prod_{j=1}^k \delta\left(u_j - \sum_{i \in F_{\ell_j}} v_i \right). \tag{2.2.9}$$

In the integrand, the δ's restrict us to

$$\sum_{j=1}^k |u_j| = \sum_{j=1}^k \left| \sum_{i \in F_{\ell_j}} v_i \right| \leq \sum_{i \in \bigcup F_{\ell_j}} |v_i|.$$

So, if $\sum_{j=1}^k |u_j| > \alpha$, then $\sum_{i \in \bigcup F_{\ell_j}} |v_i| > \alpha$. But, by the support condition on $\prod_{i=1}^n \hat{f}_i(v_i)$, $\prod_{i \in \bigcup F_{\ell_j}} \hat{f}_i(v_i) = 0$ if $\sum_{i \in \bigcup F_{\ell_j}} |v_i| > \alpha$. Hence (2.2.9) is 0 if $\sum_{j=1}^k |u_j| > \alpha$, thus the claim.

Claim 2: *Suppose that $\prod_{i=1}^n \hat{f}_i(u_i)$ is supported in $\sum_{i=1}^n |u_i| \le \alpha < 1$. Then*

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \left(\frac{-2}{\log X}\right)^k \sum_{\substack{m_i\ge 1\\ i=1,\dots,k\\ m_1\cdot\ldots\cdot m_k\neq\Box}} \left(\prod_{j=1}^k \frac{\Lambda(m_j)}{m_j^{1/2}}\chi_d(m_j)\hat{F}_{\ell_j}\left(\frac{\log m_j}{\log X}\right)\right)$$
$$= 0 \tag{2.2.10}$$

Here we are summing over all k-tuples $(m_1,\dots,m_k)$ of positive integers with $\prod_1^k m_i \notin \{1,4,9,16,\ldots\}$, and $S=\{l_1,\dots,l_k\}$.

Remark: This claim tells us that the only contributions to (2.2.7) come from perfect squares (this is dealt with in the next Claim).

Proof. Changing order of summation, applying Claim 1 and Cauchy-Shwartz, we find that the l.h.s. of (2.2.10) is

$$\ll \lim_{X\to\infty} \frac{1}{|D(X)|}\frac{1}{\log^k X}\left(\sum_{\substack{m_i\ge 1\\ \sum \log m_i \le \alpha\log X\\ m_1\cdot\ldots\cdot m_k\neq\Box}} \frac{\Lambda^2(m_1)\cdot\ldots\cdot\Lambda^2(m_k)}{m_1\cdot\ldots\cdot m_k}\right)^{1/2}\cdot$$
$$\left(\sum_{\substack{m_i\ge 1\\ \sum \log m_i \le \alpha\log X\\ m_1\cdot\ldots\cdot m_k\neq\Box}} \left|\sum_{d\in D(X)} \chi_d(m_1\cdot\ldots\cdot m_k)\right|^2\right)^{1/2} \tag{2.2.11}$$

The first bracketed term is

$$< \left(\sum_{m\le X^\alpha} \frac{\Lambda^2(m)}{m}\right)^{k/2} \ll \log^k X. \tag{2.2.12}$$

Next, the number of times we may write $m = m_1\cdot\ldots\cdot m_k$, $m_i\ge 1$, is $O\left(\sigma_0^{k-1}(m)\right) = O_\varepsilon\left(m^\varepsilon\right)$ for any $\varepsilon>0$ ($\sigma_0(m)$ being the number of divisors of m), so that the second bracketed term is

$$\ll_\varepsilon \left(X^\varepsilon \sum_{m\le X^\alpha}\left|\sum_{d\in D(X)} \chi_d(m)\right|^2\right)^{1/2}. \tag{2.2.13}$$

Applying the methods of Jutila [15], we find that the above is

$$\ll_\varepsilon \left(X^{\varepsilon+1+\alpha}\log^A X\right)^{1/2}, \qquad \text{for some constant } A \ (A=10 \text{ is admissable})$$

which, combined with (2.2.12), shows that (2.2.11) is

$$\ll_\varepsilon \lim_{X\to\infty} \frac{1}{|D(X)|} X^{\varepsilon+(1+\alpha)/2}$$

But, for ε small enough, this limit equals 0 (because $|D(X)| \sim cX$, for some constant c, and we are assuming $\alpha < 1$).

Claim 3:

$$\lim_{X\to\infty}\frac{1}{|D(X)|}\sum_{d\in D(X)}\left(\frac{-2}{\log X}\right)^k \sum_{\substack{m_i\geq 1\\ m_1\cdot\ldots\cdot m_k=\square}}\left(\prod_{j=1}^k \frac{\Lambda(m_j)}{m_j^{1/2}}\chi_d(m_j)\hat{F}_{\ell_j}\left(\frac{\log m_j}{\log X}\right)\right)$$
$$= \sum_{\substack{S_2\subseteq S\\ |S_2|\, even}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)du\right)\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du\right). \tag{2.2.14}$$

Here we are summing over all k-tuples $(m_1,\ldots,m_k)$ of positive integers with $\prod_1^k m_i \in \{1,4,9,16,\ldots\}$.

Proof. First, the $\Lambda(m_i)$'s restrict us to prime powers, $m_i = p_i^{e_i}$, so the only way that $\prod_1^k m_i$ can equal a perfect square is if some of the e_i's are even, and the rest of the $p_i^{e_i}$'s match up to produce squares.

We can focus our attention on $e_i = 1$ or 2, since the sum over $e_i \geq 3$ contributes 0 as $X\to\infty$.

Also note, in (2.2.14), that $\chi_d(\prod_1^k m_i) = 1$ since $\prod_1^k m_i$ is restricted to perfect squares. Hence the l.h.s. of (2.2.14) is

$$\lim_{X\to\infty}\sum_{\substack{S_2\subseteq S\\ |S_2|\text{even}}}\sum_{\substack{p_\ell\\ \ell\in S_2\\ \prod_{\ell\in S_2} p_\ell=\square}}\left(\frac{-2}{\log X}\right)^{|S_2|}\prod_{i\in S_2}\frac{\log(p_i)}{p_i^{1/2}}\hat{F}_i\left(\frac{\log p_i}{\log X}\right)$$
$$\cdot\sum_{\substack{p_\ell\\ \ell\in S_2^c}}\left(\frac{-2}{\log X}\right)^{|S_2^c|}\prod_{i\in S_2^c}\frac{\log(p_i)}{p_i}\hat{F}_i\left(\frac{2\log p_i}{\log X}\right)$$

(we have dropped the $(1/|D(X)|)\sum_{d\in D(X)}$ since the terms in the sum don't depend on d). The sum over $\ell \in S_2$ corresponds to the e_ℓ's that are equal to one (and which pair up to produce squares), while the sum over $\ell \in S_2^c$ corresponds to the e_ℓ's that are equal to two. To complete the proof of this Claim and hence of Lemma 1, we establish the two Subclaims below.

Subclaim 3.1:

$$\lim_{X\to\infty} \sum_{\substack{p_\ell \\ \ell\in S_2^c}} \left(\frac{-2}{\log X}\right)^{|S_2^c|} \prod_{i\in S_2^c} \frac{\log(p_i)}{p_i} \hat{F}_i\left(\frac{2\log p_i}{\log X}\right)$$
$$= \left(\frac{-1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)du \tag{2.2.15}$$

Proof. The l.h.s. of (2.2.15) factors

$$\prod_{\ell\in S_2^c} \left(\frac{-2}{\log X}\sum_p \frac{\log(p)}{p}\hat{F}_\ell\left(\frac{2\log p}{\log X}\right)\right)$$

which, summing by parts, equals

$$\prod_{\ell\in S_2^c} \frac{2}{\log X}\int_1^\infty \sum_{p\le t}\frac{\log(p)}{p}\left(\hat{F}_\ell\left(\frac{2\log t}{\log X}\right)\right)' dt.$$

The sum $\sum_{p\le t}\log(p)/p$ can be evaluated elementarily (see [13, pg 22]), and the above becomes

$$\prod_{\ell\in S_2^c} \frac{2}{\log X}\int_1^\infty (\log t + O(1))\left(\hat{F}_\ell\left(\frac{2\log t}{\log X}\right)\right)' dt$$
$$= \prod_{\ell\in S_2^c}\left(\frac{-2}{\log X}\int_1^\infty \hat{F}_\ell\left(\frac{2\log t}{\log X}\right)\frac{dt}{t} + O\left(\frac{1}{\log X}\right)\right), \tag{2.2.16}$$

the last step from integration by parts, and using the fact that $\hat{F}_\ell^{(1)}(u)$ is supported in $|u|\le\alpha$. Changing variables $u = 2\log t/\log X$ and noting that all the $\hat{F}_\ell$'s are even (since all the f_i's are), we thus find that the limit in (2.2.15) is

$$\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)du$$

Subclaim 3.2:

$$\lim_{X\to\infty} \sum_{\substack{p_\ell \\ \ell\in S_2 \\ \prod_{\ell\in S_2} p_\ell=\square}} \left(\frac{-2}{\log X}\right)^{|S_2|} \prod_{i\in S_2} \frac{\log(p_i)}{p_i^{1/2}} \hat{F}_i\left(\frac{\log p_i}{\log X}\right)$$

$$= \sum_{(A;B)} 2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \int_{\mathbb{R}} |u|\, \hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du \qquad (2.2.17)$$

Proof. In (2.2.17), $\prod_{\ell\in S_2} p_\ell = \square$ implies that the p_ℓ's pair up to produce squares. So, the l.h.s. of (2.2.17) equals

$$\lim_{X\to\infty} \sum_{(A;B)} \sum_{\substack{p_i \\ i=1,\dots,|S_2|/2}} \prod_{j=1}^{|S_2|/2} \frac{4}{\log^2(X)} \frac{\log^2(p_j)}{p_j} \hat{F}_{a_j}\left(\frac{\log p_j}{\log X}\right)\hat{F}_{b_j}\left(\frac{\log p_j}{\log X}\right) \qquad (2.2.18)$$

The sum over $(A;B)$ accounts for all ways of pairing up primes in (2.2.17). Note that there will a bit of overlap produced in (2.2.18), but this overlap contributes 0 as $X\to\infty$. For example, if $S_2 = \{1,2,5,7\}$, then the 3 ways of pairing up p_1,p_2,p_5,p_7 are: $p_1 = p_5$ and $p_2 = p_7$, $p_1 = p_7$ and $p_2 = p_5$, $p_1 = p_2$ and $p_5 = p_7$. So the sum over $p_1 = p_2 = p_5 = p_7$ will be counted 3 times in (2.2.18) whereas it is only counted once in the l.h.s. of (2.2.17). Such diagonal sums don't bother us since there are $O_k(1)$ such sums, and a typical $p_{j_1} = p_{j_2} = \ldots = p_{j_{2r}}$, $r \geq 2$, contributes to (2.2.18) a term with a factor that is

$$\ll \lim_{X\to\infty} \frac{1}{\log^{2r} X} \sum_p \frac{\log^{2r} p}{p^r} \ll \lim_{X\to\infty} \frac{1}{\log^{2r} X} = 0.$$

Now, (2.2.18) can be written as

$$\lim_{X\to\infty} \sum_{(A;B)} \prod_{j=1}^{|S_2|/2} \left(\frac{4}{\log^2(X)} \sum_p \frac{\log^2(p)}{p} \hat{F}_{a_j}\left(\frac{\log p}{\log X}\right)\hat{F}_{b_j}\left(\frac{\log p}{\log X}\right)\right).$$

Summing by parts, we find that the bracketed term is

$$4\int_0^\infty u\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du + O\left(1/\log X\right).$$

Recalling that the $\hat{F}$'s are even, we obtain the Subclaim.

Lemma 2: *Let*

$$a_\ell(d) = \sum_{\gamma_d} F_\ell\left(L\gamma_d^{(j)}\right)$$

where $F_\ell(x) = \prod_{i\in F_\ell} f_i(x)$*, and* f_i *as in Theorem 2.1. Then,*

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{\nu(\underline{F})} a_\ell(d)$$
$$= \lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{\nu(\underline{F})} \left(a_\ell(d) + O(1/\log X)\right)$$

Remark : *This Lemma justifies dropping the* $O(1/\log X)$ *when plugging (2.2.5) into (2.2.3).*

Proof. The proof is by induction. We consider

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{k} \left(a_\ell(d) + O(1/\log X)\right) \tag{2.2.19}$$

for $k = 1, 2, \ldots, \nu(\underline{F})$. When $k = 1$, this clearly equals

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} a_\ell(d).$$

Now, consider the general case. Multiplying out the product in (2.2.19) we get

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{k} a_\ell(d) \quad + \text{ remainder}$$

where the remainder consists of $2^k - 1$ terms, each of which is of the form

$$O\left(\frac{1}{\log^r(X)} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{j=1}^{k_2} \left|a_{\ell_j}(d)\right|\right) \tag{2.2.20}$$

with $r \geq 1$, $k_2 < k$. Now, if $F_\ell(x) \geq 0$ for all x, then $\left|a_{\ell_j}(d)\right| = a_{\ell_j}(d)$, and, by our inductive hypothesis combined with Lemma 1, the O term above tends to 0 as $X \to \infty$.

If $F_\ell(x)$ is not ≥ 0 for all x, we can show that the O term in (2.2.20) tends to 0 as $X \to \infty$ by replacing each $f_i(x)$ $(i = 1, \ldots, n)$ with a function $g_i(x)$, that is postitive and bigger in absolute value that $f_i(x)$, and which satisfies the conditions of Theorem 2.1, i.e. we require

· $g_i(x) \geq |f_i(x)|$.

· $g_i(x)$ even and in $S(\mathbb{R})$.

· $\prod_{i=1}^n \hat{g}_i(u_i)$ is supported $\sum_{i=1}^n |u_i| < 1$

That there exists g_i's satisyfying the required conditions can be seen as follows. Let

$$h(t) = \begin{cases} K\exp(-1/(1-t^2)), & |t| < 1; \\ 0, & |t| \geq 1. \end{cases}$$

where K is chosen so that

$$\int_{-1}^{1} h(t)dt = 1,$$

let

$$\theta_\beta(t) = \frac{1}{\beta} h(t/\beta) \tag{2.2.21}$$

(so that θ_β approximates the δ function when β is small) and consider

$$\Psi_\beta(x) = (\theta_\beta * \theta_\beta)\hat{\ }(x) = (\hat{\theta}_\beta(x))^2. \tag{2.2.22}$$

Now

$$\begin{aligned} \hat{\theta}_\beta(x) &= \frac{1}{\beta}\int_{-\beta}^{\beta} h(t/\beta)\cos(2\pi x t)dt \\ &= \int_{-1}^{1} h(u)\cos(2\pi\beta u x)du \end{aligned} \tag{2.2.23}$$

But when $|x| \leq 1/(8\beta)$, we have

$$\hat{\theta}_\beta(x) > \frac{\sqrt{2}}{2}\int_{-1}^{1} h(u)du = \frac{\sqrt{2}}{2}$$

(since, when $|x| \le 1/(8\beta)$, $|u| \le 1$, we get, $|2\pi\beta u x| \le \pi/4$). Hence

$$\Psi_\beta(x) > 1/2 \qquad \text{when } |x| \le 1/(8\beta)$$

(so, Ψ_β is bounded away from 0 for long stretches when β is small), and from (2.2.22)

$$\Psi_\beta(x) \ge 0 \qquad \text{for all } x.$$

Also note that Ψ_β is even and in $S(\mathbb{R})$ (since $h(t)$ enjoys these properties) and that $\hat{\Psi}_\beta(t) = (\theta_\beta * \theta_\beta)(t)$ is supported in $[-2\beta, 2\beta]$. We use $\Psi_\beta(x)$'s to construct a $g_i(x)$ satisfying the three required properties.

Let

$$M_f(c,d) = \max_{c \le |x| \le d} |f(x)|$$

and let

$$\beta_j^{-1} = \begin{cases} 2n + j, & j \ge 1; \\ 0, & j = 0. \end{cases}$$

(the $j = 0$ case is only for notational convenience). Then

$$g_i(x) = 2 \sum_{j=0}^{\infty} M_{f_i}((8\beta_j)^{-1}, (8\beta_{j+1})^{-1}) \Psi_{\beta_{j+1}}(x)$$

has the required properties.

2.3 r.h.s.

Our goal is to express

$$\int_{\mathbb{R}^n} f(x) W_{\mathrm{USp}}^{(n)}(x) dx,$$

in such a manner that will allow us to easily see how to match terms with (2.2.6).

We consider the more general

$$\int_{\mathbb{R}^n} f(x) W_\varepsilon(x) dx \tag{2.3.1}$$

where $\varepsilon \in \{-1, 1\}$ and

$$W_\varepsilon(x_1, \ldots, x_n) = \det\left(K_\varepsilon(x_j, x_k)\right)_{\substack{1 \le j \le n \\ 1 \le k \le n}}$$
$$K_\varepsilon(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \varepsilon \frac{\sin(\pi(x+y))}{\pi(x+y)},$$

because it will be needed when we study analogous questions for $GL_M/\mathbb{Q}$.

Write,

$$W_\varepsilon(x_1, \ldots, x_n) = \sum_\sigma \operatorname{sgn}(\sigma) \prod_{j=1}^n K_\varepsilon(x_j, x_{\sigma(j)}).$$

Here, σ is over all permutations of n elements. Express σ as a product of disjoint cycles

$$\sigma \in \bigsqcup_{\underline{F}} S^*(F_1) \times \ldots \times S^*(F_{\nu(\underline{F})}) \tag{2.3.2}$$

where $\underline{F}$ is over set partitions of $\{1, \ldots, n\}$ (as in Section 2.2) and $S^*(F_\ell)$ denotes the set of all $(|F_\ell| - 1)!$ cyclic permutations of the elements of F_ℓ. Notice that $\operatorname{sgn}(\sigma) = \prod_{\ell=1}^{\nu(\underline{F})} (-1)^{|F_\ell|-1}$.

For example, if $n = 7$ and $\underline{F} = [\{1,3,4,6\}, \{2,5,7\}]$, then $S^*(\{1,3,4,6\}) \times S^*(\{2,5,7\})$ is the set of 12 permutations:

$$\{(1\ 3\ 4\ 6)(2\ 5\ 7), (1\ 3\ 6\ 4)(2\ 5\ 7), (1\ 4\ 3\ 6)(2\ 5\ 7), (1\ 4\ 6\ 3)(2\ 5\ 7),$$
$$(1\ 6\ 3\ 4)(2\ 5\ 7), (1\ 6\ 4\ 3)(2\ 5\ 7), (1\ 3\ 4\ 6)(2\ 7\ 5), (1\ 3\ 6\ 4)(2\ 7\ 5),$$
$$(1\ 4\ 3\ 6)(2\ 7\ 5), (1\ 4\ 6\ 3)(2\ 7\ 5), (1\ 6\ 3\ 4)(2\ 7\ 5), (1\ 6\ 4\ 3)(2\ 7\ 5)\}.$$

We will be applying Parseval to (2.3.1), and thus need to determine $\hat{W}_\varepsilon(u)$. So, for each cycle $(i_1, \ldots, i_m)$, we evaluate the fourier transform

$$\int_{\mathbb{R}^m} K_\varepsilon(x_{i_1}, x_{i_2}) K_\varepsilon(x_{i_2}, x_{i_3}) \cdot \ldots \cdot K_\varepsilon(x_{i_m}, x_{i_1}) e^{2\pi i \sum_{j=1}^m u_{i_j} x_{i_j}} dx_{i_1} \ldots dx_{i_m}. \tag{2.3.3}$$

Expanding the product of K_ε's, we obtain 2^m terms

$$\int_{\mathbb{R}^m} \sum_{\boldsymbol{a}} \varepsilon^{\beta(\boldsymbol{a})} \frac{\sin(\pi(x_{i_1} - a_1 x_{i_2}))}{\pi(x_{i_1} - a_1 x_{i_2})} \cdots \frac{\sin(\pi(x_{i_m} - a_m x_{i_1}))}{\pi(x_{i_m} - a_m x_{i_1})} e^{2\pi i \sum_{j=1}^m u_{i_j} x_{i_j}} dx_{i_1} \ldots dx_{i_m}. \tag{2.3.4}$$

Here $\boldsymbol{a}$ ranges over all 2^m m-tuples $(a_1, \dots, a_m)$ with $a_j \in \{1, -1\}$, and $\beta(\boldsymbol{a}) = \#\{j \mid a_j = -1\}$.

According to Lemma 3 (below), if $\sum \left|u_{i_j}\right| < 1$, then (2.3.4) is

$$2^{m-2}\varepsilon + \sum_{\boldsymbol{c}} \delta\left(\sum_{j=1}^{m} c_j u_{i_j}\right)\left(1 - V\left(c_1 u_{i_1}, \dots, c_m u_{i_m}\right)\right) \tag{2.3.5}$$

where $\boldsymbol{c}$ is over all 2^{m-1} m-tuples $(c_1, \dots, c_m)$ with $c_j \in \{1, -1\}$, and $c_m = 1$ and where

$$\begin{aligned} V(\boldsymbol{y}) &= M(\boldsymbol{y}) - m(\boldsymbol{y}) \\ M(\boldsymbol{y}) &= \max\{s_k(\boldsymbol{y}), k = 1, \dots, n\} \\ m(\boldsymbol{y}) &= \min\{s_k(\boldsymbol{y}), k = 1, \dots, n\} \\ s_j(\boldsymbol{y}) &= \sum_{j=1}^{k} y_j. \end{aligned} \tag{2.3.6}$$

Applying Parseval to (2.3.1), and recalling the assumption that the support of $\prod_{i=1}^n \hat{f}_i(u_i)$ is in $\sum_{i=1}^n |u_i| < 1$ (so in the integral below, we are restricted to the region where Lemma 3 applies), we find that (2.3.1) equals

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} du_i \hat{f}_i(u_i)\right) \sum_{\underline{F}} \prod_{\ell=1}^{\nu(\underline{F})} (-1)^{|F_\ell|-1} \\ &\sum_{\{i|i\in F_\ell\}}{}' \left(2^{|F_\ell|-2}\varepsilon + \sum_{\boldsymbol{c}} \delta\left(\sum_{j=1}^{|F_\ell|} c_j u_{i_j}\right)\left(1 - V\left(c_1 u_{i_1}, \dots, c_{|F_\ell|} u_{i_{|F_\ell|}}\right)\right)\right) \end{aligned} \tag{2.3.7}$$

where $\sum_{\{i|i\in F_\ell\}}{}'$ is over all $(|F_\ell| - 1)!$ cyclic permutations of the elements of F_ℓ.

Next, in the inner sum, change variables $w_{i_j} = c_j u_{i_j}$. Recalling that the $\hat{f}$'s are assumed to be even functions, we find that the above becomes

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} dw_i \hat{f}_i(w_i)\right) \sum_{\underline{F}} \prod_{\ell=1}^{\nu(\underline{F})} (-2)^{|F_\ell|-1} \\ &\sum_{\{i|i\in F_\ell\}}{}' \left(\frac{\varepsilon}{2} + \delta\left(\sum_{j=1}^{|F_\ell|} w_{i_j}\right)\left(1 - V\left(w_{i_1}, \dots, w_{i_{|F_\ell|}}\right)\right)\right) \end{aligned}$$

Applying [33, (4.35)], we get

$$\int_{\mathbb{R}^n}\left(\prod_{i=1}^n dw_i \hat{f}_i(w_i)\right)\sum_{\underline{F}}\prod_{\ell=1}^{\nu(\underline{F})}(-2)^{|F_\ell|-1}$$
$$\left((|F_\ell|-1)!\frac{\varepsilon}{2}+\delta\left(\sum_{i\in F_\ell} w_i\right)\left((|F_\ell|-1)!-\sum_{[H,H^c]}(|H|-1)!\,(|F_\ell|-1-|H|)!\left|\sum_{k\in H} w_k\right|\right)\right).$$

Here, $[H,H^c]$ runs over all $\left(2^{|F_\ell|}-2\right)/2$ ways of decomposing F_ℓ into two disjoint proper subsets: $H\cup H^c=F_\ell$, $H\cap H^c=\emptyset$, with $H\neq\emptyset, F_\ell$. Since $\sum|F_\ell|=n$, we can rewrite the above as

$$\boxed{\begin{aligned}&\int_{\mathbb{R}^n}\left(\prod_{i=1}^n du_i \hat{f}_i(u_i)\right)\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\prod_{\ell=1}^{\nu(\underline{F})}\\&\left((|F_\ell|-1)!\frac{\varepsilon}{2}+\delta\left(\sum_{i\in F_\ell} u_i\right)\left((|F_\ell|-1)!-\sum_{[H,H^c]}(|H|-1)!\,(|F_\ell|-1-|H|)!\left|\sum_{k\in H} u_k\right|\right)\right)\end{aligned}} \tag{2.3.8}$$

We now prove the Lemma that was required in deriving the above.

Lemma 3: *Let* $\sum_{j=1}^m |u_j|<1$. *Then*

$$\int_{\mathbb{R}^m}\sum_{\boldsymbol{a}}\varepsilon^{\beta(\boldsymbol{a})}\frac{\sin(\pi(x_1-a_1x_2))}{\pi(x_1-a_1x_2)}\cdots\frac{\sin(\pi(x_m-a_mx_1))}{\pi(x_m-a_mx_1)}e^{2\pi i u\cdot x}dx$$
$$=2^{m-2}\varepsilon+\sum_{\boldsymbol{c}}\delta\left(\sum_{j=1}^m c_ju_j\right)\left(1-V\left(c_1u_1,\ldots,c_mu_m\right)\right) \tag{2.3.9}$$

The notation here is defined between (2.3.4) and (2.3.6). Note: in the degenerate case $m=1$, *the above should be read as*

$$\int_{\mathbb{R}}\left(1+\varepsilon\frac{\sin 2\pi x}{2\pi x}\right)e^{2\pi i u x}dx=\frac{1}{2}\varepsilon+\delta(u),\qquad |u|<1.$$

Proof. The $m = 1$ case is easy to check and follows from $\frac{1}{2}\chi_{[-1,1]}(u) = \int_{\mathbb{R}} \frac{\sin 2\pi x}{2\pi x} e^{2\pi i u x}\,dx$. So assume that $m \geq 2$ and consider a typical

$$\int_{\mathbb{R}^m} \frac{\sin(\pi(x_1 - a_1 x_2))}{\pi(x_1 - a_1 x_2)} \cdots \frac{\sin(\pi(x_m - a_m x_1))}{\pi(x_m - a_m x_1)} e^{2\pi i u\cdot x}\,dx. \tag{2.3.10}$$

Let

$$\begin{aligned} t_i &= x_i - a_i x_{i+1}, \qquad\qquad i = 1, \ldots, m-1 \\ t_m &= x_m, \end{aligned} \tag{2.3.11}$$

so that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_1 a_2 & a_1 a_2 a_3 & \ldots & a_1 \cdot \ldots \cdot a_{m-1} \\ 0 & 1 & a_2 & a_2 a_3 & \ldots & a_2 \cdot \ldots \cdot a_{m-1} \\ 0 & 0 & 1 & a_3 & \ldots & a_3 \cdot \ldots \cdot a_{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots\ldots & 0 & 1 & a_{m-1} & \\ 0 & \ldots\ldots\ldots\ldots & & 0 & 1 & \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix}$$

Let

$$K(y) \stackrel{\text{def}}{=} \sin(\pi y)/(\pi y).$$

Changing variables, (2.3.10) is

$$\int_{\mathbb{R}^m} K(t_1) \ldots K(t_{m-1}) K(t_m - a_m(t_1 + a_1 t_2 + a_1 a_2 t_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1} t_m)) \\ e^{2\pi i (t_1 s_1 + \cdots + t_m s_m)} dt_1 \ldots dt_m \tag{2.3.12}$$

where

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= a_1 u_1 + u_2 \\ s_3 &= a_1 a_2 u_1 + a_2 u_2 + u_3 \\ &\vdots \\ s_k &= a_1 \cdot \ldots \cdot a_{k-1} u_1 + a_2 \cdot \ldots \cdot a_{k-1} u_2 + \cdots + a_{k-1} u_{k-1} + u_k \\ &\vdots \end{aligned} \tag{2.3.13}$$

Now, $K(y) = K(-y)$, so, because $a_m \in \{1, -1\}$, we find that (2.3.12) equals

$$\int_{\mathbb{R}^m} K(t_1)\dots K(t_{m-1})K(a_m t_m - t_1 - a_1 t_2 - a_1 a_2 t_3 - \cdots - a_1 \cdot \ldots \cdot a_{m-1} t_m)) \\ e^{2\pi i(t_1 s_1 + \cdots + t_m s_m)} dt_1 \dots dt_m.$$

Applying [33, (4.28)] (to the variable t_1 with $\tau = -a_m t_m + a_1 t_2 + a_1 a_2 t_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1} t_m$) the above becomes

$$\int_{\mathbb{R}^m} \chi_{[-\frac{1}{2},\frac{1}{2}]}(v)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(v + s_1)\, e^{2\pi i v(-a_m t_m + a_1 t_2 + a_1 a_2 t_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1} t_m)} \\ K(t_2)\dots K(t_{m-1}) e^{2\pi i(t_2 s_2 + \cdots + t_m s_m)} dv dt_2 \dots dt_m.$$

Integrating over $t_2, \dots, t_{m-1}$, we get

$$\int_{\mathbb{R}^2} \chi_{[-\frac{1}{2},\frac{1}{2}]}(v)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(v + s_1)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(a_1 v + s_2)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(a_1 a_2 v + s_3) \cdot \ldots \\ \cdot \chi_{[-\frac{1}{2},\frac{1}{2}]}(a_1 \cdot \ldots \cdot a_{m-2} v + s_{m-1})\, e^{2\pi i t_m (s_m + v(a_1 \cdot \ldots \cdot a_{m-1} - a_m))} dv dt_m. \tag{2.3.14}$$

Now, if $\beta(\boldsymbol{a}) = \#\{i \mid a_i = -1\}$ is even, then $a_1 \cdot \ldots \cdot a_m = 1$, so $a_1 \cdot \ldots \cdot a_{m-1} = a_m$ and thus $a_1 \cdot \ldots \cdot a_{m-1} - a_m = 0$. Hence the integral over t_m pulls out a $\delta(s_m)$ from the integral.

Next, if $\beta(\boldsymbol{a})$ is odd, then $a_1 \cdot \ldots \cdot a_m = -1$, so $a_1 \cdot \ldots \cdot a_{m-1} = -a_m$ and thus $a_1 \cdot \ldots \cdot a_{m-1} - a_m = -2a_m$. Hence the integral over t_m gives us a $\delta(s_m - 2a_m v)$, which, when integrated over v pulls out a product of characteristic functions.

Hence, we find that (2.3.14) (and hence (2.3.10)) is

$$\delta(s_m) \int_{\mathbb{R}} \chi_{[-\frac{1}{2},\frac{1}{2}]}(v)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(v + s_1)\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(a_1 v + s_2) \cdot \ldots \\ \cdot \chi_{[-\frac{1}{2},\frac{1}{2}]}(a_1 \cdot \ldots \cdot a_{m-2} v + s_{m-1})\, dv \qquad \text{if } \beta(\boldsymbol{a}) \text{ is even} \tag{2.3.15}$$

$$\frac{1}{2}\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{s_m}{2a_m}\right) \chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{s_m}{2a_m} + s_1\right) \chi_{[-\frac{1}{2},\frac{1}{2}]}\left(a_1 \frac{s_m}{2a_m} + s_2\right) \cdot \ldots \\ \cdot \chi_{[-\frac{1}{2},\frac{1}{2}]}\left(a_1 \cdot \ldots \cdot a_{m-2} \frac{s_m}{2a_m} + s_{m-1}\right) \qquad \text{if } \beta(\boldsymbol{a}) \text{ is odd.} \tag{2.3.16}$$

We require the following two Claims.

Claim 4: *Let $\beta(\boldsymbol{a})$ be odd, and assume that $\sum_{i=1}^m |u_i| < 1$. Then,*

$$\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(a_1 \cdot \ldots \cdot a_{k-1}\frac{s_m}{2a_m} + s_k\right) = 1, \qquad k = 1, \ldots, m-1. \tag{2.3.17}$$

Thus, (2.3.16) equals $1/2$.

Proof. Because $a_k \in \{1, -1\}$, we have, from (2.3.13),

$$s_k = a_1 \cdot \ldots \cdot a_{k-1}\left(u_1 + a_1u_2 + a_1a_2u_3 + \cdots + a_1 \cdot \ldots \cdot a_{k-1}u_k\right) \tag{2.3.18}$$

So the coefficient of u_j in (2.3.17) is

$$\frac{(a_1 \cdot \ldots \cdot a_{k-1})(a_1 \cdot \ldots \cdot a_{m-1})(a_1 \cdot \ldots \cdot a_{j-1})}{2a_m} + (a_1 \cdot \ldots \cdot a_{k-1})(a_1 \cdot \ldots \cdot a_{j-1}). \tag{2.3.19}$$

When $\beta(\boldsymbol{a})$ is odd, $\prod_{i=1}^m a_i = -1$, hence (2.3.19) equals

$$\frac{(a_1 \cdot \ldots \cdot a_{k-1})(a_1 \cdot \ldots \cdot a_{j-1})}{2} \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}$$

so

$$\left|a_1 \cdot \ldots \cdot a_{k-1}\frac{s_m}{2a_m} + s_k\right| < 1/2$$

(since we are assuming $\sum_{i=1}^m |u_i| < 1$), hence the Claim.

Claim 5: *Let $\beta(\boldsymbol{a})$ be even, and assume that $\sum_{i=1}^m |u_i| < 1$. Then (2.3.15) equals*

$$\delta(s_m)\left(1 - V(u_1, a_1u_2, \ldots, a_1 \cdot \ldots \cdot a_{m-1}u_m)\right)$$

with $V(\boldsymbol{y})$ defined in (2.3.6).

Proof. In (2.3.15), we have, by (2.3.18),

$$\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(a_1 \cdot \ldots \cdot a_{k-1}v + s_k\right) =$$
$$\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(a_1 \cdot \ldots \cdot a_{k-1}\left(v + u_1 + a_1u_2 + a_1a_2u_3 + \cdots + a_1 \cdot \ldots \cdot a_{k-1}u_k\right)\right)$$

and we can drop the $a_1 \cdot \ldots \cdot a_{k-1} \in \{1, -1\}$ since $\chi_{[-\frac{1}{2},\frac{1}{2}]}(y)$ is even. Furthermore, the $\delta(s_m)$ restricts us to $u_1 + a_1u_2 + a_1a_2u_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1}u_m = 0$. And because we are assuming $\sum_{i=1}^{m} |u_i| < 1 < 2$, we may apply Lemma 4.3 of [33], obtaining the Claim. *Note: in [33, (4.32)] n could read $n-1$ without affecting the truth of the equation, since, in the notation of that paper, $f_2(v)f_2(v+u_1+\cdots+u_n) = f_2(v)$.*

We are now ready to complete the proof of this Lemma. By Claim 4, the contribution to (2.3.9) from $\boldsymbol{a}$ with $\beta(\boldsymbol{a})$ odd is

$$\sum_{\substack{\boldsymbol{a} \\ \beta(\boldsymbol{a}) \text{ odd}}} \frac{1}{2}\varepsilon^{\beta(\boldsymbol{a})}$$

But we are assuming $\varepsilon \in \{1, -1\}$, so the above is

$$2^{m-2}\varepsilon. \tag{2.3.20}$$

The contribution to (2.3.9) from $\boldsymbol{a}$ with $\beta(\boldsymbol{a})$ even is, by Claim 5

$$\sum_{\substack{\boldsymbol{a} \\ \beta(\boldsymbol{a}) \text{ even}}} \delta(s_m)\left(1 - V(u_1, a_1u_2, \ldots, a_1 \cdot \ldots \cdot a_{m-1}u_m)\right). \tag{2.3.21}$$

Now,

$$\begin{aligned} s_m &= a_1 \cdot \ldots \cdot a_{m-1}\left(u_1 + a_1u_2 + a_1a_2u_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1}u_m\right) \\ &= a_mu_1 + a_ma_1u_2 + a_ma_1a_2u_3 + \cdots + a_ma_1 \cdot \ldots \cdot a_{m-1}u_m \end{aligned}$$

because $\prod_{i=1}^{m} a_i = 1$ when $\beta(\boldsymbol{a})$ is even. Let

$$\boldsymbol{c} = (c_1, \ldots, c_m) = (a_m, a_ma_1, a_ma_1a_2, \ldots, a_ma_1 \cdot \ldots \cdot a_{m-1}).$$

Now, because $\prod_{i=1}^{m} a_i = 1$, $\boldsymbol{c}$ ranges over all m tuples with $c_j \in \{1, -1\}$ and $c_m = 1$. So, summing over such $\boldsymbol{c}$ we find that (2.3.21) equals

$$\sum_{\boldsymbol{c}} \delta\left(\sum_{j=1}^{m} c_ju_j\right)\left(1 - V\left(a_mc_1u_1, \ldots, a_mc_mu_m\right)\right). \tag{2.3.22}$$

But, because $V(-\boldsymbol{y}) = V(\boldsymbol{y})$, the above is (regardless of the value of $a_m = \pm 1$)

$$\sum_{\boldsymbol{c}} \delta\left(\sum_{j=1}^{m} c_ju_j\right)\left(1 - V\left(c_1u_1, \ldots, c_mu_m\right)\right). \tag{2.3.23}$$

This in combination with (2.3.20) establishes the Lemma.

2.4 l.h.s. = r.h.s.

Lemma 4:

$$\int_{\mathbb{R}^{|F_\ell|}} \prod_{i\in F_\ell} du_i \hat{f}_i(u_i) = \int_{\mathbb{R}} \hat{F}_\ell(u)du.$$

Proof. Both are equal, by Fourier inversion, to $\prod_{i\in F_\ell} f_i(0)$.

Lemma 5:

$$\int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i\in F_\ell} du_i \hat{f}_i(u_i)\right) \delta\left(\sum_{i\in F_\ell} u_i\right) = \int_{\mathbb{R}} F_\ell(x)dx.$$

Proof. Parseval.

Lemma 6: *Let $H \subset F_\ell$, $H \neq \emptyset$. Then*

$$\int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i\in F_\ell} du_i \hat{f}_i(u_i)\right) \delta\left(\sum_{i\in F_\ell} u_i\right) \left|\sum_{k\in H} u_k\right| = \int_{\mathbb{R}} \left(\widehat{\prod_{i\in H} f_i}\right)(u) \left(\widehat{\prod_{i\in H^c} f_i}\right)(u)\,|u|\,du.$$

Proof. Parseval.

Now, $W^{(n)}_{\mathrm{USp}} = W_{-1}$, so we need to compare (2.3.8), with $\varepsilon = -1$, to (2.2.6). By Lemmas 4 - 6 write (2.3.8) as

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \prod_{\ell=1}^{\nu(\underline{F})} (P_\ell + Q_\ell + R_\ell) \tag{2.4.1}$$

with

$$P_\ell = (|F_\ell|-1)! \left(\frac{-1}{2}\right) \int_{\mathbb{R}} \hat{F}_\ell(u)du$$

$$Q_\ell = (|F_\ell|-1)! \int_{\mathbb{R}} F_\ell(x)dx$$

$$R_\ell = -\sum_{[H,H^c]} (|H|-1)!\,(|F_\ell|-1-|H|)! \int_{\mathbb{R}} \left(\widehat{\prod_{i\in H} f_i}\right)(u) \left(\widehat{\prod_{i\in H^c} f_i}\right)(u)\,|u|\,du. \tag{2.4.2}$$

Expanding the product over ℓ, we get

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \sum_S \left(\prod_{\ell\in S^c} Q_\ell\right) \sum_{T\subseteq S} \left(\prod_{\ell\in T^c} P_\ell\right) \left(\prod_{\ell\in T} R_\ell\right) \tag{2.4.3}$$

where S ranges over all subsets of $\{1,\ldots,\nu(\underline{F})\}$. (we take empty products to be 1). Expanding the product $\prod_{\ell\in T} R_\ell$, we find that (2.4.3) is

$$\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\sum_{S}\left(\prod_{\ell\in S^c} Q_\ell\right)\sum_{T\subseteq S}\left(\prod_{\ell\in T^c} P_\ell\right)$$
$$\left((-1)^{|T|}\sum_{\boldsymbol{H}}\prod_{j=1}^{|T|}(|H_j|-1)!\left(\left|F_{\ell_j}\right|-1-|H_j|\right)!\int_{\mathbb{R}}\widehat{\left(\prod_{i\in H_j} f_i\right)}(u)\widehat{\left(\prod_{i\in H_j^c} f_i\right)}(u)\,|u|\,du.\right) \tag{2.4.4}$$

where $\sum_{\boldsymbol{H}}$ is over all $|T|$-tuples $\left([H_1,H_1^c],\ldots,\left[H_{|T|},H_{|T|}^c\right]\right)$ and where $T=\left\{\ell_1,\ldots,\ell_{|T|}\right\}$. (if $T=\emptyset$, we take the large bracketed factor to be 1. And if $T\neq\emptyset$, but $\sum_{\boldsymbol{H}}$ contains no terms, we take it to be 0). We have thus expressed, in (2.4.4), the r.h.s. of (2.1.4) in a form that can easily be compared with the l.h.s., as expressed in (2.2.6).

More precisely, a typical term in (2.2.6) is specified by $\underline{F}_{\text{l.h.s.}}$, $S_{\text{l.h.s.}}$, S_2, $(A;B)$. The sum over $\underline{F}$ arises from combinatorial sieving, and the sum over $S\subseteq\{1,\ldots,\nu(\underline{F})\}$ from multiplying out the explicit formula (2.2.5). The sum over $S_2\subseteq S$ comes from deciding which prime powers are paired up to produce squares, and which are already squares (S_2^c). $(A;B)$ accounts for all ways of pairing up S_2. The contribution to (2.2.6) from a typical term is

$$(-2)^{n-\nu(\underline{F}_{\text{l.h.s.}})}\left(\prod_{\ell\in S^c_{\text{l.h.s.}}}(|F_\ell|-1)!\int_{\mathbb{R}}F_\ell(x)dx\right)\left(\prod_{\ell\in S_2^c}(|F_\ell|-1)!\left(\frac{-1}{2}\right)\int_{\mathbb{R}}\hat{F}_\ell(u)du\right)$$
$$\left(2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\left(\left|F_{a_j}\right|-1\right)!\left(\left|F_{b_j}\right|-1\right)!\int_{\mathbb{R}}\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,|u|\,du.\right)$$
$$=(-2)^{n-\nu(\underline{F}_{\text{l.h.s.}})}\left(\prod_{\ell\in S^c_{\text{l.h.s.}}}Q_\ell\right)\left(\prod_{\ell\in S_2^c}P_\ell\right)$$
$$\left(2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\left(\left|F_{a_j}\right|-1\right)!\left(\left|F_{b_j}\right|-1\right)!\int_{\mathbb{R}}\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,|u|\,du.\right) \tag{2.4.5}$$

On the other hand, in (2.4.4), a typical term is specified by $\underline{F}_{\text{r.h.s.}}$, $S_{\text{r.h.s.}}$, T,

$\left([H_1, H_1^c], \ldots, \left[H_{|T|}, H_{|T|}^c\right]\right)$. Set

$$\begin{aligned}
&\underline{F}_{\text{r.h.s.}} = \{F_\ell \mid \ell \in S^c_{\text{l.h.s.}}\} \bigcup \{F_\ell \mid \ell \in S_2^c\} \bigcup \left\{F_{a_j} \cup F_{b_j} \mid j = 1, \ldots, |S_2|/2\right\} \\
&H_1 = F_{a_1}, \qquad\qquad H_1^c = F_{b_1} \\
&\vdots \qquad\qquad\qquad\quad\;\; \vdots \\
&H_{|S_2|/2} = F_{a_{|S_2|/2}}, \qquad H^c_{|S_2|/2} = F_{b_{|S_2|/2}}.
\end{aligned} \tag{2.4.6}$$

$S_{\text{r.h.s.}}$ and T are chosen in the obvious way (so that both products of Q's match, and both products of P's match). Notice that $|T| = |S_2|/2$ and that $\nu(\underline{F}_{\text{l.h.s.}}) = \nu(\underline{F}_{\text{r.h.s.}}) + |S_2|/2$.

The contribution to (2.4.4) from this term is thus

$$\begin{aligned}
&(-2)^{n-\nu(\underline{F}_{\text{l.h.s.}})+|S_2|/2} \left(\prod_{\ell \in S^c_{\text{l.h.s.}}} Q_\ell\right) \left(\prod_{\ell \in S_2^c} P_\ell\right) \\
&\left((-1)^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \left(\left|F_{a_j}\right| - 1\right)! \left(\left|F_{b_j}\right| - 1\right)! \int_{\mathbb{R}} \hat{F}_{a_j}(u) \hat{F}_{b_j}(u) \, |u| \, du.\right)
\end{aligned} \tag{2.4.7}$$

which is equal, because $|S_2|$ is even, to (2.4.5).

So every term on the l.h.s. has a corresponding term on the r.h.s. .

Conversely, this method of matching (*i.e.* (2.4.6)) produces for every term on the r.h.s. it's corresponding term on the l.h.s. . (with the convention that we disregard, on the r.h.s. , any term with $T \geq 1$ but $\sum_{\boldsymbol{H}}$ empty. We can do so since these terms contribute nothing to (2.4.4).)

Thus (2.2.6) = (2.3.8).

$\square_{\text{Theorem 2.1}}$

2.5 Examples

2.5.1 One term for $n = 17$

Let $n = 17$ and let

$$\begin{aligned}
&\underline{F}_{\text{l.h.s.}} = [F_1, F_2, F_3, F_4, F_5, F_6, F_7] \\
&= [\{1,2,13\}, \{4\}, \{3,6,7,9,17\}, \{8,10,11\}, \{5,12\}, \{14\}, \{15,16\}] \\
&S_{\text{l.h.s.}} = \{1,2,3,5,6\}, \qquad S^c_{\text{l.h.s.}} = \{4,7\} \\
&S_2 = \{1,2,5,6\}, \qquad S^c_2 = \{3\} \\
&(A;B) = (1,5;2,6).
\end{aligned} \tag{2.5.1}$$

This corresponds on the r.h.s. to

$$\begin{aligned}
&\underline{F}_{\text{r.h.s.}} = [\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5] \\
&\mathcal{F}_1 = F_4, \mathcal{F}_2 = F_7, \mathcal{F}_3 = F_3, \mathcal{F}_4 = F_1 \cup F_2, \mathcal{F}_5 = F_5 \cup F_6 \\
&S_{\text{r.h.s.}} = \{3,4,5\}, \qquad S^c_{\text{r.h.s.}} = \{1,2\} \\
&T = \{4,5\}, \qquad T^c = \{3\} \\
&H_1 = F_1, \qquad H^c_1 = F_2 \\
&H_2 = F_5, \qquad H^c_2 = F_6.
\end{aligned} \tag{2.5.2}$$

2.5.2 $n = 1, 2, 3$

Tables 2.1 - 2.2 show the correspondence between terms on the l.h.s. (as expressed in (2.4.5)) and the r.h.s. (as expressed in (2.4.7)).

2.6 Analogous results for $GL_M/\mathbb{Q}$

Let $L(s,\pi)$ be the L-function attached to a self contragredient ($\pi = \tilde{\pi}$) automorphic cuspidal representation of GL_M over $\mathbb{Q}$. Such an L-function is given initially (for $\Re s$ sufficiently large) as an Euler product of the form

$$L(s,\pi) = \prod_p L(s,\pi_p) = \prod_p \prod_{j=1}^{M} (1 - \alpha_\pi(p,j)p^{-s})^{-1}.$$

The condition $\pi = \tilde{\pi}$ implies that $\alpha_\pi(p,j) \in \mathbb{R}$. The Rankin-Selberg L-function $L(s, \pi \otimes \tilde{\pi})$ factors as the product of the symmetric and exterior square L-functions [5]

$$L(s, \pi \otimes \tilde{\pi}) = L(s, \pi \otimes \pi) = L(s, \pi, \vee^2) L(s, \pi, \wedge^2)$$

and has a simple pole at $s = 1$ which is carried by one of the two factors. Write the order of the pole of $L(s, \pi, \wedge^2)$ as $(\delta(\pi) + 1)/2$ (so that $\delta(\pi) = \pm 1$).

We desire to generalize Theorem 2.1 to the zeros of $L(s, \pi \otimes \chi_d)$ whose Euler product is given by

$$L(s, \pi \otimes \chi_d) = \prod_p \prod_{j=1}^{M} (1 - \chi_d(p) \alpha_\pi(p,j) p^{-s})^{-1}.$$

Now, when $\pi = \tilde{\pi}$, $L(s, \pi \otimes \chi_d)$ has a functional equation of the form

$$\begin{aligned} \Phi(s, \pi \otimes \chi_d) &:= \pi^{-Ms/2} \prod_{j=1}^{M} \Gamma\left((s + \mu_{\pi \otimes \chi_d}(j))/2\right) L(s, \pi \otimes \chi_d) \\ &= \varepsilon(s, \pi \otimes \chi_d) \Phi(1 - s, \pi \otimes \chi_d) \end{aligned}$$

where the $\mu_{\pi \otimes \chi_d}(j)$'s are complex numbers that are known to satisfy

$$\Re\left(\mu_{\pi \otimes \chi_d}(j)\right) > -1/2$$

(and are conjectured to satisfy $\Re\left(\mu_{\pi \otimes \chi_d}(j)\right) \geq 0$). We also have

$$\varepsilon(s, \pi \otimes \chi_d) = \varepsilon(\pi \otimes \chi_d) Q_{\pi \otimes \chi_d}^{-s+1/2} = \pm Q_{\pi \otimes \chi_d}^{-s+1/2}$$

with $\varepsilon(\pi \otimes \chi_d) = \chi'(d)$ where χ' is a quadratic character that depends only on π. When $\delta(\pi) = -1$, all twists have $\varepsilon(\pi \otimes \chi_d) = 1$. If $\delta(\pi) = 1$, then half the $L(s, \pi \otimes \chi_d)$'s have $\varepsilon(\pi \otimes \chi_d) = 1$ and the other half have $\varepsilon(\pi \otimes \chi_d) = -1$ (with the corresponding d's lying in fixed arithmetic progressions to the modulus of the character χ').

When $\varepsilon(\pi \otimes \chi_d) = 1$, we write the non-trivial zeros of $L(s, \pi \otimes \chi_d)$ as

$$1/2 + i\gamma_{\pi \otimes \chi_d}^{(j)}, \qquad j = \pm 1, \pm 2, \pm 3, \ldots$$

with

$$\ldots \Re\gamma_{\pi \otimes \chi_d}^{(-2)} \leq \Re\gamma_{\pi \otimes \chi_d}^{(-1)} \leq 0 \leq \Re\gamma_{\pi \otimes \chi_d}^{(1)} \leq \Re\gamma_{\pi \otimes \chi_d}^{(2)} \leq \ldots$$

and

$$\gamma^{(-k)}_{\pi\otimes\chi_d} = -\gamma^{(k)}_{\pi\otimes\chi_d}.$$

When $\varepsilon(\pi\otimes\chi_d) = -1$, $\gamma = 0$ is a zero of $L(s, \pi\otimes\chi_d)$, and we index the zeros as

$$1/2 + i\gamma^{(j)}_{\pi\otimes\chi_d}, \qquad j \in \mathbb{Z}$$

with

$$\dots \Re\gamma^{(-2)}_{\pi\otimes\chi_d} \le \Re\gamma^{(-1)}_{\pi\otimes\chi_d} \le \gamma^{(0)}_{\pi\otimes\chi_d} = 0 \le \Re\gamma^{(1)}_{\pi\otimes\chi_d} \le \Re\gamma^{(2)}_{\pi\otimes\chi_d} \le \dots$$

and

$$\gamma^{(-k)}_{\pi\otimes\chi_d} = -\gamma^{(k)}_{\pi\otimes\chi_d}.$$

Next, let $D(X)$ be as in (2.1.1) and let

$$D_{\pi,+}(X) = \{d \in D(X) : \varepsilon(\pi\otimes\chi_d) = 1\}$$
$$D_{\pi,-}(X) = \{d \in D(X) : \varepsilon(\pi\otimes\chi_d) = -1\}.$$

Then, assuming, for $M \ge 4$, the Ramanujan conjecture

$$|\alpha_\pi(p,j)| \le 1,$$

we have

Theorem 2.2: *Let $f(x_1,\dots,x_n) = \prod_{i=1}^n f_i(x_i)$ be even in all its variables with each f_i in $S(\mathbb{R})$. Assume further that $\hat{f}(u_1,\dots,u_n)$ is supported in $\sum_{i=1}^n |u_i| < 1/M$. Then, if $\delta(\pi) = 1$*

$$\lim_{X\to\infty} \frac{1}{|D_{\pi,\pm}(X)|} \sum_{d\in D_{\pi,\pm}(X)} \sideset{}{^*}\sum_{j_1,\cdots,j_n} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M\gamma^{(j_2)}_{\pi\otimes\chi_d}\dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right)$$
$$= \int_{\mathbb{R}^n} f(x) W^{(n)}_{\pm,O}(x)dx, \tag{2.6.1}$$

and, if $\delta(\pi) = -1$ (so that all twists have $\varepsilon(\pi\otimes\chi_d) = 1$), then

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \sideset{}{^*}\sum_{j_1,\cdots,j_n} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M\gamma^{(j_2)}_{\pi\otimes\chi_d}\dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right)$$
$$= \int_{\mathbb{R}^n} f(x) W^{(n)}_{USp}(x)dx, \tag{2.6.2}$$

where

$$L_M = \frac{M\log X}{2\pi}$$

$$W^{(n)}_{USp}(x_1,\dots,x_n) = \det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}}$$

$$W^{(n)}_{+,O}(x_1,\dots,x_n) = \det\left(K_{1}(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}}$$

$$W^{(n)}_{-,O}(x_1,\dots,x_n) = \det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}} + \sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\ne\nu\le n\\1\le k\ne\nu\le n}}$$

$$K_\varepsilon(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \varepsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}$$

($W^{(1)}_{-,O}(x) = 1-\sin(2\pi x)/(2\pi x)+\delta(x)$) and where $\sum^*_{j_1,\dots,j_n}$ *is over* $j_k = (0),\pm1,\pm2,\dots,$ *with* $j_{k_1}\ne\pm j_{k_2}$ *if* $k_1\ne k_2$.

Remark : *Again, as in Theorem 2.1 the assumptions f_i even and f of the form $\prod f_i$ can be removed.*

Proof. The proof is similar to that of Theorem 2.1. The main difference is in the explicit formula which, for $L(s,\pi\otimes\chi_d)$, reads

$$\sum_{\gamma_{\pi\otimes\chi_d}} F_\ell\left(L_M\gamma_{\pi\otimes\chi_d}\right) = \int_{\mathbb{R}} F_\ell(x)dx + O(1/\log X) - \frac{2}{M\log X}\sum_{m=1}^{\infty}\frac{\Lambda(m)a_\pi(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{M\log X}\right) \tag{2.6.3}$$

where

$$a_\pi(p^k) = \sum_{j=1}^{M}\alpha^k_\pi(p,j).$$

We consider the two cases, $\delta(\pi)=-1$ and $\delta(\pi)=1$, seperately.

For both cases we require the estimates

$$\begin{aligned}\sum_{m\le T}\left|a_\pi(m)\Lambda(m)\right|^2/m &\sim \log^2(T)/2\\ \sum_{p\le T}a_\pi(p^2)\log p &\sim -\delta(\pi)T\\ \sum_{p\le T}\left|a_\pi(p)\log p\right|^2/p &\sim \log^2(T)/2\end{aligned} \tag{2.6.4}$$

(see [33] and [16]. For these, and $M \geq 4$, the Ramanujan conjecture is assumed. These three are needed in the analogs of Claim 2, Subclaim 2.2.15, and Subclaim 2.2.17).

When $\delta(\pi) = -1$, all twists have $\varepsilon(\pi \otimes \chi_d) = 1$. The combinatorics work out exactly the same. The smaller support of $\hat{f}$ compensates for the presence of the M in the explicit formula.

When $\delta(\pi) = 1$, we need to examine the two sub-cases, $\varepsilon(\pi \otimes \chi_d) = 1$ and $\varepsilon(\pi \otimes \chi_d) = -1$, seperately.

As the analog of Lemma 1, we have:

Lemma 7: *When* $\delta(\pi) = 1$

$$\lim_{X\to\infty} \frac{1}{|D_{\pi,+}(X)|} \sum_{d\in D_{\pi,+}(X)} \left(\frac{-2}{M\log X}\right)^k \prod_{j=1}^{k} \sum_{m=1}^{\infty} \frac{\Lambda(m)a_\pi(m)}{m^{1/2}} \chi_d(m) \hat{F}_{\ell_j}\left(\frac{\log m}{M\log X}\right)$$
$$= \sum_{\substack{S_2\subseteq S \\ |S_2|\ even}} \left(\left(\frac{1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)du\right) \left(\sum_{(A;B)} 2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \int_{\mathbb{R}} |u|\, \hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du\right) \tag{2.6.5}$$

where $S = \{l_1, \ldots, l_k\}$. $\sum_{\substack{S_2\subseteq S \\ |S_2|\ even}}$ *is over all subsets* S_2 *of* S *whose size is even.* $\sum_{(A;B)}$ *is over all ways of pairing up the elements of* S_2. $F_\ell(x)$ *is defined in (2.2.4).*

Proof. Notice that the only difference in the r.h.s. of this Lemma as compared to Lemma 1 is in the factor

$$\left(\frac{1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)du$$

The difference in sign is accounted for by the opposite sign in 2.6.4.

□

So , we have that the l.h.s. of (2.6.1), for $D_{\pi,+}$, tends, as $X \to \infty$, to

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \left(\prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell| - 1)!\right) \sum_S \left(\prod_{\ell\in S^c} \int_{\mathbb{R}} F_\ell(x)dx\right)$$
$$\sum_{\substack{S_2\subseteq S \\ |S_2|\text{even}}} \left(\left(\frac{1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)du\right) \left(\sum_{(A;B)} 2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \int_{\mathbb{R}} |u|\, \hat{F}_{a_j}(u)\hat{F}_{b_j}(u)du\right)$$

This matches (2.3.8) with $\varepsilon = 1$, i.e. equals (in the notation of Section 2.3)

$$\int_{\mathbb{R}^n} f(x)W_+(x)dx = \int_{\mathbb{R}^n} f(x)W^{(n)}_{+,\mathrm{O}}(x)dx.$$

For the $\varepsilon(\pi\otimes\chi_d) = -1$ case, there is always a zero at $s = 1/2$

$$\gamma^{(0)}_{\pi\otimes\chi_d} = 0,$$

and, before applying the combinatorial sieving of Section 2.2, we need to isolate this zero. Now

$$\begin{aligned}
&\sum_{j_1,\dots,j_n}^{*} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M\gamma^{(j_2)}_{\pi\otimes\chi_d} \dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right) \\
&= \sum_{j_1\neq 0,\dots,j_n\neq 0}^{*} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M\gamma^{(j_2)}_{\pi\otimes\chi_d} \dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right) \\
&+ \sum_{\nu=1}^{n} \sum_{\substack{j_\nu=0 \\ j_k\neq 0, k\neq\nu}}^{*} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, \dots, L_M\gamma^{(j_{\nu-1})}_{\pi\otimes\chi_d}, 0, L_M\gamma^{(j_{\nu+1})}_{\pi\otimes\chi_d}, \dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right).
\end{aligned}$$

We only focus on the first sum on the r.h.s. above. The same technique applies to the remaining sums.

By combinatorial sieving and the explicit formula, we find that

$$\begin{aligned}
&\sum_{j_1\neq 0,\dots,j_n\neq 0}^{*} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M\gamma^{(j_2)}_{\pi\otimes\chi_d} \dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right) \\
&= \sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell|-1)! \\
&\quad \cdot \left(\int_{\mathbb{R}} F_\ell(x)dx - \frac{2}{M\log X}\sum_{m=1}^{\infty} \frac{\Lambda(m)a_\pi(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{M\log X}\right) - F_\ell(0) + O(1/\log X)\right).
\end{aligned} \tag{2.6.6}$$

But, by 2.6.4

$$-F_\ell(0) = \lim_{X\to\infty} \frac{4}{M\log X}\sum_p \frac{\Lambda(p^2)a_\pi(p^2)}{p}\hat{F}_\ell\left(\frac{2\log p}{M\log X}\right)$$

and this has the effect, in (2.6.6) of changing the sign of the contribution from the squares of primes.

2.7 Other Families

For twists by characters of order $\kappa \geq 3$, one expects to feel the presence of $U_\kappa(\infty)$, and we outline in this section what happens for these families.

Let $D^{(\kappa)}$ denote the set of primitive characters with $\chi^\kappa(n) = 1$ for all n, and $\chi^{\kappa_2}(n) \neq 1$ if $\kappa_2 < \kappa$. Let

$$D^{(\kappa)}(X) = \left\{\chi \in D^{(\kappa)} : \ \chi \text{ a character to the modulus } q,\ X/2 \leq q < X\right\}.$$

For π self contragredient, Write the non-trivial zeros of $L(s, \pi \otimes \chi)$ as

$$1/2 + i\gamma_{\pi\otimes\chi}^{(j)}, \qquad j = \pm 1, \pm 2, \pm 3, \ldots$$

with

$$\ldots \Re\gamma_{\pi\otimes\chi}^{(-2)} \leq \Re\gamma_{\pi\otimes\chi}^{(-1)} < 0 \leq \Re\gamma_{\pi\otimes\chi}^{(1)} \leq \Re\gamma_{\pi\otimes\chi}^{(2)} \leq \ldots$$

Theorem 2.3: *Let $f(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i)$ with f_i in $S(\mathbb{R})$. Assume further some appropriate support condition on $\hat{f}(u_1, \ldots, u_n)$ (i.e. supported in a small neighborhood of the origin). Then*

$$\lim_{X\to\infty} \frac{1}{|D^{(\kappa)}(X)|} \sum_{\{\chi\}\in D^{(\kappa)}(X)} \sum_{\substack{j_1,\ldots,j_n \\ \text{distinct}}} f\left(L_M\gamma_{\pi\otimes\chi}^{(j_1)}, L_M\gamma_{\pi\otimes\chi}^{(j_2)} \ldots, L_M\gamma_{\pi\otimes\chi}^{(j_n)}\right)$$
$$= \int_{\mathbb{R}^n} f(x) W_U^{(n)}(x)dx, \qquad (2.7.1)$$

where

$$L_M = \frac{M \log X}{2\pi}$$
$$W_U^{(n)}(x_1, \ldots, x_n) = \det\left(K_0(x_j, x_k)\right)_{\substack{1\leq j\leq n \\ 1\leq k\leq n}}.$$

Remark : *The support condition for $\hat{f}$ is vague since it would depend of what could be proven for the analog of Jutila's theorem (used in Claim 2).*

Outline. For quadratic twists we found, after invoking the explicit formula, a main contribution coming from terms of the form $m_1 \cdot \cdots \cdot m_k = \Box$. This led us to consider, in the explicit formula, the contribution from the matching of primes (as in Subclaim 3.2), and from the squares of primes (as in Subclaim 3.1).

Here, the explicit formula reads

$$\sum_{\gamma_{\pi\otimes\chi}} F_\ell\left(L_M\gamma_{\pi\otimes\chi}\right) = \int_{\mathbb{R}} F_\ell(x)dx + O(1/\log X)$$
$$-\frac{2}{M\log X}\sum_{m=1}^{\infty}\frac{\Lambda(m)a_\pi(m)}{m^{1/2}}\Re\left(\chi(m)\hat{F}_\ell\left(\frac{\log m}{M\log X}\right)\right) \qquad (2.7.2)$$

We get no contribution from perfect κth powers, since, for $\kappa \geq 3$, $r \geq 1$,

$$\sum_p \log^r(p)/p^{\kappa/2}$$

converges.

However, we still get a contribution from the matching up of m_i's in pairs, because

$$\left(\Re\chi(m)\hat{F}_\ell\left(\frac{\log m}{M\log X}\right)\right)^2$$

is always ≥ 0.

In the end, we get something that resembles (2.2.6) but without the

$$\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)du$$

factor, and this corresponds to setting $\varepsilon = 0$ on the r.h.s..

n	$\underline{F}_{\text{l.h.s.}}$	$S_{\text{l.h.s.}}$	S_2	$(A;B)$	$\underline{F}_{\text{r.h.s.}}$	$S_{\text{r.h.s.}}$	T	$\boldsymbol{H}$
1	$[\{1\}]$	$\emptyset$	$\emptyset$	—	$[\{1\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
2	$[\{1,2\}]$	$\emptyset$	$\emptyset$	—	$[\{1,2\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
	$[\{1\},\{2\}]$	$\emptyset$	$\emptyset$	—	$[\{1\},\{2\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2\}]$	$\{1\}$	$\{1\}$	$[\{1\},\{2\}]$
3	$[\{1,2,3\}]$	$\emptyset$	$\emptyset$	—	$[\{1,2,3\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
	$[\{1,2\},\{3\}]$	$\emptyset$	$\emptyset$	—	$[\{1,2\},\{3\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2,3\}]$	$\{1\}$	$\{1\}$	$[\{1,2\},\{3\}]$
	$[\{1,3\},\{2\}]$	$\emptyset$	$\emptyset$	—	$[\{1,3\},\{2\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2,3\}]$	$\{1\}$	$\{1\}$	$[\{1,3\},\{2\}]$
	$[\{2,3\},\{1\}]$	$\emptyset$	$\emptyset$	—	$[\{2,3\},\{1\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2,3\}]$	$\{1\}$	$\{1\}$	$[\{2,3\},\{1\}]$
	$[\{1\},\{2\},\{3\}]$	$\emptyset$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{3\}$	$\emptyset$	—		$\{3\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2\},\{3\}]$	$\{1\}$	$\{1\}$	$[\{1\},\{2\}]$
		$\{1,3\}$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\{1,3\}$	$\emptyset$	—
			$\{1,3\}$	$(1;3)$	$[\{1,3\},\{2\}]$	$\{1\}$	$\{1\}$	$[\{1\},\{3\}]$
		$\{2,3\}$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\{2,3\}$	$\emptyset$	—
			$\{2,3\}$	$(2;3)$	$[\{2,3\},\{1\}]$	$\{1\}$	$\{1\}$	$[\{2\},\{3\}]$
		$\{1,2,3\}$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2\},\{3\}]$	$\{1,2\}$	$\{1\}$	$[\{1\},\{2\}]$
			$\{1,3\}$	$(1;3)$	$[\{1,3\},\{2\}]$	$\{1,2\}$	$\{1\}$	$[\{1\},\{3\}]$
			$\{2,3\}$	$(2;3)$	$[\{2,3\},\{1\}]$	$\{1,2\}$	$\{1\}$	$[\{2\},\{3\}]$

Table 2.1: Matching the l.h.s. with the r.h.s. for $n = 1, 2, 3$. Here $S_{\text{l.h.s.}} \subseteq \{1, \ldots, \nu(\underline{F}_{\text{l.h.s.}})\}$, $S_2 \subseteq S_{\text{l.h.s.}}$, with $|S_2|$ even. $(A;B)$ accounts for all ways of pairing up S_2. Further, $S_{\text{r.h.s.}} \subseteq \{1, \ldots, \nu(\underline{F}_{\text{r.h.s.}})\}$, $T \subseteq S_{\text{r.h.s.}}$, and $\boldsymbol{H}$ is over all $|T|$-tuples $\left([H_1, H_1^c], \ldots, \left[H_{|T|}, H_{|T|}^c\right]\right)$. The matching is as described in (2.4.6).

n	$\underline{F}_{\text{r.h.s.}}$	$S_{\text{r.h.s.}}$	T	$\boldsymbol{H}$
1	$[\{1\}]$	$\{1\}$	$\{1\}$	none
2	$[\{1\},\{2\}]$	$\{1\}$	$\{1\}$	none
	$[\{1\},\{2\}]$	$\{2\}$	$\{2\}$	none
	$[\{1\},\{2\}]$	$\{1,2\}$	$\{1,2\}$	none
3	$[\{1,2\},\{3\}]$	$\{2\}$	$\{2\}$	none
	$[\{1,2\},\{3\}]$	$\{1,2\}$	$\{2\}$	none
	$[\{1,2\},\{3\}]$	$\{1,2\}$	$\{1,2\}$	none
	$[\{1,3\},\{2\}]$	$\{2\}$	$\{2\}$	none
	$[\{1,3\},\{2\}]$	$\{1,2\}$	$\{2\}$	none
	$[\{1,3\},\{2\}]$	$\{1,2\}$	$\{1,2\}$	none
	$[\{2,3\},\{1\}]$	$\{2\}$	$\{2\}$	none
	$[\{2,3\},\{1\}]$	$\{1,2\}$	$\{2\}$	none
	$[\{2,3\},\{1\}]$	$\{1,2\}$	$\{1,2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1\}$	$\{1\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{2\}$	$\{2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{3\}$	$\{3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2\}$	$\{1\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2\}$	$\{2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2\}$	$\{1,2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,3\}$	$\{1\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,3\}$	$\{3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,3\}$	$\{1,3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{2,3\}$	$\{2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{2,3\}$	$\{3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{2,3\}$	$\{2,3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{1\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{1,2\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{1,3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{2,3\}$	none
	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\{1,2,3\}$	none

Table 2.2: Terms on the r.h.s. that are discarded since they contribute nothing to (2.4.4).

Chapter 3

A Recipe for Computing L-Functions

3.1 Preliminaries

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

be a Dirichlet series that converges absolutely in a half plane, $\Re(s) > \sigma_1$ (hence converges uniformly in any half plane $\Re(s) \geq \sigma_2 > \sigma_1$ by comparison with the series for $L(\sigma_2)$).

Let

$$\Lambda(s) = Q^s \prod_{j=1}^{a} \Gamma(\gamma_j s + \lambda_j) L(s), \qquad Q \in \mathbb{R}^+, \gamma_j \in \{1/2, 1\}, \Re\lambda_j \geq 0, \tag{3.1.1}$$

and assume that:

1. $\Lambda(s)$ has a meromorphic continuation to all of $\mathbb{C}$ with simple poles at $s_1, \ldots, s_\ell$ and corresponding residues $r_1, \ldots, r_\ell$.

2. (functional equation) $\Lambda(s) = \omega\overline{\Lambda(1 - \overline{s})}$ for some $\omega \in \mathbb{C}$, $\omega \neq 0$.

3. For any $\alpha \leq \beta$, $L(\sigma + it) = O(\exp t^A)$ for some $A > 0$, as $|t| \to \infty$, $\alpha \leq \sigma \leq \beta$, with A and the constant in the 'Oh' notation depending on α and β.

The above class of Dirichlet series includes all the L-functions that we were interested in computing (and more!). Note that if $b(n), \lambda_j \in \mathbb{R}$, then the second assumption reads $\Lambda(s) = \omega\Lambda(1-s)$.

In all examples that arise in number theory, it seems that the γ_j's can all be taken to equal $1/2$. (It is useful to know the Legendre duplication formula

$$\Gamma(s) = (2\pi)^{-1/2}2^{s-1/2}\Gamma(s/2)\Gamma((s+1)/2) \tag{3.1.2}$$

). However, it is sometimes more convenient to work with (3.1.1), and we avoid specializing prematurely to $\gamma_j = 1/2$.

Section 3.2 gives a formula ((3.2.4)) that can be taken as a starting point for computing $L(s)$. It requires choosing an auxilary function $g(s)$ that aids in controlling the size of $\Lambda(s)$. The choice of this function is described in Section 3.3.1. In Section 3.3.2 we work out in detail the remaining aspects of the algorithm for the case $a = 1$. In Section 3.3.3 we describe the analogous formulae for the case $a \geq 2$, $\lambda_j = 1/2$, but we avoid details since only the case $a = 1$ was needed for the computations in this thesis.

Remarks : *a) The 3rd condition, $L(\sigma + it) = O(\exp t^A)$, is very mild. Using the fact that $L(s)$ is bounded in $\Re s \geq \sigma_2 > \sigma_1$, the functional equation and the estimate (3.2.9), and the Phragmén-Lindelöf Theorem [32] we can show that in any vertical strip $\alpha \leq \sigma \leq \beta$,*

$$L(s) = O(t^b), \qquad \textit{for some } b > 0 \tag{3.1.3}$$

where both b and the constant in the 'Oh' notation depend on α and β.
b) In (3.1.1), Q does not stand for the conductor of $L(s)$ (though it is easily related to it) and a does not stand for the degree of $L(s)$ (since $\gamma_j \in \{1/2, 1\}$).

3.2 A formula

Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function that, for fixed s, satisfies

$$\left|\Lambda(z+s)g(z+s)z^{-1}\right| \to 0$$

as $|\Im z| \to \infty$, in vertical strips, $-\alpha \le \Re z \le \alpha$. Then

Theorem 3.1: *For $s \notin \{s_1, \ldots, s_\ell\}$, and $L(s)$, $g(s)$ as above,*

$$\Lambda(s)g(s) = \sum_{k=1}^{\ell} \frac{r_k g(s_k)}{s - s_k} + Q^s \sum_{n=1}^{\infty} \frac{b(n)}{n^s} f_1(s,n)$$
$$+\omega Q^{1-s} \sum_{n=1}^{\infty} \frac{\overline{b}(n)}{n^{1-s}} f_2(1-s,n) \tag{3.2.4}$$

where

$$f_1(s,n) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \prod_{j=1}^{a} \Gamma(\gamma_j(z+s) + \lambda_j) z^{-1} g(s+z)(Q/n)^z dz$$
$$f_2(s,n) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \prod_{j=1}^{a} \Gamma(\gamma_j(z+s) + \overline{\lambda_j}) z^{-1} g(1-s-z)(Q/n)^z dz \tag{3.2.5}$$

with $\nu > \max\{0, -\Re(\lambda_1/\gamma_1 + s), \ldots, -\Re(\lambda_a/\gamma_a + s)\}$.

Remark : *The third condition on $L(s)$ is not required for this Theorem. It is only required if we wish to allow certain $g(s)$'s. See Section 3.3.1.*

Proof.

Let C be the rectangle with verticies $(-\alpha, -iT)$, $(\alpha, -iT)$, (α, iT), $(-\alpha, iT)$, let $s \in \mathbb{C} - \{s_1, \ldots, s_\ell\}$, and consider

$$\frac{1}{2\pi i} \int_C \Lambda(z+s) g(z+s) z^{-1} dz. \tag{3.2.6}$$

(integrated counter-clockwise). α and T are chosen big enough so that all the poles of the integrand are contained within the rectangle. We will also require, soon, that $\alpha > \sigma_1 - \Re s$. On the one hand (3.2.6) equals

$$\Lambda(s)g(s) + \sum_{k=1}^{\ell} \frac{r_k g(s_k)}{s_k - s} \tag{3.2.7}$$

since the poles of the integrand are included in the set $\{0, s_1 - s, \ldots, s_\ell - s\}$, and are all simple (typically, the set of poles will coincide with this set. However, if

$\Lambda(s)g(s) = 0$, then $z = 0$ is no longer a pole of the integrand. But then $\Lambda(s)g(s)$ contributes nothing to (3.2.7) and the equality remains valid. And if $g(s_k) = 0$, then there is no pole at $z = s_k - s$ but also no contribution from $r_k g(s_k)/(s_k - s)$).

On the other hand, we may break the integral over C into four integrals:

$$\begin{aligned}\int_C &= \int_{\alpha-iT}^{\alpha+iT} + \int_{\alpha+iT}^{-\alpha+iT} + \int_{-\alpha+iT}^{-\alpha-iT} + \int_{-\alpha-iT}^{\alpha-iT} \\ &= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}.\end{aligned}$$

The integral over C_1, assuming that α is big enough to write $L(s+z)$ in terms of its Dirichlet series (i.e. $\alpha > \sigma_1 - \Re s$), is

$$Q^s \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \prod_{j=1}^{a} \Gamma(\gamma_j(z+s)+\lambda_j) z^{-1} g(s+z)(Q/n)^z dz$$

(we are justified in rearranging summation and integration since the series for $L(z+s)$ converges uniformly on C_1). Further, by the functional equation, the integral over C_3 equals

$$\begin{aligned}&\frac{\omega}{2\pi i}\int_{-\alpha+iT}^{-\alpha-iT} \overline{\Lambda(1-\overline{z+s})} g(z+s) z^{-1} dz \\ &= \omega Q^{1-s} \sum_{n=1}^{\infty} \frac{\overline{b}(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{-\alpha+iT}^{-\alpha-iT} \prod_{j=1}^{a} \Gamma(\gamma_j(1-s-z)+\overline{\lambda_j}) z^{-1} g(s+z)(Q/n)^{-z} dz \\ &= \omega Q^{1-s} \sum_{n=1}^{\infty} \frac{\overline{b}(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \prod_{j=1}^{a} \Gamma(\gamma_j(1-s+z)+\overline{\lambda_j}) z^{-1} g(s-z)(Q/n)^{z} dz. \quad (3.2.8)\end{aligned}$$

Letting $T \to \infty$, the integrals over C_2 and C_4 tend to zero by our assumption on the rate of growth of $g(s)$, hence the theorem (the integrals in (3.2.5) are, by Cauchy's Theorem, independent of the choice of ν, so long as
$\nu > \max\{0, -\Re(\lambda_1/\gamma_1 + s), \ldots, -\Re(\lambda_a/\gamma_a + s)\}$).

□

Formulae of the form (3.2.4) are well known. Usually, one finds it in the literature with $g(s) = 1$. For example, for the Riemann zeta function this leads to Riemann's formula [31, pg 179] [37, pg 22]

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = -\frac{1}{s} - \frac{1}{1-s} + \pi^{-s/2}\sum_{n=1}^{\infty}\frac{1}{n^s}\Gamma(s/2,\pi n^2) + \pi^{(s-1)/2}\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\Gamma((1-s)/2,\pi n^2)$$

where $\Gamma(s,w)$ is the incomplete gamma function (see Section 3.3.2).

However, the choice $g(s) = 1$ is not well suited for computing $\Lambda(s)$ as $|\Im(s)|$ grows. By Stirling's formula

$$|\Gamma(s)| \sim (2\pi)^{1/2}|s|^{\sigma-1/2}e^{-|t|\pi/2}, \qquad s = \sigma + it \tag{3.2.9}$$

(see Appendix A , Estimate 1) as $|t| \to \infty$, and so decreases very quickly as $|t|$ increases. Hence, with $g(s) = 1$, the l.h.s. of (3.2.4) is extremely small for large $|t|$ and fixed σ. On the other hand, one can show that the terms on the r.h.s., though decrease as $n \to \infty$, start off relatively large compared to the l.h.s.. Hence a **tremendous** amount of cancelation must occur on the r.h.s. and and we would need an unreasonable amount of precision in our computations. This difficulty has been noted by several other authors, including, Lagarias and Odlyzko [21] (Artin L-functions), Tollis [38] (Dedekind zeta functions), Fermigier [11] (elliptic curve L-functions), and Spira [35] (Ramanujan τ L-function). The former suggested a choice of $g(s)$ which does not have this difficulty (of cancelation) but did not implement it since it led to complications regarding the computation of (3.2.5), though we are able to overcome this difficulty. See the discussion in the next section.

3.3 Computing $\Lambda(s)$

We assume, for now, that only a single evaluation of $\Lambda(s)$ is to be performed at $s = s_0 = \sigma_0 + it_0$, with $\sigma_0 \geq 1/2$ (this assumption is reasonable by the functional equation). Later, we will describe precomputations that should be carried out if many evaluations of $\Lambda(s)$ are desired.

3.3.1 Choosing $g(s)$

In order to control the exponentially small size of $\Lambda(s)$, we choose

$$g(s) = \delta^{-s}$$

where

$$\delta = \begin{cases} 1, & \text{if } |t_0| \leq 2c/\pi; \\ \exp\left(i\theta\left(\operatorname{sgn}(t_0)\pi/2 - c/t_0\right)\right), & \text{if } |t_0| > 2c/\pi. \end{cases} \tag{3.3.10}$$

Here

$$\theta = \sum_{j=1}^{a} \gamma_j$$

and $c > 0$ is a constant that will be choosen depending on how many digits of precision we have at our disposal (the more precision, the larger we may choose c). Hence

$$|g(s_0)| = \begin{cases} 1, & \text{if } |t_0| \leq 2c/\pi; \\ \exp\left(\theta\,|t_0|\,\pi/2 - c\theta\right), & \text{if } |t_0| > 2c/\pi. \end{cases}$$

By Estimate 1 of Appendix A, and (3.3.10) we get

$$|\Lambda(s_0)g(s_0)| = \begin{cases} * \cdot \exp\left(-\theta\,|t_0|\,\pi/2\right)|L(s_0)|, & \text{if } |t_0| \leq 2c/\pi; \\ * \cdot \exp\left(-c\theta\right)|L(s_0)|, & \text{if } |t_0| > 2c/\pi. \end{cases} \tag{3.3.11}$$

where

$$* = Q^{\sigma_0}(2\pi)^{a/2} \prod_{j=1}^{a} |\gamma_j s_0 + \lambda_j|^{\gamma_j\sigma_0 + \Re\lambda_j - 1/2}\, h(\gamma_j s_0 + \lambda_j) \tag{3.3.12}$$

($h(s)$ is defined in Appendix A). Note that, when $\sigma_0 \geq 1/2$, $\Re(\gamma_j s_0 + \lambda_j) \geq 1/4$ (since $\lambda_j \in \{1/2, 1\}$, $\Re\lambda_j \geq 0$), and Estimate 1 gives

$$(2\pi)^{1/2}h(\gamma_j s_0 + \lambda_j) \geq (2\pi)^{1/2}e^{-1/(6/4)-1/4} = (2\pi)^{1/2}e^{-11/12} > 1 \tag{3.3.13}$$

so

$$* > Q^{\sigma_0} \prod_{j=1}^{a} |\gamma_j s_0 + \lambda_j|^{\gamma_j\sigma_0 + \Re\lambda_j - 1/2}. \tag{3.3.14}$$

We have thus managed to control the exponentially small size of $\Lambda(s)$ up to a factor of $\exp(-c\theta)$ which we can regulate via the choice of c. (In practice, I chose $c\theta = 17$, or smaller).

Remark : *The choice $g(s) = \delta^{-s}$ meets the criteria of Theorem 3.1. It is entire, and, and by (3.3.10) and (3.1.3), we have $\left|\Lambda(z+s)\delta^{-(z+s)}z^{-1}\right| \to 0$ in vertical strips (in fact, exponentially fast) as $|\Im z| \to \infty$.*

3.3.2 The functions $f_1(s,n), f_2(s,n)$, for the case $a = 1$

We treat the case $a = 1$ seperately since it provides a model for the case $a \geq 2$ and since many popular L-functions have $a = 1$.

Here we are assuming that

$$\Lambda(s) = Q^s\Gamma(\gamma s + \lambda)L(s), \qquad Q \in \mathbb{R}^+, \gamma \in \{1/2, 1\}, \Re\lambda \geq 0.$$

Hence, when $g(s) = \delta^{-s}$ and $a = 1$, the function $f_1(s,n)$ that appears in Theorem 3.1 equals

$$\begin{aligned} f_1(s,n) &= \frac{\delta^{-s}}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \Gamma(\gamma(z+s)+\lambda)z^{-1}\left(Q/(n\delta)\right)^z dz \\ &= \frac{\delta^{-s}}{2\pi i}\int_{\gamma\nu-i\infty}^{\gamma\nu+i\infty} \Gamma(u+\gamma s+\lambda)u^{-1}\left(Q/(n\delta)\right)^{u/\gamma} du \end{aligned} \tag{3.3.15}$$

(we have substituted $u = \gamma z$). Now

$$\Gamma(v+u)u^{-1} = \int_0^\infty \Gamma(v,t)t^{u-1}dt, \qquad \Re u > 0, \quad \Re(v+u) > 0 \tag{3.3.16}$$

where

$$\Gamma(z,w) = \int_w^\infty e^{-x}x^{z-1}dx = w^z\int_1^\infty e^{-wx}x^{z-1}dx, \qquad \Re(w) > 0. \tag{3.3.17}$$

$\Gamma(z,w)$ is known as the incomplete gamma function. So, by Mellin inversion

$$f_1(s,n) = \delta^{-s}\Gamma\left(\gamma s + \lambda, (n\delta/Q)^{1/\gamma}\right). \tag{3.3.18}$$

Similarly

$$f_2(s,n) = \delta^{s-1}\Gamma\left(\gamma s + \overline{\lambda}, (n/(\delta Q))^{1/\gamma}\right). \tag{3.3.19}$$

We may thus express, when $a = 1$ and $g(s) = \delta^{-s}$, (3.2.4) as

$$\boxed{\begin{aligned} Q^s\Gamma(\gamma s+\lambda)L(s)\delta^{-s} = \sum_{k=1}^{\ell}\frac{r_k\delta^{-s_k}}{s-s_k} &+ (\delta/Q)^{\lambda/\gamma}\sum_{n=1}^{\infty} b(n)n^{\lambda/\gamma}G\left(\gamma s+\lambda,(n\delta/Q)^{1/\gamma}\right) \\ &+\frac{\omega}{\delta}(Q\delta)^{-\overline{\lambda}/\gamma}\sum_{n=1}^{\infty}\overline{b}(n)n^{\overline{\lambda}/\gamma}G\left(\gamma(1-s)+\overline{\lambda},(n/(\delta Q))^{1/\gamma}\right)\end{aligned}} \tag{3.3.20}$$

where

$$G(z,w) = w^{-z}\Gamma(z,w) = \int_1^{\infty} e^{-wx}x^{z-1}dx, \qquad \Re(w) > 0. \tag{3.3.21}$$

Remark : *From (3.3.10) with $a = 1$, we have that $\Re\delta^{1/\gamma} > 0$, so both $(n\delta/Q)^{1/\gamma}$ and $(n/(\delta Q))^{1/\gamma}$ have positive $\Re$ part.*

Examples

1) Riemann zeta function, $\zeta(s)$: the necessary background can be found in [37] Formula (3.3.20), for $\zeta(s)$, is

$$\begin{aligned}\pi^{-s/2}\Gamma(s/2)\zeta(s)\delta^{-s} = -\frac{1}{s} - \frac{\delta^{-1}}{1-s} &+ \sum_{n=1}^{\infty} G\left(s/2,\pi n^2\delta^2\right) \\ &+\delta^{-1}\sum_{n=1}^{\infty} G\left((1-s)/2,\pi n^2/\delta^2\right)\end{aligned} \tag{3.3.22}$$

2) Dirichlet L-functions, $L(s,\chi)$: (see [6, chapter 9]). When χ is primitive and even, $\chi(-1) = 1$, we get

$$\begin{aligned}\left(\frac{q}{\pi}\right)^{s/2}\Gamma(s/2)L(s,\chi)\delta^{-s} &= \sum_{n=1}^{\infty}\chi(n)G\left(s/2,\pi n^2\delta^2/q\right) \\ &+\frac{\tau(\chi)}{\delta q^{1/2}}\sum_{n=1}^{\infty}\overline{\chi}(n)G\left((1-s)/2,\pi n^2/(\delta^2 q)\right)\end{aligned} \tag{3.3.23}$$

and when χ is primitive and odd, $\chi(-1) = -1$, we get

$$\left(\frac{q}{\pi}\right)^{s/2} \Gamma(s/2+1/2) L(s,\chi)\delta^{-s} = \delta\left(\frac{\pi}{q}\right)^{1/2} \sum_{n=1}^{\infty} \chi(n) n G\left(s/2+1/2, \pi n^2\delta^2/q\right) + \frac{\tau(\chi)\pi^{1/2}}{iq\delta^2} \sum_{n=1}^{\infty} \overline{\chi}(n) n G\left((1-s)/2+1/2, \pi n^2/(\delta^2 q)\right) \tag{3.3.24}$$

Here, $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

3) Cusp form L-functions: (see [27]). Let $f(z)$ be a cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$, k a positive even integer:

1. $f(z)$ is entire on $\mathbb{H}$, the upper half plane.
2. $f(\sigma z) = (cz+d)^k f(z)$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H}$.
3. $\lim_{t\to\infty} f(it) = 0$.

Assume further that f is a Hecke eigenform (i.e. an eigenfunction of the Hecke operators). We may expand f in a Fourier series

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}, \qquad \Im(z) > 0$$

and associate to $f(z)$ the Dirichlet series

$$L_f(s) := \sum_{1}^{\infty} \frac{a_n}{n^{(k-1)/2}} n^{-s}.$$

We normalize f so that $a_1 = 1$. This series converges absolutely when $\Re(s) > 1$ because, as proven by Deligne [8],

$$|a_n| \le \sigma_0(n) n^{(k-1)/2}, \tag{3.3.25}$$

where $\sigma_0(n) := \sum_{d|n} 1$ ($= O(n^\epsilon)$ for any $\epsilon > 0$).

$L_f(s)$ admits to an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation

$$\Lambda_f(s) := (2\pi)^{-s}\Gamma(s+(k-1)/2)L_f(s) = (-1)^{k/2}\Lambda_f(1-s).$$

With our normalization, $a_1 = 1$, the a_n's are real since they are eigenvalues of self adjoint operators (the Hecke operators with respect to the Petersson inner product) (see [27, III-12]). Furthermore, the required rate of growth on $L_f(s)$ (condition 3. on page 62) follows from the modularity of f.

Hence, in this example, formula (3.3.20) is

$$\begin{aligned}(2\pi)^{-s}\Gamma(s+(k-1)/2)L_f(s)\delta^{-s} &= (\delta 2\pi)^{(k-1)/2}\sum_{n=1}^{\infty} a_n G\left(s+(k-1)/2, 2\pi n\delta\right) \\ &+ \frac{(-1)^{k/2}}{\delta}\left(\frac{2\pi}{\delta}\right)^{(k-1)/2}\sum_{n=1}^{\infty} a_n G\left(1-s+(k-1)/2, 2\pi n/\delta\right)\end{aligned} \tag{3.3.26}$$

Remark : *The assumption that $a_1 = 1$ is not needed, since (3.3.26) remains true if we scale f by a constant. However, the way we have formulated Theorem 3.1 assumes a functional equation of the form $\Lambda(s) = \omega\overline{\Lambda(1-\overline{s})}$ which, in the case of $\Lambda_f(s)$, only holds if the a_n's are real.*

4) Twists of cusp forms: $L_f(s,\chi)$, χ primitive, $f(z)$ as in the previous example. $L_f(s,\chi)$ is given by the Dirichlet series

$$L_f(s,\chi) = \sum_1^{\infty} \frac{a_n}{n^{(k-1)/2}}\chi(n)n^{-s}.$$

$L_f(s,\chi)$ extends to an entire function, which satisfies the functional equation

$$\Lambda_f(s,\chi) := \left(\frac{q}{2\pi}\right)^s \Gamma(s+(k-1)/2)L_f(s,\chi) = (-1)^{k/2}\chi(-1)\frac{\tau(\chi)}{\tau(\overline{\chi})}\Lambda_f(1-s,\overline{\chi}).$$

In this example, formula (3.3.20) is

$$\left(\frac{q}{2\pi}\right)^s \Gamma(s+(k-1)/2)L_f(s,\chi)\delta^{-s} = \left(\frac{2\pi\delta}{q}\right)^{(k-1)/2} \sum_{n=1}^{\infty} a_n\chi(n)G\left(s+(k-1)/2, 2\pi n\delta/q\right)$$
$$+\frac{(-1)^{k/2}}{\delta}\chi(-1)\frac{\tau(\chi)}{\tau(\overline{\chi})}\left(\frac{2\pi}{q\delta}\right)^{(k-1)/2} \sum_{n=1}^{\infty} a_n\overline{\chi}(n)G\left(1-s+(k-1)/2, 2\pi n/(\delta q)\right) \tag{3.3.27}$$

5) Elliptic curve L-functions: (see [20, especially chapters X,XII]). Let E be an elliptic curve over $\mathbb{Q}$, which we write in global minimal Weierstrass form

$$y^2 + c_1xy + c_3y = x^3 + c_2x^2 + c_4x + c_6$$

where the c_j's are integers and the disciminant Δ is minimal.

To the elliptic curve E we may associate an Euler product

$$L_E(s) := \prod_{p|\Delta}(1-a_pp^{-1/2-s})^{-1}\prod_{p\nmid\Delta}(1-a_pp^{-1/2-s}+p^{-2s})^{-1} \tag{3.3.28}$$

where $a_p = p+1-\#E_p(\mathbb{Z}_p)$, with $\#E_p(\mathbb{Z}_p)$ being the number of points (x,y) in $\mathbb{F}_p\mathrm{x}\mathbb{F}_p$ on the curve E (considered modulo p).

When $p|\Delta$, a_p is either 1, -1, or 0 If $p \nmid \Delta$, a theorem of Hasse states that $|a_p| < 2p^{1/2}$. Hence, (3.3.28) converges when $\Re(s) > 1$, and for these values of s we may expand $L_E(s)$ in an absolutely convergent Dirichlet series

$$L_E(s) = \sum_1^{\infty} \frac{a_n}{n^{1/2}}n^{-s}.$$

The Hasse-Weil conjecture asserts that $L_E(s)$ extends to an entire function and has the functional equation

$$\Lambda_E(s) := \left(\frac{N^{1/2}}{2\pi}\right)^s \Gamma(s+1/2)L_E(s) = -\varepsilon\Lambda_E(1-s).$$

where N is the conductor of E, and ε, which depends on E, is either ± 1. The Hasse-Weil conjecture and also the required rate of growth on $L_E(s)$ follows from the Shimura-Taniyama-Weil conjecture, which has been proven by Wiles and Taylor [36] [40] for elliptic curves with square free conductor (and, apparently, has been extended, in recent work of Conrad, Diamond, and Taylor to N's which are not divisable by 27).

Hence, assuming that $L_E(s)$ satisfies the S-T-W conjecture, we have

$$\left(\frac{N^{1/2}}{2\pi}\right)^s \Gamma(s+1/2)L_E(s)\delta^{-s} = \left(\frac{2\pi\delta}{N^{1/2}}\right)^{1/2}\sum_{n=1}^{\infty} a_n G\left(s+1/2, 2\pi n\delta/N^{1/2}\right)$$
$$-\frac{\varepsilon}{\delta}\left(\frac{2\pi}{N^{1/2}\delta}\right)^{1/2}\sum_{n=1}^{\infty} a_n G\left(1-s+1/2, 2\pi n/(\delta N^{1/2})\right) \tag{3.3.29}$$

6) Twists of elliptic curve L-functions: $L_E(s,\chi)$, χ a primitive character of conductor q, $(q,N)=1$. Here $L_E(s,\chi)$ is given by the Dirichlet series

$$L_E(s,\chi) = \sum_1^{\infty} \frac{a_n}{n^{1/2}}\chi(n)n^{-s}.$$

The Hasse-Weil conjecture asserts, here, that $L_E(s)$ extends to an entire function and satisfies

$$\Lambda_E(s,\chi) := \left(\frac{qN^{1/2}}{2\pi}\right)^s \Gamma(s+1/2)L_E(s,\chi) = -\varepsilon\chi(-N)\frac{\tau(\chi)}{\tau(\overline{\chi})}\Lambda_E(1-s,\overline{\chi})$$

(here N and ε are the same as for E). In this example the formula is

$$\left(\frac{qN^{1/2}}{2\pi}\right)^s \Gamma(s+1/2)L_E(s)\delta^{-s} = \left(\frac{2\pi\delta}{qN^{1/2}}\right)^{1/2}\sum_{n=1}^{\infty} a_n\chi(n) G\left(s+1/2, 2\pi n\delta/(qN^{1/2})\right)$$
$$-\frac{\varepsilon}{\delta}\chi(-N)\frac{\tau(\chi)}{\tau(\overline{\chi})}\left(\frac{2\pi}{qN^{1/2}\delta}\right)^{1/2}\sum_{n=1}^{\infty} a_n\overline{\chi}(n) G\left(1-s+1/2, 2\pi n/(\delta qN^{1/2})\right) \tag{3.3.30}$$

We have reduced the computation of $\Lambda(s)$ to one of evaluating two sums of incomplete gamma functions (the $\Gamma(\gamma s+\lambda)\delta^{-s}$ factor on the left of (3.3.20) and elsewhere is easily evaluated using several terms of Stirling's formula and also the recurrence $\Gamma(z+1)=z\Gamma(z)$ applied a few times. The second step is needed for small z. Some care needs to be taken to absorb the $e^{-\pi\gamma|t|/2}$ factor of $\Gamma(\gamma s+\lambda)$ into the $e^{\pi\gamma|t|/2}$ factor of δ^{-s}. Otherwise our efforts to control the size of $\Gamma(\gamma s+\lambda)\delta^{-s}$ will have been in vain, and lack of precision will wreak havoc on the numerics).

We now have several issues to deal with regarding the r.h.s. of (3.3.20):

1. when to truncate the sums over n.

2. how to compute $G(z, w)$.

3. how to perform precomputations to speed up the algorithm (when many evaluations of $\Lambda(s)$ are desired).

4. how much cancelation (and hence loss of precision) can occur.

Issues 1 and 4. Precision and truncating the sum

The decision on when to truncate the sums depends on how much precision is available and on how accurate we wish to be. My programs were written in C++ and run on machines which follow the IEEE Standard 754 to represent floating point numbers. Double precision (i.e. 64 bits instead of 32 bits) was used. According to IEEE-754 [14], of these 64 bits, 52 bits are reserved for the mantissa, 11 bits are used for the exponent, and one bit is used to determine the sign of the number. Let $s \in \{0,1\}$ be the sign bit, let M be the value of the 52 bit mantissa, $0 \le M \le 2^{52}-1 \approx 4.5 \cdot 10^{15}$, and let E be the value of the exponent $0 \le E \le 2^{11}-1 = 2047$. The 64 bits in question represent the number $(-1)^s M 10^{E-1023}$. This means, that in the applicable range, we can represent numbers to 15 or 16 decimal places accuracy (starting at the first non-zero digit).

In truncating the two sums in (3.3.20) we need to consider two things. First, it would be pointless to sum terms whose contribution are rendered meaningless due to lack of precision. Second, we would need to show that the contribution from all the neglected terms is in fact negligible.

A rigorous discussion would require not just analytic bounds for the $G(\cdot,\cdot)$'s but also a detailed knowledge of minutiae related to the particular computing device (for example, how round off errors are handled). One would also need to consider factors such as how big the accumulated round off errors can be, and this would require bounding the total number of operations used. With regard to how much cancelation can occur in (3.3.20), we would need to take into account how the $G(\cdot,\cdot)$'s are computed and not just estimate their size.

In principle, many such issues can be dealt with theoretically or even computationally, for example, by keeping track in our computations of how many operations are performed or of how much cancelation has occured. In practice, I was satisfied with the estimates that follow, experimentation, and intuition.

With 15 digits precision, my goal was to truncate the sums in (3.3.20) when the contribution from their tails was estimated to be smaller than $|Q^{s_0}\Gamma(\gamma s_0 + \lambda)\delta^{-s_0}|10^{-15}$. To do so required estimating the size of $G(z,w)$. A rough bound is provided by

$$|G(z,w)| \le e^{-\Re(w)} \int_0^\infty e^{-(\Re(w)-\Re(z)+1)t} dt = \frac{e^{-\Re(w)}}{\Re(w)-\Re(z)+1}, \qquad \Re(w) > \Re(z) - 1. \tag{3.3.31}$$

(we have put $t = x - 1$ in (3.3.21) and have used $t + 1 \le e^t$). This tells us that the terms in (3.3.20) decrease exponentially fast once n is sufficiently large, and do so uniformly in vertical strips.

More precise information is available. We use Estimate 2 of Appendix A to bound the contribution from the tail end of the sums (we only work out the details for one sum). From Estimate 2

$$\left|\sum_{n=B}^{\infty} b(n) n^{\lambda/\gamma} G\left(\gamma s_0 + \lambda, (n\delta/Q)^{1/\gamma}\right)\right| < 4 \sum_{n=B}^{\infty} \frac{|b(n)|\, n^{\Re\lambda/\gamma} e^{-\Re\left((n\delta/Q)^{1/\gamma}\right)}}{|n/Q|^{1/\gamma} - |\gamma s_0 + \lambda| - \Re(\gamma s_0 + \lambda) - 1} \tag{3.3.32}$$

assuming that $|B/Q|^{1/\gamma} > |\gamma s_0 + \lambda| + |\Re(\gamma s_0 + \lambda)| + 1$ (this restricts our choice of B). Note from (3.3.10) that

$$\Re\left((n\delta/Q)^{1/\gamma}\right) = \begin{cases} (n/Q)^{1/\gamma}, & \text{if } |t_0| \le 2c/\pi; \\ (n/Q)^{1/\gamma} \sin(c/|t_0|), & \text{if } |t_0| > 2c/\pi. \end{cases}$$

Hence, assuming further that

$$\frac{1}{2}|B/Q|^{1/\gamma} > |\gamma s_0 + \lambda| + |\Re(\gamma s_0 + \lambda)| + 1 \tag{3.3.33}$$

then (3.3.32) is

$$<\begin{cases} 8Q^{1/\gamma}\sum_{n=B}^{\infty}|b(n)|\,n^{(\Re\lambda-1)/\gamma}e^{-(n/Q)^{1/\gamma}}, & \text{if } |t_0|\le 2c/\pi;\\ 8Q^{1/\gamma}\sum_{n=B}^{\infty}|b(n)|\,n^{(\Re\lambda-1)/\gamma}e^{-(n/Q)^{1/\gamma}.58c/|t_0|}, & \text{if } |t_0|>2c/\pi.\end{cases}$$

(we have also used $\sin(x)>x-x^3/3!=x(1-x^2/6)>.58x$ when $0<x<\pi/2$)

$$=\begin{cases} 8Q^{1/\gamma}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^\alpha}n^{\alpha+(\Re\lambda-1)/\gamma}e^{-(n/Q)^{1/\gamma}}, & \text{if } |t_0|\le 2c/\pi;\\ 8Q^{1/\gamma}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^\alpha}n^{(\Re\lambda-1)/\gamma}e^{-(n/Q)^{1/\gamma}.58c/|t_0|}, & \text{if } |t_0|>2c/\pi.\end{cases}$$

where $\alpha>\sigma_1$, the abscissa of absolute convergence of $L(s)$ (so that $\sum_{n=B}^{\infty}|b(n)|\,n^{-\alpha}$ converges). This is

$$\le\begin{cases} 8Q^{1/\gamma}B^{\alpha+(\Re\lambda-1)/\gamma}e^{-(B/Q)^{1/\gamma}}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^\alpha}, & \text{if } |t_0|\le 2c/\pi;\\ 8Q^{1/\gamma}B^{\alpha+(\Re\lambda-1)/\gamma}e^{-(B/Q)^{1/\gamma}.58c/|t_0|}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^\alpha}, & \text{if } |t_0|>2c/\pi.\end{cases} \tag{3.3.34}$$

assuming that B is chosen big enough so that the sequence

$$\begin{array}{rll} n^{\alpha+(\Re\lambda-1)/\gamma}e^{-(n/Q)^{1/\gamma}}, & n\ge B, & \text{if } |t_0|\le 2c/\pi;\\ n^{\alpha+(\Re\lambda-1)/\gamma}e^{-(n/Q)^{1/\gamma}.58c/|t_0|}, & n\ge B, & \text{if } |t_0|>2c/\pi.\end{array}$$

is decreasing. If $\alpha+(\Re\lambda-1)/\gamma\le 0$, the sequence is decreasing regardless of the choice of B. And if $\alpha+(\Re\lambda-1)/\gamma>0$, then, by calculus, it is decreasing so long as

$$B>\begin{cases} Q(\alpha\gamma+\Re\lambda-1)^\gamma & \text{if } |t_0|\le 2c/\pi;\\ Q\,|t_0|^\gamma\,(\alpha\gamma+\Re\lambda-1)^\gamma/(.58c)^\gamma & \text{if } |t_0|>2c/\pi.\end{cases} \tag{3.3.35}$$

Hence, assuming (both for $\gamma s+\lambda$ and $\gamma(1-s)+\overline{\lambda}$) that B is big enough so that (3.3.33) holds, and also so that the aforementioned sequence is decreasing, we find, from (3.3.34), that the contribution (including the factors outside the sums) to (3.3.20) coming from $n\ge B$, is, in norm,

$$<\begin{cases} 8Q^{(1-\Re\lambda)/\gamma}(1+|\omega|)B^{\alpha+(\Re\lambda-1)/\gamma}e^{-(B/Q)^{1/\gamma}}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^\alpha}, & \text{if } |t_0|\le 2c/\pi;\\ 8Q^{(1-\Re\lambda)/\gamma}(1+|\omega|)B^{\alpha+(\Re\lambda-1)/\gamma}e^{-(B/Q)^{1/\gamma}.58c/|t_0|}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^\alpha}, & \text{if } |t_0|>2c/\pi.\end{cases} \tag{3.3.36}$$

Now, we wish to choose B so that this is less than

$$\begin{aligned}&|Q^{s_0}\Gamma(\gamma s_0+\lambda)\delta^{-s_0}|10^{-15}\\&>\begin{cases}e^{-\gamma|t_0|\pi/2}Q^{\sigma_0}\,|\gamma s_0+\lambda|^{\gamma\sigma_0+\Re\lambda-1/2}\,10^{-15} & \text{if } |t_0|\le 2c/\pi;\\ e^{-\gamma c}Q^{\sigma_0}\,|\gamma s_0+\lambda|^{\gamma\sigma_0+\Re\lambda-1/2}\,10^{-15} & \text{if } |t_0|> 2c/\pi.\end{cases}\end{aligned} \tag{3.3.37}$$

(we have used (3.3.14)). If we assume, as was the case for all my computations, that $|b(n)|\le\sigma_0(n)$, then, choosing $\alpha=2$, we have, by Estimate 3,

$$\begin{aligned}\sum_{n=B}^{\infty}\frac{|b(n)|}{n^2}\le\sum_{n=B}^{\infty}\frac{\sigma_0(n)}{n^2}&<(B-1)^{-1}\left(\log(B-1)+(B-1)^{-1}+2.2221\ldots\right), \qquad B\ge 2\\&<1.1B^{-1}\left(\log(B)+2.4\right), \qquad B\ge 11\end{aligned} \tag{3.3.38}$$

(here $2.2221497\ldots=$ Euler's constant$+\zeta(2)$. We have also used $(B-1)^{-1}=(B(1-B^{-1}))^{-1}<1.1B^{-1}$ when $B\ge 11$, and $\log(1-B^{-1})+(B-1)^{-1}+2.2221\ldots<2.4$ when $B\ge 11$). So if we let

$$B\ge\begin{cases}Q\,|t_0|^{\gamma}\,v_1^{\gamma} & \text{if } |t_0|\le 2c/\pi;\\ Q\,|t_0|^{\gamma}\,v_2^{\gamma} & \text{if } |t_0|>2c/\pi.\end{cases} \tag{3.3.39}$$

(where v_1 and v_2 are yet to be determined), we find (with $|b(n)|\le\sigma_0(n)$, $\alpha=2$, and also assuming, as was the case for all my computations, that $|\omega|=1$) that (3.3.36) is

$$<\begin{cases}17.6\left(\log Q+\gamma\left(\log(|t_0|\,v_1)\right)+2.4\right)Q(|t_0|\,v_1)^{\gamma+\Re\lambda-1}e^{-|t_0|v_1} & \text{if } |t_0|\le 2c/\pi;\\ 17.6\left(\log Q+\gamma\left(\log(|t_0|\,v_2)\right)+2.4\right)Q(|t_0|\,v_2)^{\gamma+\Re\lambda-1}e^{-.58cv_2} & \text{if } |t_0|>2c/\pi.\end{cases} \tag{3.3.40}$$

($17.6=1.1\cdot 16$). We want to choose v_1 and v_2 so that the ratio (3.3.40):(3.3.37)<1. In principle, we could write down an explicit choice of v_1 and v_2 in terms of Q,γ,λ,c,s_0. However, in practice, we used (3.3.40) for specific examples, and so we only illustrate the choice of v_1 and v_2 for a particular example. Let $L_\tau(s)$ be Ramanujan's L-function attached to the cusp form of weight 12

$$e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24}. \tag{3.3.41}$$

Here, $Q = (2\pi)^{-1}$, $\gamma = 1$, $\lambda = 11/2$. With $c = 17$, $\sigma_0 = 1/2$, and $|t_0| < 10^{100}$, we get, by choosing

$$\begin{aligned} v1 &= \pi/2 + 25\log(10)/|t_0| \\ v2 &= 1/.58 + 25\log(10)/(.58c) = 7.56, \end{aligned} \tag{3.3.42}$$

that (3.3.40) is easily smaller than (3.3.37).

Remarks : *a) The assumption* $|t_0| < 10^{100}$ *is much larger than what was needed for my computations of* $L_\tau(s_0)$*. In our example this assumption was only used to simplify describing* v_1 *and* v_2 *(by making the logarithmic term in (3.3.40) bounded).*
b) one can obtain stronger bounds than those above, for example by assuming that there is cancelation in sums of the form $\sum_N^{N+M} b(n)$*. Rather than (3.3.32) we could sum by parts and exploit such cancelation. Improvements can also be obtained but at the expense of extra writing. For instance, we used* $\sin(c/|t_0|) > .58c/|t_0|$*, if* $|t_0| > 2c/\pi$*. However, as* $|t_0|$ *grows, the* $.58$ *can be replaced by any* $1-\varepsilon$*,* $\varepsilon > 0$*, so long as* $|t_0|$ *is big enough. This would have a significant impact on (3.3.40). This fact is reflected in the discontinuity of the choice of* B *at* $|t_0| = 2c/\pi$*.*

In practice, slightly improved inequalities were used which led to faster computer programs.

Issue 2, computing $G(z, w)$

There are many expressions and identities involving the incomplete gamma function, but none that I know of are well suited for evaluating $\Gamma(z, w)$ for a wide range of points in $\mathbb{C}$x$\mathbb{C}$. However, using three different expressions we can readily compute $\Gamma(z, w)$. Let

$$\gamma(z, w) := \Gamma(z) - \Gamma(z, w) = \int_0^w e^{-x} x^{z-1} dx, \qquad \Re z > 0, \quad |\arg w| < \pi$$

be the complimentary incomplete gamma function, and set

$$g(z, w) = w^{-z}\gamma(z, w) = \int_0^1 e^{-wt} t^{z-1} dt.$$

so that $G(z,w)+g(z,w)=w^{-z}\Gamma(z)$.

Then the following were used to compute $G(z,w)$:

$$g(z,w)=e^{-w}\sum_{j=0}^{M-1}\frac{w^j}{(z)_{j+1}}+R_M(z,w), \qquad \Re z>-M \tag{3.3.43}$$

$$\gamma(z,w+d)=\gamma(z,w)+w^{z-1}e^{-w}\sum_{j=0}^{\infty}\frac{(1-z)_j}{(-w)^j}(1-e^{-d}e_j(d)), \qquad |d|<|w| \tag{3.3.44}$$

$$G(z,w)=\frac{e^{-w}}{w}\sum_{j=0}^{M-1}\frac{(1-z)_j}{(-w)^j}+\epsilon_M(z,w), \tag{3.3.45}$$

where

$$\begin{aligned}
(z)_j &= \begin{cases} z(z+1)\dots(z+j-1) & \text{if } j>0;\\ 1 & \text{if } j=0.\end{cases}\\
R_M(z,w) &= \frac{w^M}{(z)_M}g(z+M,w)\\
e_j(d) &= \sum_{m=0}^{j}\frac{d^m}{m!}\\
\epsilon_M(z,w) &= \frac{(1-z)_M}{(-w)^M}G(z-M,w).
\end{aligned}$$

Both (3.3.43) and (3.3.45) are well known and are obtained via integration by parts. The second expression (3.3.44) is less known and due to Nielsen (a proof can be found in [10]). It is especially well suited when $\gamma=1$ since then, in (3.3.20), the $G(\cdot,\cdot)$'s have their second variable in arithmetic progression and this fact can be exploited to speed up the algorithm by performing some precomputations.

Table 3.1 describes when the three expressions (3.3.43) - (3.3.45) were used to compute $G(z,w)$. The third expression is well suited when $|w|$ is somewhat bigger than $|z|$. The first works well when $|w|$ is small regardless of the value of z, since then the $(z)_{j+1}$'s quickly overpower the w^j's, and also for larger $|w|$, so long as $|w|<|z|$. In my computations, it was used for

$$|w|\le|\Im z|-|\Im z|^{1/2}.$$

Finally one can also use (3.3.43) when $w > 0$ is real and $|z|$ is small (say $|z| \leq 10$, since, in this case, not to much cancelation can occur in the sum in (3.3.43).

Nielsen's expression can be used for those values of w where the other two fail. Note, however, that because (3.3.44) is an inductive relationship that relates $G(z, w+d)$ to $G(z,w)$, it could not be used, in (3.3.20) for the $n = 1$ term. Furthermore, in order to use it for n, the condition $|d| < |w|$ in (3.3.44) is

$$(n/Q)^{1/\gamma} - ((n-1)/Q)^{1/\gamma} < ((n-1)/Q)^{1/\gamma}$$

i.e. $n/(n-1) < 2^\gamma$. If $\gamma = 1/2$ this leads to $n \geq 4$, and if $\gamma = 1$ this implies $n \geq 3$. Fortunately, (3.3.43) could always be used, in my computations, for at least the $n = 1$ term, since I always had $Q \geq (2\pi)^{-1}$ when $\gamma = 1$, and $Q \geq \pi^{-1/2}$ when $\gamma = 1/2$ (so in both cases, $|w|$ for the $n = 1$ term is $\leq 2\pi$, and the terms in (3.3.43) do not have a chance to blow up). One could also still use (3.3.43) for the $n = 2$, $\gamma = 1$ term, and the $n = 2, 3$, $\gamma = 1/2$ terms if Q is big enough, or (if not) at the expense of some precision. Alternatively, for these first values of n, one could still use Nielsen's expansion, but by breaking up the gap $d = (n/Q)^{1/\gamma} - ((n-1)/Q)^{1/\gamma}$ into smaller steps.

Details related to (3.3.43)

When $|w| \leq |\Im z| - |\Im z|^{1/2}$, the number of terms needed is

$$M = \lceil 15 \log(10)/ \log |\Im(z)/w| \rceil \tag{3.3.46}$$

(here we have used $\left|w^M/(z)_M\right| \leq |w/\Im(z)|^M$ and also the fact that $g(z+M, w)$ is comparable, if not smaller, in size to $g(z,w)$). Note, if $\Re z \leq 0$, we should also make sure that M is big enough so that $\Re z + M > 0$.

Details for (3.3.44)

In computing (3.3.44) some care needs to be taken to avoid dangerous numerical pitfalls. One pitfall is that, as j grows, $e^{-d} e_j(d) \to 1$. So once $\left|1 - e^{-d} e_j(d)\right| < 10^{-15}$, then the error in computation of $1-e^{-d}e_j(d)$ is bigger than its value. So in computing $((1-z)_j/(-w)^j)\,(1-e^{-d}e_j(d))$ one must avoid the temptation to view this as a product of $(1-z)_j/(-w)^j$ and $1-e^{-d}e_j(d)$. Instead, we let

$$a_j(z,w,d) = \frac{(1-z)_j}{(-w)^j}(1-e^{-d}e_j(d)).$$

Now, $1-e^{-d}e_j(d) = e^{-d}(e^d - e_j(d))$, and we get

$$\begin{aligned} a_{j+1}(z,w,d) &= a_j(z,w,d)\frac{z-(j+1)}{w}\left(\sum_{j+2}^{\infty} d^m/m!\right) \bigg/ \left(\sum_{j+1}^{\infty} d^m/m!\right) \\ &= a_j(z,w,d)\frac{z-(j+1)}{w}\left(1-1/\beta_j(d)\right), \qquad j=1,2,3,\ldots \end{aligned}$$

$(a_0(z,w,d) = 1-e^{-d})$
where

$$\beta_j(d) = \sum_{m=0}^{\infty} d^m/(j+2)_m. \tag{3.3.47}$$

Notice in (3.3.20) that, when $\gamma = 1$, the $G(\cdot,\cdot)$'s have their second variable in arithmetic progression (with $d = \delta/Q$ the common difference between terms). This means that in using Nielsen's expansion, we need only compute the $\beta_j(\delta/Q)$'s once (and store them). These values can then be used for all the n's for which Nielsen's expansion is invoked to compute $G(\gamma s+\lambda, n\delta/Q)$. Furthermore, these same $\beta_j(\delta/Q)$'s can be used for any value of s so long as δ is the same (see Issue 3). Also note, that $|d| = |\delta/Q| = 1/Q$ was never large in my computations (in fact it was always $\le 2\pi$) and thus few terms were needed to compute $\beta_j(d)$.

Remark : *One might be tempted to compute the $\beta_j(d)$'s using the recursion*

$$\beta_{j+1}(d) = (\beta_j(d)-1)(j+2)/d$$

but this leads to numerical instability. The $\beta_j(d)$'s are all equal to $1+O_d(1/(j+2))$ and are thus all roughly of comparable size. Hence, a small error (due to roundoff) in $\beta_j(d)$ is turned into a much larger error in $\beta_{j+1}(d)$ ($(j+2)/|d|$ times larger), and this quickly destroys the numerics.

Our luck is not as great for the $\gamma = 1/2$ case since then the terms in (3.3.20) have $G(\cdot,\cdot)$'s with the second variable equal to $(n\delta/Q)^2$. Thus, the difference between two consecutive terms is

$$d = (2n+1)\delta^2/Q^2$$

which depends on n and hence the savings described for the $\gamma = 1$ case do not apply as nicely here (storing and reusing the $\beta_j(d)$'s in the $\gamma = 1/2$ case only makes sense if we plan on many evaluations of $\Lambda(s)$ (using the same δ), and if we have sufficient memory). Furthermore, we are using Nielsen's expansion when $|z| \sim |w|$, i.e. with $n \sim |z|^{1/2} Q$. Hence

$$|d| \sim 2\,|z|^{1/2}/Q.$$

Thus as $|z|^{1/2}/Q$ grows, we are faced with the problem that the terms in (3.3.47) explode. One can overcome this problem, at the expense of time, by breaking up the jump $d = (2n+1)\delta^2/Q^2$ into smaller steps of bounded size.

Now, Nielsen's expansion is to be used for $|w| \sim |z|$, as $|z| \to \infty$ (see Table 3.1). Furthermore

$$\beta_j(d) - 1 \sim d/(j+2), \qquad \text{as } |d|/j \to 0.$$

Hence, for $|w| \sim |z|$, we *roughly* have (as $|d|/j \to 0$)

$$\left|\frac{z-(j+1)}{w}(1 - 1/\beta_j(d))\right| \le \left(1 + \frac{j+1}{|w|}\right)\frac{|d|}{j+2} \le \frac{|d|}{j+2} + \frac{|d|}{|w|}.$$

Thus, because $|d/w| < 1$, we have, for j big enough, that the above is < 1, and so the sum in (3.3.44) converges geometrically fast, and hence only a handful of terms, which we denote by M_2 are required.

Remark : *As an alternative to Nielsen's expansion, there seems to be some potential in an asymptotic expression due to Ramanujan [1, pg 193, entry 6]*

$$G(z,w) \sim w^{-z}\Gamma(z)/2 + e^{-w}\sum_{k=0}^{M} p_k(w-z+1)/w^{k+1}, \qquad \textit{as } |z| \to \infty,$$

with $|w-z|$ relatively small, where $p_k(v)$ is a polynomial in v of degree $2k+1$, though this potential has not been investigated substantially. This expansions is relevant since *we are using Nielsen's expansion to compute $G(z,w)$ for w's that satisfy $|z-w| = O\left(|z|^{1/2}\right)$.*

We list the first few $p_k(v)$'s here:

$$p_0(v) = -v + 2/3$$
$$p_1(v) = -\frac{v^3}{3} + \frac{v^2}{3} - \frac{4}{135}$$
$$p_2(v) = -\frac{v^5}{15} + \frac{v^3}{9} - \frac{2\,v^2}{135} - \frac{4\,v}{135} + \frac{8}{2835}$$
$$p_3(v) = -\frac{v^7}{105} - \frac{v^6}{45} + \frac{v^5}{45} + \frac{7\,v^4}{135} - \frac{8\,v^3}{405} - \frac{16\,v^2}{567} + \frac{16\,v}{2835} + \frac{16}{8505}$$
$$p_4(v) = -\frac{v^9}{945} - \frac{2\,v^8}{315} - \frac{2\,v^7}{315} + \frac{8\,v^6}{405} + \frac{11\,v^5}{405} - \frac{62\,v^4}{2835} - \frac{32\,v^3}{1215} + \frac{16\,v^2}{1701} + \frac{16\,v}{2835} - \frac{8992}{12629925}$$
$$p_5(v) = -\frac{v^{11}}{10395} - \frac{v^{10}}{945} - \frac{2\,v^9}{567} - \frac{2\,v^8}{2835} + \frac{43\,v^7}{2835} + \frac{41\,v^6}{2835} - \frac{968\,v^5}{42525} - \frac{68\,v^4}{2835} + \frac{368\,v^3}{25515}$$
$$+ \frac{138064\,v^2}{12629925} - \frac{35968\,v}{12629925} - \frac{334144}{492567075}$$

Details for (3.3.45)

By the sentence following (A.0.1) of Appendix A

$$\left|(1-z)_M/w^M\right| \le e^{-M(|w|-|z|)/(2|w|)}, \qquad M \le |w| - |z| - 1.$$

In order for this to be smaller than 10^{-15} we need

$$M \ge 30\log(10)\,|w|\,/(|w|-|z|)$$

(note: $G(z-M,w)$ is comparable, if not smaller, in size to $G(z,w)$, so in (3.3.45), we stop when $\left|(1-z)_M/w^M\right| \le 10^{-15}$).

We can find an M between two real numbers if their difference is at least 1, and this leads to the inequality

$$|w| - |z| - 1 \ge 30\log(10)\,|w|\,/(|w|-|z|) + 1$$

i.e.

$$(|w|-|z|)^2 - 2(|w|-|z|) - 30\log(10)\,|w| \ge 0.$$

This is an inequality in $|w|-|z|$ which is satisfied if $|w|$ is greater than or equal to

$$|z|+(15\log(10)+1)+\sqrt{(15\log(10)+1)^2+4(15\log(10)+1)\,|z|} \tag{3.3.48}$$

and we set

$$M=\lceil 30\log(10)\,|w|\,/(|w|-|z|)\rceil\,. \tag{3.3.49}$$

Notice that the largest M is when $|w|$ is smallest, i.e.

$$|w|=|z|+(15\log(10)+1)+\sqrt{(15\log(10)+1)^2+4(15\log(10)+1)\,|z|}$$

and, for this value of $|w|$, the number of terms, M, needed is of size $O\left(|z|^{1/2}\right)$, as $|z|\to\infty$.

Summary for a single evaluation of $\Lambda(s)$

We have written

$$\begin{aligned}Q^s\Gamma(\gamma s+\lambda)L(s)\delta^{-s}=\sum_{k=1}^{\ell}\frac{r_k\delta^{-s_k}}{s-s_k}&+(\delta/Q)^{\lambda/\gamma}\sum_{n=1}^{B}b(n)n^{\lambda/\gamma}G\left(\gamma s+\lambda,(n\delta/Q)^{1/\gamma}\right)\\&+\frac{\omega}{\delta}(Q\delta)^{-\overline{\lambda}/\gamma}\sum_{n=1}^{B}\overline{b}(n)n^{\overline{\lambda}/\gamma}G\left(\gamma(1-s)+\overline{\lambda},(n/(\delta Q))^{1/\gamma}\right)\\&+\text{remainder}\end{aligned}$$

with $|\text{remainder}|\le|Q^s\Gamma(\gamma s+\lambda)\delta^{-s}|\,10^{-15}$. The choice of δ is described in (3.3.10), and of B on pages 75 - 78 (especially (3.3.39)). In my computations I used a value of $c=17/\gamma$, or smaller, which led, at most, to cancelation of size $e^{-17}=0.000000041\ldots$ (i.e. a loss in precision of at most roughly 8 decimal places).

To compute the sums over n the three expressions (3.3.43) - (3.3.45) are used according to Table 3.1. For the $\gamma=1$ case, the values of $\beta_j(d)$ are computed and stored ahead of time. They are reused for each invokation of Nielsen's expansion. (For the $\gamma=1/2$ case, these numbers are only stored if many evaluations of $\Lambda(s)$ are desired and if we have sufficient memory to store these numbers).

(3.3.43)	$n=1$, or $\lvert w\rvert \le \lvert\Im z\rvert - \lvert\Im z\rvert^{1/2}$, or ($\delta = 1$ and $\lvert z\rvert \le 10$)
(3.3.44)	otherwise
(3.3.44)	$\lvert w\rvert \ge \lvert z\rvert + (15\log(10)+1) + \sqrt{(15\log(10)+1)^2 + 4(15\log(10)+1)\lvert z\rvert}$

Table 3.1: A table indicating when (3.3.43) - (3.3.45) are used to compute the terms in (3.3.20). Here $w = (n\delta/Q)^{1/\gamma}$ or $(n/(\delta Q))^{1/\gamma}$, $z = \gamma s_0 + \lambda$ or $\gamma(1-s_0)+\overline{\lambda}$.

Issue 3, Precomputations

In this subsection we use the notation $s = \sigma + it$, and we assume that $\sigma \ge 1/2$.

When many evaluations of $\Lambda(s)$ are desired, it makes sense to use the same value of δ for each evaluation and to make sure that all sums are of the same length (i.e. the same B and same M_1, M_2, M_3 in (3.3.43) - (3.3.45). This allows us to rearrange order of summation, compute and store certain numbers in advance, and greatly enhance the efficiency of our program.

However, for a fixed value of δ, we run into cancelation problems as $|t|$ grows, as described following (3.2.9). This restricts the set of s's for which we may use a fixed δ.

Note in (3.3.10) that we are already using just one value of δ when $|t_0| \le 2c/\pi$ (namely, $\delta = 1$), so what follows is for the case $|t_0| > 2c/\pi$.

With the choice of δ described in (3.3.10), we have

$$\left|\delta^{-s}\right| = \exp\left(\gamma\,|t|\,\pi/2 - c\gamma\,|t/t_0|\right),$$

assuming that $\mathrm{sgn}(t) = \mathrm{sgn}(t_0)$. Combining this with Estimate 1 and (3.3.13) we get

$$\left|\Lambda(s)\delta^{-s}\right| > Q^{\sigma}\,|\gamma s + \lambda|^{\gamma\sigma + \Re\lambda - 1/2}\, e^{-c\gamma|t/t_0|}\,|L(s)|\,, \qquad \mathrm{sgn}(t) = \mathrm{sgn}(t_0)$$

hence, the exponential factor $e^{-c\gamma|t/t_0|}$ is no worse than it is for $s = s_0$, so long as $|t| \le |t_0|$. Thus the choice of B described on pages 75 - 78 works for these t as well.

Thus, we can use the same choice of δ and B so long as $\mathrm{sgn}(t) = \mathrm{sgn}(t_0)$ and $|t| \le |t_0|$. However, we would not want to do so when $|t|$ is much smaller than $|t_0|$, since then the choice of B is much larger than it need be. We could, for example, use the same δ for dyadic intervals. However, this would necessitate using Nielsen's expansion too often. Instead, one should use the same δ for intervals of length $|t_0|^{1/2}$.

Next, we describe precomputations that should be carried out. Consider a typical sum

$$\sum_{n=1}^{\infty} b(n) n^{\lambda/\gamma} G\left(\gamma s+\lambda, (n\delta/Q)^{1/\gamma}\right)$$
$$= \left(\sum_{1}^{B_1} + \sum_{B_1+1}^{B_2} + \sum_{B_2+1}^{B}\right) b(n) n^{\lambda/\gamma} G\left(\gamma s+\lambda, (n\delta/Q)^{1/\gamma}\right) \tag{3.3.50}$$

(details for the sum with $\gamma s + \lambda$ replaced by $\gamma(1-s) + \overline{\lambda}$ and $n\delta/Q$ replaced by $n/(\delta Q)$ are similar). We use (3.3.43) - (3.3.45), respectively for the three sums $\sum_1^{B_1}$, $\sum_{B_1+1}^{B_2}$, $\sum_{B_2+1}^{B}$. B_1 and B_2 are determined according to Table 3.2, where we also list the number of terms M_1, M_2, M_3 used in (3.3.43) - (3.3.45).

i	B_i	M_i
1	$Q\left(\left(\lvert t_0\rvert - \lvert t_0\rvert^{1/2}\right) - \left(\lvert t_0\rvert - \lvert t_0\rvert^{1/2}\right)^{1/2}\right)^{\gamma}$	$\left\lceil 15\log(10)/\log\left(\lvert t_0\rvert/(\lvert t_0\rvert - \lvert t_0\rvert^{1/2})\right)\right\rceil$
2	QW^{γ}	$O(1)$
3	—	$\lceil 30\log(10)W/(W-Z)\rceil$

Table 3.2: Computing (3.3.50) for many values of s (i.e. $|t_0| - |t_0|^{1/2} \le |t| \le |t_0|$, and $|\gamma s+\lambda| \le Z$) using the three expressions (3.3.43) - (3.3.45). Here, $W = Z + (15\log(10)+1) + \sqrt{(15\log(10)+1)^2 + 4(15\log(10)+1)Z}$. In the case, $\delta = 1$ and, say, $|\gamma s+\lambda| \le 10$, we can improve the efficiency of the algorithm by simply using (3.3.43) (i.e. $B_1 = B$ and $M_1 = O(1)$).

In what follows, we will require many instances of the numbers $\left\{\exp\left(-(n\delta/Q)^{1/\gamma}\right)\right\}_{n=1}^{B}$. It makes sense to compute and store these numbers ahead of time.

Precomputations on the $\sum_1^{B_1}$ sum

This sum (without the remainder term) equals

$$\sum_{n=1}^{B_1} b(n) n^{\lambda/\gamma}\left(\frac{\Gamma(\gamma s+\lambda)}{(n\delta/Q)^{s+\lambda/\gamma}} - \exp\left(-(n\delta/Q)^{1/\gamma}\right)\sum_{j=0}^{M_1-1}\frac{(n\delta/Q)^{j/\gamma}}{(\gamma s+\lambda)_{j+1}}\right)$$
$$= \Gamma(\gamma s+\lambda)(Q/\delta)^{s+\lambda/\gamma}\sum_{n=1}^{B_1}\frac{b(n)}{n^s} - \sum_{j=0}^{M_1-1}\frac{(B_1\delta/Q)^{j/\gamma}}{(\gamma s+\lambda)_{j+1}}\sum_{n=1}^{B_1}\frac{b(n)n^{(\lambda+j)/\gamma}}{B_1^{j/\gamma}}\exp\left(-(n\delta/Q)^{1/\gamma}\right) \tag{3.3.51}$$

Here we have reaaranged order of summation, and have also inserted a factor of $B_1^{j/\gamma}$ in the outer sum and of $B_1^{-j/\gamma}$ in the inner sum. This was done to control the large size of $n^{j/\gamma}$ in the inner sum and the small size of $Q^{-j/\gamma}/(\gamma s+\lambda)_{j+1}$ in the outer sum.

Now the inner sum

$$c_j := \sum_{n=1}^{B_1} \frac{b(n)n^{(\lambda+j)/\gamma}}{B_1^{j/\gamma}} \exp\left(-\,(n\delta/Q)^{1/\gamma}\right), \qquad j=0,\ldots,M_1-1$$

does not depend on s. So by precomputing and storing the c_j's, we may use them repeatedly, and thus rewrite (3.3.51) as

$$\Gamma(\gamma s+\lambda)(Q/\delta)^{s+\lambda/\gamma}\sum_{n=1}^{B_1}\frac{b(n)}{n^s} - \sum_{j=0}^{M_1-1} c_j \frac{(B_1\delta/Q)^{j/\gamma}}{(\gamma s+\lambda)_{j+1}} \tag{3.3.52}$$

The number of operations required to compute the c_j's is

$$O(M_1B_1) = O(|s|^{1/2+\gamma}\, Q)$$

and for computing the 2nd sum in (3.3.52) is

$$O(M_1) = O(|s|^{1/2}).$$

Precomputations for the $\sum_{B_1+1}^{B_2}$ sum

Here the sum (without the remainder term) is

$$\Gamma(\gamma s+\lambda)(Q/\delta)^{s+\lambda/\gamma}\sum_{n=B_1+1}^{B_2}\frac{b(n)}{n^s} - (Q/\delta)^{s+\lambda/\gamma}\sum_{n=B_1+1}^{B_2}\frac{b(n)}{n^s}\gamma\left(\gamma s+\lambda,(n\delta/Q)^{1/\gamma}\right). \tag{3.3.53}$$

Nielsen's expansion is used for the second sum as described on pages 80 - 82. Assuming that the $\beta_j(d)$'s have been precomputed, the number of operations required depends on γ. If $\gamma=1$, the number of operations required is proportional to B_2-B_1, i.e. is $O(|s|^{1/2}\, Q)$. For each n we require

$$\begin{cases} 1 & \text{if } |s|^{1/2}/Q \le \kappa; \\ O\left(|s|^{1/2}/(\kappa Q)\right), & \text{if } |s|^{1/2}/Q > \kappa, \end{cases}$$

invokations of (3.3.44), where κ is a constant (see page 82).So the total number of operations involving (3.3.44), when $\gamma = 1/2$, is

$$\begin{cases} O(Q) & \text{if } |s|^{1/2}/Q \le \kappa; \\ O(|s|^{1/2}/\kappa) & \text{if } |s|^{1/2}/Q > \kappa. \end{cases}$$

Precomputations for the $\sum_{B_2+1}^{B}$ sum

Here the sum (without the remainder) equals

$$\begin{aligned} &\sum_{n=B_2+1}^{B} b(n) n^{\lambda/\gamma} \frac{\exp\left(-(n\delta/Q)^{1/\gamma}\right)}{(n\delta/Q)^{1/\gamma}} \sum_{j=0}^{M_3-1} \frac{(1-\gamma s-\lambda)_j}{(n\delta/Q)^{j/\gamma}} \\ &= (Q/\delta)^{1/\gamma} \sum_{j=0}^{M_3-1} \frac{(1-\gamma s-\lambda)_j}{(B_2\delta/Q)^{j/\gamma}} \sum_{n=B_2+1}^{B} b(n) n^{(\lambda-1)/\gamma} (B_2/n)^{j/\gamma} \exp\left(-(n\delta/Q)^{1/\gamma}\right) \end{aligned} \tag{3.3.54}$$

Here we have inserted a $B_2^{-j/\gamma}$ in the outer sum and a $B_2^{j/\gamma}$ in the inner sum to control the size of the terms that appear in both. Thus, letting

$$d_j := \sum_{n=B_2+1}^{B} b(n) n^{(\lambda-1)/\gamma} (B_2/n)^{j/\gamma} \exp\left(-(n\delta/Q)^{1/\gamma}\right)$$

and precomputing and storing these numbers, we have that (3.3.54) equals

$$(Q/\delta)^{1/\gamma} \sum_{j=0}^{M_3-1} d_j \frac{(1-\gamma s-\lambda)_j}{(B_2\delta/Q)^{j/\gamma}}. \tag{3.3.55}$$

The number of operations required for the precomputations is, by (3.3.39) and Table 3.2, $O((B-B_2)M_3) = O(Q\,|s|^{\gamma+1/2}\,v)$, and for the computation of (3.3.55) is $O(M_3) = O(|s|^{1/2})$.

Precomputations for the 'main sum' $\sum_1^{B_2} b(n)n^{-s}$

Notice that

$$\sum_{n=1}^{B_2} \frac{b(n)}{n^s} = \sum_{n=1}^{B_2} \frac{b(n)}{n^{s_1}} e^{(s_1-s)\log(n)}. \tag{3.3.56}$$

If $\log(B_2)\,|s-s_1|$ is bounded (say $\le h$), then we can compute $e^{(s_1-s)\log(n)}$ to within a certain precision using only a fixed number, $K = K(h)$, of terms of the taylor expansion of e^z

$$e^{(s_1-s)\log(n)} = \sum_{j=0}^{K} ((s_1-s)\log(n))^j/j! + \text{remainder}.$$

Hence, (3.3.56) may be written (without the remainder) as

$$\begin{aligned}&\sum_{n=1}^{B_2}\frac{b(n)}{n^{s_1}}\sum_{j=0}^{K}((s_1-s)\log(n))^j/j!\\ &=\sum_{j=0}^{K}((s_1-s)\log(B_2))^j/j!\sum_{n=1}^{B_2}\frac{b(n)}{n^{s_1}}(\log(n)/\log(B_2))^j.\end{aligned} \tag{3.3.57}$$

Here we have inserted a $(\log(B_2))^j$ in the outer sum and a $(\log(B_2))^j$ in the inner sum in order to control the size of both. Hence, precomputing and storing

$$\alpha_j(s_1) := \sum_{n=1}^{B_2}\frac{b(n)}{n^{s_1}}(\log(n)/\log(B))^j$$

allows us to compute the main sum very efficiently (in constant time) for all $|s-s_1| \le h/\log(B_2)$.

3.3.3 The functions $f_1(s,n)$, $f_2(s,n)$, for the case $a \ge 2$, $\gamma_j = 1/2$

In this case, the function $f_1(s,n)$ that appears in Theorem 3.1 is

$$f_1(s,n) = \frac{\delta^{-s}}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\prod_{j=1}^{a}\Gamma((z+s)/2+\lambda_j)z^{-1}\,(Q/(\delta n))^z\,dz. \tag{3.3.58}$$

This is a special case of the Meijer G function and we develop some of its properties.

Let $\mathcal{M}\left(\phi(t);z\right)$ denote the Mellin transform of ϕ

$$\mathcal{M}\left(\phi(t);z\right) = \int_0^\infty \phi(t)t^{z-1}.$$

We will express $\prod_{j=1}^{a}\Gamma((z+s)/2+\lambda_j)z^{-1}$ as a Mellin transform analogous to (3.3.16).

Letting $\phi_1 * \phi_2$ denote the convolution of two functions

$$(\phi_1 * \phi_2)(v) = \int_0^\infty \phi_1(v/t)\phi_2(t)\frac{dt}{t}$$

we have (under certain conditions on ϕ_1, ϕ_2)

$$\mathcal{M}\left(\phi_1 * \phi_2; z\right) = \mathcal{M}\left(\phi_1; z\right) \cdot \mathcal{M}\left(\phi_2; z\right).$$

Thus

$$\prod_{j=1}^{a} \mathcal{M}\left(\phi_j; z\right) = \int_0^\infty (\phi_1 * \cdots * \phi_a)(t) t^{z-1} dt, \tag{3.3.59}$$

with

$$(\phi_1 * \cdots * \phi_a)(v) = \int_0^\infty \cdots \int_0^\infty \phi_1(v/t_1)\phi_2(t_1/t_2)\ldots\phi_{a-1}(t_{a-2}/t_{a-1})\phi_a(t_{a-1})\frac{dt_1}{t_1}\ldots\frac{dt_{a-1}}{t_{a-1}}.$$

Now

$$\prod_{j=1}^{a} \Gamma((z+s)/2+\lambda_j)z^{-1} = \left(\prod_{j=1}^{a-1} \Gamma((z+s)/2+\lambda_j)\right)\left(\Gamma((z+s)/2+\lambda_a)z^{-1}\right).$$

But

$$\Gamma((z+s)/2+\lambda) = \mathcal{M}\left(2e^{-t^2}t^{2\lambda+s}; z\right),$$

and (3.3.16) gives

$$\Gamma((z+s)/2+\lambda)z^{-1} = \mathcal{M}\left(\Gamma(s/2+\lambda, t^2); z\right).$$

So letting

$$\phi_j(t) = \begin{cases} 2e^{-t^2}t^{2\lambda_j+s} & j = 1, \ldots a-1; \\ \Gamma(s/2+\lambda_a, t^2) & j = a, \end{cases}$$

and applying Mellin inversion, we find that (3.3.58) equals

$$f_1(s,n) = \delta^{-s}(\phi_1 * \cdots * \phi_a)(n\delta/Q), \tag{3.3.60}$$

where

$$(\phi_1 * \cdots * \phi_a)(v) = v^{2\lambda_1+s} \int_0^\infty \cdots \int_0^\infty 2^{a-1} \prod_{j=1}^{a-1} t_j^{2(\lambda_{j+1}-\lambda_j)} e^{-\left(\frac{v^2}{t_1^2}+\frac{t_1^2}{t_2^2}+\cdots+\frac{t_{a-2}^2}{t_{a-1}^2}\right)} \left(\int_1^\infty e^{-t_{a-1}^2 x} x^{s/2+\lambda_a-1} dx\right) \frac{dt_1}{t_1} \cdots \frac{dt_{a-1}}{t_{a-1}}.$$

Substituting $u_j = \frac{(v^2x)^{j/a}}{v^2} t_j^2$ and rearranging order of integration this becomes

$$v^{2\mu+s} \int_1^\infty E_{\boldsymbol{\lambda}}\left(xv^2\right) x^{s/2+\mu-1} dx,$$

where

$$\mu = \frac{1}{a} \sum_{l=1}^{a} \lambda_j \tag{3.3.61}$$

$$E_{\boldsymbol{\lambda}}(w) = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{a-1} u_j^{\lambda_{j+1}-\lambda_j} e^{-w^{1/a}\left(\frac{1}{u_1}+\frac{u_1}{u_2}+\cdots+\frac{u_{a-2}}{u_{a-1}}+u_{a-1}\right)} \frac{du_1}{u_1} \cdots \frac{du_{a-1}}{u_{a-1}} \tag{3.3.62}$$

So, returning to (3.3.60), we find that

$$f_1(s,n) = (n\delta/Q)^{2\mu} (n/Q)^s \int_1^\infty E_{\boldsymbol{\lambda}}\left(x\,(n\delta/Q)^2\right) x^{s/2+\mu-1} dx.$$

Note that because (3.3.58) is symmetric in the λ_j's, so is $E_{\boldsymbol{\lambda}}$.

Similarly

$$f_2(s,n) = \delta^{-1} \left(n/(\delta Q)\right)^{2\overline{\mu}} (n/Q)^s \int_1^\infty E_{\overline{\boldsymbol{\lambda}}}\left(x\,(n/(\delta Q))^2\right) x^{s/2+\overline{\mu}-1} dx.$$

Hence,

$$\boxed{\begin{aligned} Q^s \prod_{j=1}^{a} \Gamma(s/2+\lambda_j) L(s) \delta^{-s} &= \sum_{k=1}^{\ell} \frac{r_k \delta^{-s_k}}{s-s_k} \\ &\quad + (\delta/Q)^{2\mu} \sum_{n=1}^{\infty} b(n) n^{2\mu} G_{\boldsymbol{\lambda}}\left(s/2+\mu, (n\delta/Q)^2\right) \\ &\quad + \frac{\omega}{\delta} (\delta Q)^{-2\overline{\mu}} \sum_{n=1}^{\infty} \overline{b}(n) n^{2\overline{\mu}} G_{\overline{\boldsymbol{\lambda}}}\left((1-s)/2+\overline{\mu}, (n/(\delta Q))^2\right) \end{aligned}} \tag{3.3.63}$$

with

$$G_{\boldsymbol{\lambda}}(z,w) = \int_1^\infty E_{\boldsymbol{\lambda}}(xw)\, x^{z-1} dx$$

(μ and $E_{\boldsymbol{\lambda}}$ are given by (3.3.61), (3.3.62)).

Examples

When $a = 2$

$$\begin{aligned} E_{\boldsymbol{\lambda}}(xw) &= \int_0^\infty t^{\lambda_2-\lambda_1} e^{-(wx)^{1/2}(1/t+t)} \frac{dt}{t} \\ &= 2K_{\lambda_2-\lambda_1}\left(2(wx)^{1/2}\right) = 2K_{\lambda_1-\lambda_2}\left(2(wx)^{1/2}\right), \end{aligned} \tag{3.3.64}$$

K being the K-Bessel function, so that $G_{\boldsymbol{\lambda}}$ is an incomplete integral of the K-Bessel function.

Note further that if $\lambda_1 = \lambda/2$, $\lambda_2 = (\lambda+1)/2$ then (3.3.64) is

$$2K_{1/2}\left(2(wx)^{1/2}\right) = \left(\pi^{1/2}/(wx)^{1/4}\right) e^{-2(wx)^{1/2}}$$

(see [10]), so $G_{(\lambda/2,(\lambda+1)/2)}(z,w) = 2(2\pi)^{1/2}(4w)^{-z}\Gamma(2z-1/2, 2w^{1/2})$, i.e. the incomplete gamma function. This is what we expect since, using (3.1.2), we can write the gamma factor $\Gamma((s+\lambda)/2)\Gamma((s+\lambda+1)/2)$ in terms of $\Gamma(s+\lambda)$, for which the $a=1$ expansion, (3.3.20), applies.

1) Maass cusp form L-functions: (background material can be found in [4]). Let f be a Maass cusp form with eigenvalue $\lambda = 1/4 - v^2$, i.e. $\Delta f = -\lambda f$, where $\Delta = y^2(\partial/\partial x^2 + \partial/\partial y^2)$, and Fourier expansion

$$f(z) = \sum_{n\neq 0} a_n y^{1/2} K_v(2\pi\,|n|\,y) e^{2\pi i n x},$$

with $a_{-n} = a_n$ for all n, or $a_{-n} = -a_n$ for all n. Let

$$L_f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}, \qquad \Re s > 1$$

(absolute convergence in this half plane can be proven via the Rankin-Selberg method), and let $\varepsilon = 0$ or 1 according to whether $a_{-n} = a_n$ or $a_{-n} = -a_n$. We have that

$$\Lambda_f(s) := \pi^{-s}\Gamma((s+\varepsilon+v)/2)\Gamma((s+\varepsilon-v)/2)L_f(s)$$

extends to an entire function and satisfies

$$\Lambda_f(s) = (-1)^\varepsilon \Lambda_f(1-s).$$

Hence, formula (3.3.63), for $L_f(s)$, is

$$\begin{aligned}
&\pi^{-s}\Gamma((s+\varepsilon+v)/2)\Gamma((s+\varepsilon-v)/2)L_f(s)\delta^{-s} = \qquad (3.3.65)\\
&(\delta\pi)^{\varepsilon}\sum_{n=1}^{\infty} a_n n^{\varepsilon} G_{((\varepsilon+v)/2,(\varepsilon-v)/2)}\left(s/2+\varepsilon/2,(n\delta\pi)^2\right)\\
&+\frac{(-1)^{\varepsilon}}{\delta}(\pi/\delta)^{\varepsilon}\sum_{n=1}^{\infty} a_n n^{\varepsilon} G_{((\varepsilon+\overline{v})/2,(\varepsilon-\overline{v})/2)}\left((1-s)/2+\varepsilon/2,(n\pi/\delta)^2\right)
\end{aligned}$$

where, by (3.3.64),

$$\begin{aligned}
G_{((\varepsilon+v)/2,(\varepsilon-v)/2)}\left(s/2+\varepsilon/2,(n\delta\pi)^2\right) &= 4\int_1^{\infty} K_v(2n\delta\pi t)t^{s+\varepsilon-1}dt\\
G_{((\varepsilon+\overline{v})/2,(\varepsilon-\overline{v})/2)}\left((1-s)/2+\varepsilon/2,(n\pi/\delta)^2\right) &= 4\int_1^{\infty} K_{\overline{v}}(2n\pi t/\delta)t^{-s+\varepsilon}dt.
\end{aligned}$$

Computing $G_{\boldsymbol{\lambda}}(z,w)$

We develop formulae analogous to (3.3.43) - (3.3.45), but for the function $G_{\boldsymbol{\lambda}}(z,w)$.

Let

$$\begin{aligned}
\Gamma_{\boldsymbol{\lambda}}(z,w) = w^z G_{\boldsymbol{\lambda}}(z,w) &= \int_w^{\infty} E_{\boldsymbol{\lambda}}(t)\,t^{z-1}dt,\\
\Gamma_{\boldsymbol{\lambda}}(z) &= \int_0^{\infty} E_{\boldsymbol{\lambda}}(t)\,t^{z-1}dt, \qquad (3.3.66)\\
\gamma_{\boldsymbol{\lambda}}(z,w) &= \int_0^{w} E_{\boldsymbol{\lambda}}(t)\,t^{z-1}dt,
\end{aligned}$$

with $E_{\boldsymbol{\lambda}}$ given by (3.3.62).

Lemma 8:

$$\Gamma_{\boldsymbol{\lambda}}(z) = \prod_{j=1}^{a}\Gamma(z-\mu+\lambda_j)$$

where $\mu = \frac{1}{a}\sum_{j=1}^{a}\lambda_j$.

Proof. Let $\psi_j(t) = e^{-t}t^{\lambda_j}$, $j = 1, \ldots, a$, and consider

$$
\begin{aligned}
(\psi_1 * \cdots * \psi_a)(v) &= v^{\lambda_1} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{a-1} t_j^{\lambda_{j+1}-\lambda_j} e^{-\left(\frac{v}{t_1}+\frac{t_1}{t_2}+\cdots+\frac{t_{a-2}}{t_{a-1}}+t_{a-1}\right)} \frac{dt_1}{t_1} \cdots \frac{dt_{a-1}}{t_{a-1}} \\
&= v^{\mu} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{a-1} x_j^{\lambda_{j+1}-\lambda_j} e^{-v^{1/a}\left(\frac{1}{x_1}+\frac{x_1}{x_2}+\cdots+\frac{x_{a-2}}{x_{a-1}}+x_{a-1}\right)} \frac{dx_1}{x_1} \cdots \frac{dx_{a-1}}{x_{a-1}}.
\end{aligned}
$$

(we have put $t_j = v^{1-j/a}x_j$). Thus, from (3.3.62)

$$
E_{\boldsymbol{\lambda}}(v) = v^{-\mu}(\psi_1 * \cdots * \psi_a)(v),
$$

and hence (3.3.66) equals

$$
\int_0^\infty (\psi_1 * \cdots * \psi_a)(t) t^{z-\mu-1} dt
$$

which, by (3.3.59) is $\prod_{j=1}^a \Gamma(z - \mu + \lambda_j)$.

The analogs of (3.3.43) - (3.3.45) are

$$
\begin{aligned}
w^{-z}\gamma_{\boldsymbol{\lambda}}(z,w) &= \frac{1}{z}\sum_{j=0}^{\infty} \frac{(-w)^j}{(1+z)_{j+1}} E_{\boldsymbol{\lambda}}^{(j)}(w) \\
\gamma_{\boldsymbol{\lambda}}(z,w+d) &= \gamma_{\boldsymbol{\lambda}}(z,w) + w^{z-1}\sum_{j=0}^{\infty} \frac{(1-z)_j}{(-w)^j} H_{\boldsymbol{\lambda}}(d,j,w), \qquad |d| < |w| \\
G_{\boldsymbol{\lambda}}(z,w) &\sim -\frac{1}{w}\sum_{j=0}^{M-1} \frac{(1-z)_j}{(w)^j} E_{\boldsymbol{\lambda}}^{(-j-1)}(w)
\end{aligned}
$$

where

$$
\begin{aligned}
H_{\boldsymbol{\lambda}}(d,j,w) &= \frac{1}{j!}\int_0^d E_{\boldsymbol{\lambda}}(u+w)\, u^j du \\
&= (-1)^{j+1} E_{\boldsymbol{\lambda}}^{(-j-1)}(w) + \sum_{m=0}^{j} (-1)^m \frac{d^{j-m}}{(j-m)!} E_{\boldsymbol{\lambda}}^{(-m-1)}(w+d)
\end{aligned}
$$

and $E_{\boldsymbol{\lambda}}^{(j)}(w)$ stands for the jth derivative with respect to w of $E_{\boldsymbol{\lambda}}(w)$ (anti-derivative if j is negative).

Appendix A

Messy estimates

We collect here some estimates concerning the the gamma and incomplete gamma functions.

Estimate 1: *Let $s = \sigma + it$, with $\sigma > 0$. Then,*

$$|\Gamma(s)| = (2\pi)^{1/2} |s|^{\sigma - 1/2} e^{-|t|\pi/2} h(s)$$

where

$$e^{-1/(6|s|) - \sigma^3/(3|t|^2)} \le h(s) \le e^{1/(6|s|) + \sigma^3/(3|t|^2)}, \qquad \text{if } |t| > \sigma$$
$$e^{-1/(6|s|) - \sigma} \le h(s) \le e^{1/(6|s|) + (\pi/2 - 1)\sigma}, \qquad \text{if } |t| \le \sigma.$$

Remark : *The first inequality tells us that $h(s) = 1 + O(1/|t|)$ as $|t| \to \infty$ (and σ fixed), while the second inequality tells us that, for $|t| \le \sigma$ and σ of reasonable size, $h(s)$ is neither too small or too large.*

Proof.

Using one term of Stirling's expansion we have

$$\Gamma(s) = (2\pi)^{1/2} s^{s-1/2} e^{-s} e^{R(s)}$$

where [28, pg 294],

$$|R(s)| \le \frac{1}{6|s|}$$

so that

$$(2\pi)^{1/2}|s^{s-1/2}|e^{-\sigma}e^{-1/(6|s|)} \leq |\Gamma(s)| \leq (2\pi)^{1/2}|s^{s-1/2}|e^{-\sigma}e^{1/(6|s|)}$$

But

$$|s^{s-1/2}| = |s|^{\sigma-1/2}\, e^{-t\arg(\sigma+it)}$$

and

$$\arg(\sigma+it) = \begin{cases} \pi/2 - \arctan(\sigma/t) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -\pi/2 - \arctan(\sigma/t) & \text{if } t < 0 \end{cases}$$

From the McLaurin expansion of $\arctan(x)$

$$x - |x|^3/3 \leq \arctan(x) \leq x + |x|^3/3, \qquad |x| < 1$$

we get

$$|\Gamma(s)| = (2\pi)^{1/2}\,|s|^{\sigma-1/2}\,e^{-|t|\pi/2}h(s)$$

where

$$e^{-1/(6|s|)-\sigma^3/(3|t|^2)} \leq h(s) \leq e^{1/(6|s|)+\sigma^3/(3|t|^2)}, \qquad \text{if } |t| > \sigma.$$

If $0 < |t| \leq \sigma$, then $0 < t\arctan(\sigma/t) \leq \sigma\pi/2$ (since $|t| \leq \sigma$, $|\arctan(\sigma/t)| \leq \pi/2$, and because $t\arctan(\sigma/t) \geq 0$ when $\sigma > 0$). Hence, in this case

$$e^{-1/(6|s|)-\sigma} \leq h(s) \leq e^{1/(6|s|)+(\pi/2-1)\sigma}, \qquad \text{if } 0 < |t| \leq \sigma.$$

For the case $t = 0$, $\arg(\sigma+it) = 0$, and

$$e^{-1/(6|s|)-\sigma} \leq h(s) \leq e^{1/(6|s|)-\sigma}, \qquad \text{if } t = 0.$$

Estimate 2: *Let* $|w| > |z| + |\Re(z)| + 2$, *and* $\Re(w) > 0$. *Then*

$$|G(z,w)| < 4\frac{e^{-\Re(w)}}{|w| - |z| - |\Re(z)| - 1}.$$

Proof.

Let $j \geq 1$. We have

$$\left|\frac{(1-z)_j}{w^j}\right| = \left|\frac{(1-z)\dots(j-z)}{w^j}\right| \leq \prod_{m=1}^{j}\left(\frac{m}{|w|}+\left|\frac{z}{w}\right|\right)$$
$$\leq e^{-j(1-|z/w|)+j(j+1)/(2|w|)} \tag{A.0.1}$$

which follows from $1+x \leq e^x$ (applied to $(m+|z|)/|w| = 1+(m/|w|+|z|/|w|-1)$). Now if $(j+1)/|w| \leq 1-|z/w|$ then the above is $\leq e^{-j(1-|z/w|)/2}$. Hence, from (3.3.45) and (3.3.31),

$$|G(z,w)| \leq \frac{e^{-\Re(w)}}{|w|}\sum_{j=0}^{M-1} e^{-j(1-|z/w|)/2}$$
$$+\frac{e^{-\Re(w)}}{\Re(w)-\Re(z)+M+1}e^{-M(1-|z/w|)/2}, \qquad \Re(z-w)-1 < M \leq |w|-|z|-1.$$

(the condition $M > \Re(z-w)-1$ is required by (3.3.31)). Now, we are also implicitly assuming $M \geq 1$. We can definitely find an $M \geq 1$ satisfying

$$\Re(z-w)-1 < M \leq |w|-|z|-1$$

if $|w|-|z|-1 \geq 1$, i.e.

$$|w| \geq |z|+2 \tag{A.0.2}$$

and if the difference between the two bounds is ≥ 1, i.e.

$$|w|-|z| \geq \Re(z-w)+1. \tag{A.0.3}$$

Both (A.0.2) and (A.0.3) are satisfied if

$$|w| \geq |z|+|\Re(z)|+2$$

(because we are assuming $\Re(w) > 0$).

Note that the conditions of the estimate imply that $|z/w| < 1$, hence

$$\sum_{j=0}^{M-1} e^{-j(1-|z/w|)/2} = \frac{1-e^{-M(1-|z/w|)/2}}{1-e^{-(1-|z/w|)/2}} \leq \frac{1}{1-e^{-(1-|z/w|)/2}}$$
$$\leq \frac{8}{3(1-|z/w|)}, \qquad |z| < |w| \tag{A.0.4}$$

(the last step because $1 - e^{-x} \geq x - x^2/2 \geq 3x/4$ when $0 \leq x \leq 1/2$), and so

$$\begin{aligned}|G(z,w)| \leq & \frac{8e^{-\Re(w)}}{3(|w|-|z|)} \\ & + \frac{e^{-\Re(w)}}{\Re(w)-\Re(z)+M+1} e^{-M(1-|z/w|)/2}, \qquad \Re(z-w)-1 < M \leq |w|-|z|-1.\end{aligned}$$

Choosing M as large as possible, i.e. $M = \lfloor |w|-|z|-1 \rfloor > |w|-|z|-2$ we finally get

$$\begin{aligned}|G(z,w)| &\leq \frac{8e^{-\Re(w)}}{3(|w|-|z|)} + \frac{e^{-\Re(w)}}{\Re(w)-\Re(z)+(|w|-|z|-2)+1} \\ &\leq 4\frac{e^{-\Re(w)}}{|w|-|z|-|\Re(z)|-1}. \end{aligned} \tag{A.0.5}$$

(the last step because we've assumed that $\Re(w) > 0$, and $|w| \geq |z| + |\Re(z)| + 2$, so the final denominator is positive).

Estimate 3: *Let x be a positive integer, $\alpha > 1$. Then,*

$$\sum_{n>x} \frac{\sigma_0(n)}{n^\alpha} < \frac{x^{1-\alpha}}{\alpha-1}\left(\log(x)+\gamma+\zeta(\alpha)+x^{-1}\right)$$

where $\gamma = .57721566\ldots$ is Euler's constant.

Proof. Summing by parts, we get

$$\sum_{n \leq x} \frac{1}{n^\alpha} = \zeta(\alpha) - \frac{x^{1-\alpha}}{\alpha-1} + R(x), \qquad \alpha > 0, \alpha \neq 1,$$

where

$$0 < R(x) = \alpha \int_x^\infty \{t\}\, t^{-\alpha-1} dt < x^{-\alpha}.$$

Here $\{t\} = t - \lfloor t \rfloor$ denotes the fractional part of t. So

$$\begin{aligned}\sum_{n \leq x} \frac{\sigma_0(n)}{n^\alpha} &= \sum_{d_1 \leq x} \frac{1}{d_1^\alpha} \sum_{d_2 \leq x/d_1} \frac{1}{d_2^\alpha} \\ &= \sum_{d_1 \leq x} \frac{1}{d_1^\alpha} \zeta(\alpha) - \frac{(x/d_1)^{1-\alpha}}{\alpha-1} + R(x/d_1) \\ &= \zeta(\alpha)\left(\zeta(\alpha) - \frac{x^{1-\alpha}}{\alpha-1} + R(x)\right) - \frac{x^{1-\alpha}}{\alpha-1}\sum_{d \leq x}\frac{1}{d} + R_2(x) \end{aligned} \tag{A.0.6}$$

where

$$0 < R_2(x) = \sum_{d \le x} \frac{1}{d^\alpha} R(x/d) < x^{1-\alpha}.$$

Summing by parts,

$$\sum_{d \le x} \frac{1}{d} = \log(x) + \gamma + R_3(x)$$

with

$$0 < R_3(x) = \int_x^\infty \{t\}\, t^{-2} dt < x^{-1}.$$

Hence, (A.0.6) equals

$$\zeta^2(\alpha) - \frac{x^{1-\alpha}}{\alpha - 1} \left(\log(x) + \gamma + \zeta(\alpha) + R_3(x)\right) + \zeta(\alpha) R(x) + R_2(x).$$

Now,

$$\sum_{n > x} \frac{\sigma_0(n)}{n^\alpha} = \zeta^2(\alpha) - \sum_{n \le x} \frac{\sigma_0(n)}{n^\alpha}$$

which, from the above, is less than

$$\frac{x^{1-\alpha}}{\alpha - 1} \left(\log(x) + \gamma + \zeta(\alpha) + x^{-1}\right)$$

(here we have also used $\zeta(\alpha) > 0$ when $\alpha > 1$).

Bibliography

[1] Bruce C. Berndt, *Ramanujan's notebooks. Part II*, Springer-Verlag, New York, 1989.

[2] M. V. Berry and J. P. Keating, *A new asymptotic representation for $\zeta(\frac{1}{2}+it)$ and quantum spectral determinants*, Proc. Roy. Soc. London Ser. A **437** (1992), no. 1899, 151–173.

[3] B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. I*, J. Reine Angew. Math. **212** (1963), 7–25.

[4] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997.

[5] Daniel Bump and David Ginzburg, *Symmetric square L-functions on GL(r)*, Ann. of Math. (2) **136** (1992), no. 1, 137–205.

[6] Harold Davenport, *Multiplicative number theory*, second ed., Graduate Texts in Mathematics, vol. 74, Springer-Verlag, New York, 1980, Revised by Hugh L. Montgomery.

[7] D. Davies and C. B. Haselgrove, *The evaluation of Dirichlet L-functions*, Proc. Roy. Soc. Ser. A **264** (1961), 122–132.

[8] Pierre Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.

[9] H. M. Edwards, *Riemann's zeta function*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974, Pure and Applied Mathematics, Vol. 58.

[10] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi, *Higher transcendental functions. Vol. II*, Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981, Based on notes left by Harry Bateman, Reprint of the 1953 original.

[11] Stéfane Fermigier, *Zéros des fonctions L de courbes elliptiques*, Experiment. Math. **1** (1992), no. 2, 167–173.

[12] Stephen Gelbart, *An elementary introduction to the Langlands program*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), no. 2, 177–219.

[13] A. E. Ingham, *The distribution of prime numbers*, Cambridge University Press, Cambridge, 1990, Reprint of the 1932 original, With a foreword by R. C. Vaughan.

[14] Institute of Electrical and Electronics Engineers, *IEEE standard for binary floating-point arithmetic, ANSI/IEEE standard 754-1985*, August 1985.

[15] Matti Jutila, *On the mean value of $L(1/2, \chi)$ for real characters*, Analysis **1** (1981), no. 2, 149–161.

[16] Nicholas Katz and Peter Sarnak, *Zeros of zeta functions, their spacings, and their spectral nature*, preprint, January 1997.

[17] ———, *Random matrices, Frobenius eigenvalues, and monodromy*, 1998, preprint.

[18] J. B. Keiper, *On the zeros of the Ramanujan τ-Dirichlet series in the critical strip*, Math. Comp. **65** (1996), no. 216, 1613–1619.

[19] A. W. Knapp, *Introduction to the Langlands program*, Representation theory and automorphic forms (Edinburgh, 1996), Proc. Sympos. Pure Math., vol. 61, Amer. Math. Soc., Providence, RI, 1997, pp. 245–302.

[20] Anthony W. Knapp, *Elliptic curves*, Mathematical Notes, vol. 40, Princeton University Press, Princeton, NJ, 1992.

[21] J. C. Lagarias and A. M. Odlyzko, *On computing Artin L-functions in the critical strip*, Math. Comp. **33** (1979), no. 147, 1081–1095.

[22] Madan Lal Mehta, *Random matrices*, second ed., Academic Press Inc., Boston, MA, 1991.

[23] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972) (Providence, R.I.), Amer. Math. Soc., 1973, pp. 181–193.

[24] M. Ram Murty, *A motivated introduction to the Langlands program*, Advances in number theory (Kingston, ON, 1991), Oxford Univ. Press, New York, 1993, pp. 37–66.

[25] A. M. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Math. Comp. **48** (1987), no. 177, 273–308.

[26] ———, *The 10^{20}th zero of the Riemann zeta function and 70 million of its neighbors*, A.T.&T., `http://www.research.att.com/~amo/unpublished/index.html`, 1989.

[27] Andrew Ogg, *Modular forms and Dirichlet series*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[28] F. W. J. Olver, *Asymptotics and special functions*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974, Computer Science and Applied Mathematics.

[29] Ali E. Özlük and C. Snyder, *Small zeros of quadratic L-functions*, Bull. Austral. Math. Soc. **47** (1993), no. 2, 307–319.

[30] R. B. Paris, *An asymptotic representation for the Riemann zeta function on the critical line*, Proc. Roy. Soc. London Ser. A **446** (1994), no. 1928, 565–587.

[31] Bernhard Riemann, *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge*, Springer-Verlag, Berlin, 1990, Based on the edition by

Heinrich Weber and Richard Dedekind, Edited and with a preface by Raghavan Narasimhan.

[32] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[33] Zeév Rudnick and Peter Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81** (1996), no. 2, 269–322, A celebration of John F. Nash, Jr.

[34] Robert Rumely, *Numerical computations concerning the ERH*, Math. Comp. **61** (1993), no. 203, 415–440, S17–S23.

[35] Robert Spira, *Calculation of the Ramanujan τ-Dirichlet series*, Math. Comp. **27** (1973), 379–385.

[36] Richard Taylor and Andrew Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572.

[37] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.

[38] Emmanuel Tollis, *Zeros of Dedekind zeta functions in the critical strip*, Math. Comp. **66** (1997), no. 219, 1295–1321.

[39] J. van de Lune, H. J. J. te Riele, and D. T. Winter, *On the zeros of the Riemann zeta function in the critical strip. IV*, Math. Comp. **46** (1986), no. 174, 667–681.

[40] Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551.

[41] Hiroyuki Yoshida, *On calculations of zeros of L-functions related with Ramanujan's discriminant function on the critical line*, J. Ramanujan Math. Soc. **3** (1988), no. 1, 87–95.

[42] ———, *On calculations of zeros of various L-functions*, J. Math. Kyoto Univ. **35** (1995), no. 4, 663–696.