

Sixth powers in a mean square: an accessible guide to the 11/12 argument for $\zeta(s)$

Claude Opus 5.5 and OpenAI Astra
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A note from Michael Rubinstein. These are working notes, generated by Claude and Codex to help me read the papers on the quasi-Riemann hypothesis that OpenAI released.

Abstract

We give an accessible account of an argument showing that $\zeta(s)$ has no zeros in $\Re s > 11/12$. The averages are written out in classical terms: Möbius sums over Eisenstein integers, twisted by sextic residue symbols and averaged in mean square. Rows that are sixth powers recover the coefficients of $1/(\zeta(s)L(s, \chi_{-3}))$, and the mean square is bounded using Poisson summation, Kubota's cubic theta function, a quadratic large sieve, and a descent.

The proof in six steps.

1. To show that $\zeta(s)$ has no zeros with real part above $11/12$, it is enough to prove that the smoothed Möbius sum $A_1(D) = \sum_n \mu_K(n) W(N(n)/D)$ over Eisenstein integers, whose coefficients are those of $1/(\zeta(s)L(s, \chi_{-3}))$, is at most about $D^{11/12}$.
2. That sum is embedded in a large family of similar sums $A_u(D) = \sum_n \mu_K(n)(u/n)_6 W(N(n)/D)$, each twisted by a sextic residue symbol, and the roughly $H^{1/6}$ rows where $u = \pi^6$ is the sixth power of a prime reproduce the original sum, up to a small error.
3. The mean square $\sum_{0 < N(u) \leq H} |A_u(D)|^2$ is a sum of nonnegative terms, so a bound of size about DH for it (square-root cancellation in each row, on average) controls the roughly $H^{1/6}$ copies of $|A_1(D)|^2$ it contains; this gives $|A_1(D)|^2 \lesssim DH^{5/6}$, and taking H a little larger than D gives the exponent $11/12$.
4. Poisson summation in the row variable u turns this average into a dual family with fewer rows, and here is the key trick: the Gauss sums produced by Poisson summation combine with the Möbius function to give *cubic* Gauss sums, because the cube of a cubic Gauss sum at a prime carries exactly a factor -1 , which is the Möbius sign.
5. Cubic Gauss sums are, up to fixed residue-class factors, the Fourier coefficients of Kubota and Patterson's cubic theta function, whose symmetry under a group of matrices shortens each twisted sum and turns the sextic twist into an ordinary quadratic one, so that Heath-Brown's quadratic large sieve gives the needed bound.
6. The theta function's coefficients include extra terms indexed by cubes; these are removed by Möbius inversion and by a repeated two-step Poisson argument that passes back through Möbius sums and shrinks the number of rows each time, until the problem becomes trivial.

Scope. This is a guide to the argument of [1], specialized to ζ , not a new independent proof of its analytic estimates. Sections 1–5 give the deduction from one precisely stated mean-square estimate in full. Sections 6–8 explain the estimates behind it, retaining the parameters needed to see how the argument closes. The cubic theta theorem and the quadratic large sieve are substantial inputs. In particular, Section 8 states the uniform transfer estimate used from [1] and gives the induction that closes once that estimate and the completed theta estimate are available. This separates the deduction of the zero-free region from the analytic estimates on which it rests. The finite checks in Section 9 test identities and parameter ranges; they do not independently verify those uniform estimates or certify the claimed theorem. Background on $\mathbb{Z}[\omega]$, residue symbols, Gauss sums, Poisson summation over $\mathbb{Z}[\omega]$ and the cubic theta function is in the companion notes [3].

1 The ordinary integer coefficients

Let μ be the ordinary Möbius function and put

$$\chi_{-3}(m) = \begin{cases} 0, & 3 \mid m, \\ 1, & m \equiv 1 \pmod{3}, \\ -1, & m \equiv 2 \pmod{3}. \end{cases} \quad (1.1)$$

The coefficients of the reciprocal product are completely classical:

$$c(m) = \sum_{de=m} \mu(d)\mu(e)\chi_{-3}(e), \quad \frac{1}{\zeta(s)L(s, \chi_{-3})} = \sum_{m \geq 1} \frac{c(m)}{m^s} \quad (\Re s > 1). \quad (1.2)$$

This follows by multiplying the two absolutely convergent reciprocal Dirichlet series. For primes other than 3, the local coefficients are

	$c(p)$	$c(p^2)$	$c(p^j), j \geq 3$
$p \equiv 1 \pmod{3}$	-2	1	0
$p \equiv 2 \pmod{3}$	0	-1	0

(1.3)

and c is multiplicative. Thus c differs from μ : for example, $c(7) = -2$, $c(49) = 1$, and $c(25) = -1$.

Choose a complex-valued smooth function $W \in C_c^\infty((0, \infty); \mathbb{C})$ supported in a fixed interval $[a, b] \subset (0, \infty)$. Think of $W(m/D)$ as a smooth way of restricting m to size D . The sum to be bounded is

$$A_1(D) = \sum_{\substack{m \geq 1 \\ (m, 6) = 1}} c(m)W(m/D). \quad (1.4)$$

Removing the primes 2 and 3 gives the exact identity

$$\sum_{(m, 6) = 1} \frac{c(m)}{m^s} = \frac{1}{(1 - 3^{-s})(1 - 4^{-s})\zeta(s)L(s, \chi_{-3})}. \quad (1.5)$$

At 2, the reciprocal Euler factor is $1 - 2^{-2s} = 1 - 4^{-s}$; this explains the 4 rather than 2 in this formula.

2 Why the sums use Eisenstein integers

The notes [3, Sections 2–3] give the arithmetic of $\mathbb{Z}[\omega]$ and its residue symbols in full, with proofs and tables. They write λ for $1 - \omega$; their Sections 4–5 write $\sqrt{-3}$ for the element $\lambda = 1 + 2\omega$ used here.

Put $\omega = e^{2\pi i/3}$, so $\omega^2 + \omega + 1 = 0$. An Eisenstein integer is a point $n = a + b\omega$ with $a, b \in \mathbb{Z}$, and

$$N(n) = |n|^2 = a^2 - ab + b^2. \quad (2.1)$$

These points form a triangular lattice in the plane. The ring $\mathbb{Z}[\omega]$ has unique factorization. Its six units are $\pm 1, \pm\omega, \pm\omega^2$. To avoid counting six associates, we choose the *primary* generator:

$$n \equiv 1 \pmod{3}, \quad \text{equivalently } a \equiv 1 \pmod{3}, \quad b \equiv 0 \pmod{3}. \quad (2.2)$$

Every element coprime to 3 has exactly one primary associate. In sums below, n, f, b_1 and divisors of primary elements are primary and coprime to 6, unless stated otherwise. The row variables u, k, h run over all of $\mathbb{Z}[\omega]$. A star means that the element is squarefree.

Define $\mu_K(n) = 0$ if a prime square divides n , and $\mu_K(n) = (-1)^r$ if n is a product of r distinct primary primes; $\mu_K(1) = 1$. This is the same factorization rule as for the ordinary Möbius function, applied in $\mathbb{Z}[\omega]$.

A rational prime $p \equiv 1 \pmod{3}$ splits into two conjugate prime factors of norm p . A prime $p \equiv 2 \pmod{3}$ remains prime, of norm p^2 . Consequently

$$\sum_{\substack{n \text{ primary} \\ N(n)=m}} \mu_K(n) = c(m) \quad ((m, 6) = 1). \quad (2.3)$$

For a split prime the local factor is $(1 - p^{-s})^2$; for an inert prime it is $1 - p^{-2s}$. These are exactly the factors in (1.2). Thus (1.4) can also be written

$$A_1(D) = \sum_n \mu_K(n) W(N(n)/D). \quad (2.4)$$

The two-dimensional notation keeps track of the two primes above a split rational prime separately. The norm alone loses this information.

3 The average, written out

For a prime π coprime to 6, the number $N(\pi) - 1$ is divisible by 6. Define the sextic residue symbol by

$$\left(\frac{u}{\pi}\right)_6 = \text{the sixth root of unity congruent to } u^{(N(\pi)-1)/6} \pmod{\pi} \quad (3.1)$$

when $\pi \nmid u$, and define it to be 0 when $\pi \mid u$. Extend it multiplicatively in the denominator and write $\chi_n(u) = (u/n)_6$. It is multiplicative in the numerator as well. For $n = 1$ its value is 1, including at $u = 0$. Whenever we take a power of a character, its zero values remain zero, also for negative or zero exponents.

The individual sums, and their mean square, are

$$A_u(D) = \sum_n \mu_K(n) \chi_n(u) W(N(n)/D), \quad (3.2)$$

$$M(D, H) = \sum_{0 < N(u) \leq H} |A_u(D)|^2. \quad (3.3)$$

There are about H rows u , and each row contains about D possible columns n . More explicitly, with $Q(a, b) = a^2 - ab + b^2$,

$$M(D, H) = \sum_{\substack{x, y \in \mathbb{Z} \\ 0 < Q(x, y) \leq H}} \left| \sum_{\substack{a \equiv 1 \pmod{3}, b \equiv 0 \pmod{3} \\ (a+b\omega, 2)=1}} \mu_K(a + b\omega) \left(\frac{x + y\omega}{a + b\omega} \right)_6 W\left(\frac{Q(a, b)}{D}\right) \right|^2. \quad (3.4)$$

This is an ordinary sum over integer coordinates. Coprimality to 2 means that a, b are not both even.

Expanding the square gives the requested double sum over the columns:

$$M(D, H) = \sum_{n_1, n_2} \mu_K(n_1) \mu_K(n_2) \overline{W(N(n_1)/D)} W(N(n_2)/D) S_H(n_1, n_2), \quad (3.5)$$

where the finite character correlation is

$$S_H(n_1, n_2) = \sum_{0 < N(u) \leq H} \overline{\left(\frac{u}{n_1} \right)_6} \left(\frac{u}{n_2} \right)_6. \quad (3.6)$$

Every coefficient in this average is now specified by factorization, finite-field exponentiation, and a smooth weight.

For completeness, one can group every row by ordinary integer norms:

$$c_u(m) = \sum_{\substack{n \text{ primary} \\ N(n)=m}} \mu_K(n) \chi_n(u), \quad A_u(D) = \sum_{(m,6)=1} c_u(m) W(m/D). \quad (3.7)$$

Here $c_1(m) = c(m)$ from (1.2). In general $c_u(m)$ requires the residue symbols: μ and χ_{-3} alone do not specify the separate twists at conjugate primes.

The analytic estimate to prove. For every fixed $0 < \vartheta \leq 1/10$ and every $\varepsilon > 0$, the required estimate is

$$M(D, D^{1+\vartheta}) \ll_{W, \vartheta, \varepsilon} D^{2+\vartheta+\varepsilon}. \quad (3.8)$$

The notation $\ll$ means an inequality up to a constant independent of D . The factor D^ε allows an arbitrarily small power loss. The full argument controls the constant using finitely many derivatives of W . The estimate says that the *average* squared row sum is at most $D^{1+\varepsilon}$; it does not assert such a bound for every row.

4 The diagonal and the special rows

4.1 Why HD is the natural scale

For $n_1 = n_2 = n$, the correlation is simply

$$S_H(n, n) = \#\{u \in \mathbb{Z}[\omega] : 0 < N(u) \leq H, (u, n) = 1\}. \quad (4.1)$$

For fixed n , elementary lattice counting and inclusion–exclusion give

$$S_H(n, n) = \frac{2\pi}{\sqrt{3}} H \prod_{\pi|n} \left(1 - \frac{1}{N(\pi)} \right) + O(\tau(n)(\sqrt{H} + 1)), \quad (4.2)$$

where $\tau(n)$ is the number of primary divisors of n . The area of the disc is πH and the lattice has covolume $\sqrt{3}/2$. For the proof, the simpler uniform upper bound $S_H(n, n) \ll H$ suffices. The diagonal contribution to (3.5) is therefore $O_W(HD)$. Estimate (3.8) bounds the whole mean square on this scale, up to D^ε . It controls the aggregate off-diagonal contribution; it does not bound the sum of the absolute values of individual off-diagonal terms.

4.2 Sixth powers expose the untwisted sum

If $u = v^6$, then

$$\chi_n(v^6) = \mathbf{1}_{(v,n)=1}, \quad A_{v^6}(D) = \sum_{(n,v)=1} \mu_K(n) W(N(n)/D). \quad (4.3)$$

We use prime $v = \pi$ of comparable size so that the omitted terms are few. Put $Y = H^{1/6}$ and

$$\mathcal{P}_Y = \{\pi \text{ primary prime, } \pi \nmid 6 : Y/2 < N(\pi) \leq Y\}, \quad J = |\mathcal{P}_Y|. \quad (4.4)$$

Then $N(\pi^6) \leq H$, and

$$|A_{\pi^6}(D) - A_1(D)| \ll_W D/Y. \quad (4.5)$$

Indeed, a contributing omitted n is πm with $N(m) \leq bD/N(\pi)$; a disc of squared radius T contains $O(1 + T)$ lattice points. In our range $D/Y \geq 1$. The prime number theorem in the progression 1 (mod 3) gives $J \asymp Y/\log Y$. These rows are distinct: equal sixth powers have associated bases, and primary associates are unique.

The elementary inequality $|z|^2 \leq 2|w|^2 + 2|z - w|^2$, followed by positivity of (3.3), gives the rigorous extraction

$$J|A_1(D)|^2 \leq 2 \sum_{\pi \in \mathcal{P}_Y} |A_{\pi^6}(D)|^2 + O_W(JD^2/Y^2) \leq 2M(D, H) + O_W(JD^2/Y^2). \quad (4.6)$$

The dyadic interval $Y/2 < N(\pi) \leq Y$ is useful: it makes the error D/Y uniform for every selected row.

4.3 Cubes, squares, and rational rows

There are two other useful exact identities:

$$\chi_n(v^3) = \left(\frac{v}{n}\right)_2, \quad \chi_n(v^2) = \left(\frac{v}{n}\right)_3. \quad (4.7)$$

Thus cube rows are quadratic twists and square rows are cubic twists. These classes overlap, as sixth powers illustrate. For a nonzero *rational integer* v and an odd denominator n ,

$$\left(\frac{v}{n}\right)_2 = \left(\frac{v}{N(n)}\right), \quad (4.8)$$

with the ordinary Jacobi symbol on the right. For split primes this is Euler's criterion over $\mathbb{F}_p$; for an inert prime of norm p^2 , every nonzero element of $\mathbb{F}_p$ is a square in $\mathbb{F}_{p^2}$. Both sides vanish if $p \mid v$. Consequently

$$A_{v^3}(D) = \sum_{(m,6)=1} c(m) \left(\frac{v}{m}\right) W(m/D). \quad (4.9)$$

If $\chi_v(m) = (v/m)$ on odd m denotes the corresponding quadratic Dirichlet character there, the Dirichlet series of these coefficients is

$$\prod_{p \nmid 6} (1 - \chi_v(p)p^{-s})(1 - \chi_v(p)\chi_{-3}(p)p^{-s}). \quad (4.10)$$

This states the Euler factors explicitly without conventions at 2.

A rational numerator does not generally eliminate the need for Eisenstein residue symbols in a sextic or cubic twist. Also, the rational slice $u \in \mathbb{Z}$, $0 < N(u) \leq H$, has $\asymp H^{1/2}$ rows, since $N(u) = u^2$. The whole lattice has $\asymp H$ rows. Sixth-power lattice rows number $\asymp H^{1/6}$; sixth powers with rational bases number $\asymp H^{1/12}$. These are different counts.

5 From the mean square to the zero-free region

Theorem 1. *If (3.8) holds for every smooth compactly supported W and every fixed $0 < \vartheta \leq 1/10$, then $\zeta(s)$ has no zeros in $\Re s > 11/12$.*

Proof. Take $H = D^{1+\vartheta}$ in (4.6). Absorbing a logarithm into D^ε gives

$$|A_1(D)|^2 \ll_{W,\vartheta,\varepsilon} DH^{5/6} D^\varepsilon + D^2 H^{-1/3} \quad (5.1)$$

$$= D^{11/6+5\vartheta/6+\varepsilon} + D^{5/3-\vartheta/3}. \quad (5.2)$$

Taking square roots, and then choosing ϑ and ε sufficiently small, yields, for every $\eta > 0$,

$$A_1(D) \ll_{W,\eta} D^{11/12+\eta}. \quad (5.3)$$

The exponent $11/12$ is half of $2 - 1/6$: the mean square has size roughly D^2 , and the useful rows number roughly $D^{1/6}$. Quadratic or cubic twists would have more useful rows and a better exponent; Section 7.3 explains why the argument needs sextic ones.

Here is the precise link with zeros. Write

$$F(s) = (1 - 3^{-s})(1 - 4^{-s})\zeta(s)L(s, \chi_{-3}), \quad \widehat{W}(s) = \int_0^\infty W(y)y^s \frac{dy}{y}. \quad (5.4)$$

Suppose $\zeta(\rho) = 0$ and $\Re \rho > 11/12$. Choose a nonzero nonnegative smooth function ϕ supported in $(1, 2)$ and set $W(y) = y^{-\rho}\phi(y)$. Then $\widehat{W}(\rho) > 0$. By (5.3),

$$B_W(s) = \int_0^\infty A_1(D)D^{-s} \frac{dD}{D} \quad (5.5)$$

is holomorphic in $\Re s > 11/12$. There is no difficulty at $D = 0$: the sum vanishes for $D < 1/2$. For $\Re s > 1$, absolute convergence and (1.5) give $F(s)B_W(s) = \widehat{W}(s)$. Meromorphic continuation extends this identity to $\Re s > 11/12$. Since $L(s, \chi_{-3})$ is entire and $\rho \neq 1$, $F(\rho) = 0$. Evaluation at ρ now gives $0 = \widehat{W}(\rho) > 0$, a contradiction. $\square$

The remaining question is how to prove (3.8). The next three sections explain the argument of [1], specialized to ζ , and identify its hard inputs.

6 Poisson summation changes the coefficients

Additive characters, Gauss sums and Poisson summation over $\mathbb{Z}[\omega]$, including the identities used in this section, are explained in [3, Section 4].

6.1 A smooth row cutoff

Replace the disc cutoff by a fixed nonnegative smooth radial weight $\Phi(N(u)/H)$ with $\Phi(t) \geq 1$ for $0 \leq t \leq 1$. One can choose Φ rapidly decreasing and its two-dimensional Fourier transform $\widehat{\Phi}$ compactly supported. This gives an upper bound for M . It is this smoothed average to which Poisson summation is applied.

Here is an exact version of the transformation. Put

$$\lambda = 1 + 2\omega = i\sqrt{3}, \quad e(z) = \exp(4\pi i \operatorname{Im} z / \sqrt{3}), \quad \widehat{\Phi}(N(w)) = \frac{2}{\sqrt{3}} \int_{\mathbb{C}} \Phi(|z|^2) e(-zw) dx dy. \quad (6.1)$$

Only squarefree n_1, n_2 occur because of the Möbius factors. Write $n_i = gz_i$ with $g = (n_1, n_2)$. Then z_1, z_2 are coprime and

$$\overline{\chi_{n_1}(u)} \chi_{n_2}(u) = \mathbf{1}_{(u,g)=1} \psi(u), \quad \psi = \overline{\chi_{z_1}} \chi_{z_2}. \quad (6.2)$$

The character ψ is primitive modulo $z_1 z_2$. For a primitive character modulo q , set

$$\gamma(\psi) = N(q)^{-1/2} \sum_{x \bmod q} \psi(x) e(x/q). \quad (6.3)$$

Inclusion–exclusion for $(u, g) = 1$ and lattice Poisson summation give

$$\begin{aligned} & \sum_{u \in \mathbb{Z}[\omega]} \Phi(N(u)/H) \mathbf{1}_{(u,g)=1} \psi(u) \\ &= \frac{H\gamma(\psi)}{\sqrt{N(z_1 z_2)}} \sum_{d|g} \frac{\mu_K(d) \psi(d)}{N(d)} \sum_{h \in \mathbb{Z}[\omega]} \overline{\psi(h)} \widehat{\Phi} \left(\frac{HN(h)}{N(d)N(z_1 z_2)} \right). \end{aligned} \quad (6.4)$$

The zero frequency $h = 0$ occurs only when $z_1 = z_2 = 1$, exactly the diagonal. All other frequencies satisfy

$$N(h) \ll \frac{N(d)N(z_1 z_2)}{H} \ll \frac{D^2}{HN(g)}, \quad (6.5)$$

because $d \mid g$ and $N(z_i) \asymp D/N(g)$. Thus a row range of size $H > D$ becomes a frequency range of size at most a constant times $D^2/H < D$. The divisor d in the exact formula must be retained.

6.2 The Möbius–Gauss identity

For squarefree n put

$$\gamma_j(n) = N(n)^{-1/2} \sum_{x \bmod n} \chi_n(x)^j e(x/n), \quad \alpha(n) = \frac{n}{|n|}, \quad G(n) = \overline{\chi_n(4)} \gamma_3(n). \quad (6.6)$$

The arithmetic identity that makes the method work is

$$\mu_K(n) \gamma_{-1}(n) = \chi_n(-1) G(n)^{-1} \overline{\alpha(n)} \gamma_2(n). \quad (6.7)$$

The right side contains a *cubic* Gauss sum, since $\chi_n^2 = (\cdot/n)_3$. The unit-modulus factor G depends only on the residue class of n modulo 72 (in fact modulo 4 here).

To see its effect on (3.5), let $\mathfrak{G}$ be the finite group of invertible classes modulo 72 that are 1 modulo 3. Expand the function $t \mapsto G(t^{-1})$ in its finite group characters: $G(t^{-1}) = \sum_{\xi} c_{\xi} \xi(t)$. With $a_{\xi}(n) = \overline{\alpha(n)} \gamma_2(n) \xi(n)$, reciprocity and (6.7) give, for coprime squarefree z_1, z_2 ,

$$\mu_K(z_1) \mu_K(z_2) \gamma(\overline{\chi_{z_1}} \chi_{z_2}) = \sum_{\xi} c_{\xi} a_{\xi}(z_1) \overline{a_{\xi}(z_2)}. \quad (6.8)$$

There are only finitely many ξ , independently of D . Formula (6.8) preserves the form of a mean square while replacing its Möbius coefficients by cubic Gauss coefficients.

After resolving the coprimality conditions and dividing the variables into ranges $F \leq N(f) < 2F$, the averages to be bounded are

$$\mathcal{E}(\mathcal{H}, X, F) = \frac{1}{XF} \sum_{F \leq N(f) < 2F}^* \sum_{0 < N(k) \leq \mathcal{H}} \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/X) \right|^2. \quad (6.9)$$

Here ξ, W are suppressed from the notation. Both f, n are squarefree. The fourth power in $\chi_n(f)^4$ has two sources. Writing e for the exclusion divisor in Poisson summation gives a twist $\chi_n(he^5)$. Removing the coprimality condition between the two columns introduces a common divisor v and the identity $a_\xi(vm) = a_\xi(v)a_\xi(m)\chi_m(v)^4$. With $f = ev$ and $k = eh$,

$$he^5v^4 = k(ev)^4 = kf^4. \quad (6.10)$$

Thus the v -part comes from twisted multiplicativity, and the e -part from the Poisson exclusion divisor. If $B \geq 1$ is a further divisor-size parameter, the initial ranges are

$$X = \frac{D}{BF}, \quad \Sigma := XF = \frac{D}{B}, \quad \mathcal{H} \ll \frac{D^2}{HB^2}. \quad (6.11)$$

The needed bound is $\mathcal{E} \ll D^\varepsilon \Sigma$. The normalizations and divisor sums then recover (3.8); this is the first Poisson reduction.

7 Why cubic theta and a quadratic large sieve appear

7.1 Adding the cube indices

The cubic Gauss coefficients in (6.9) occur, after a fixed normalization, in the Fourier expansion of Kubota's cubic theta function. Its indices include cube factors, powers of λ , and units. The explicit coefficient formulas are due to Patterson and are recorded in [2, Section 5.1 and Appendix]. The notes [3, Section 5] explain the theta function and state the properties used here.

Here is the normalization needed for our sum. Restrict a Fourier index ℓ to those for which $\lambda^3\ell$ is primary and coprime to 6. Among the nonzero coefficients, this selects exactly $\ell = \lambda^{-3}nb_1^3$, with n squarefree and b_1 arbitrary, both primary and coprime to 6. The coefficient of the conjugated theta function at such an index is

$$d_0(\lambda^{-3}nb_1^3) = 3^{5/2}|b_1|\tilde{g}(n) = 3^{5/2}|b_1| \overline{\left(\frac{\lambda}{n}\right)_3} \gamma_2(n). \quad (7.1)$$

Eisenstein's supplementary law says that $(\lambda/n)_3$ depends only on $n \bmod 9$. Multiplying the twist at $x = \lambda^3\ell = nb_1^3$ by $(\lambda/x)_3 = (\lambda/n)_3$ cancels this extra factor. It is absorbed into the same finite congruence-class bookkeeping used above. This selection applies to the original series; unit factors and varying powers of λ reappear in the reflected cusp expansions and are included in their estimates.

For fixed k, f , put $\Psi(n) = \xi(n)\chi_n(k)\chi_n(f)^4$. The convenient completed sum is

$$T(X; k, f) = \frac{1}{\sqrt{X}} \sum_n^* \sum_{b_1} \overline{\alpha(n)} \gamma_2(n) \Psi(n) \overline{\alpha(b_1)}^3 \Psi(b_1)^3 N(b_1)^{1/2} W\left(\frac{N(n)N(b_1)^3}{X}\right). \quad (7.2)$$

The $b_1 = 1$ part is precisely $X^{-1/2}$ times the column sum in (6.9). All sums here are finite because W has compact support.

7.2 The transformation and its purpose

The theta function obeys a transformation law relating its values at rational translates to its values near one of three cusps. Expanding Ψ in finite additive characters makes that law applicable to (7.2). Differentiating in the horizontal variable removes the constant Fourier terms; Mellin transformation then gives a reflected sum. The constant terms would otherwise produce residues, so their removal is an essential step.

The important local calculation is especially simple to state. Let $\pi \nmid 6$ be a prime at which the original twist is χ_π^j , and expand χ_π^j in additive characters modulo π . Characters are extended by zero at multiples of their prime, also when an exponent is 0.

If $1 \leq j \leq 5$, the zero frequency has coefficient 0, so every frequency with a nonzero coefficient is nonzero modulo π , and π divides the denominator of every translate. The reflected coefficient then contains, up to a factor of absolute value 1,

$$B_{\pi,j}(x) = \begin{cases} \chi_\pi(x)^{-j-2}, & j \notin \{0, 4\}, \\ N(\pi)^{-1/2}(-1 + N(\pi)\mathbf{1}_{\pi \nmid x}), & j = 4. \end{cases} \quad (7.3)$$

If $j = 0$, the twist is the principal character $\chi_\pi^0(x) = \mathbf{1}_{\pi \nmid x}$, and its expansion has two parts:

$$\mathbf{1}_{\pi \nmid x} = \left(1 - \frac{1}{N(\pi)}\right) - \frac{1}{N(\pi)} \sum_{h \not\equiv 0 \pmod{\pi}} e\left(\frac{hx}{\pi}\right). \quad (7.4)$$

The constant part (the *inactive* branch) gives translates whose denominators are not divisible by π . It contributes only the factor $1 - N(\pi)^{-1}$, with no reflected local factor at π . The terms with $h \not\equiv 0$ (the *active* branch) give, up to a factor of absolute value 1,

$$B_{\pi,0}(x) = N(\pi)^{-1/2} \chi_\pi(x)^{-2}. \quad (7.5)$$

Both branches occur. In particular,

$$B_{\pi,1}(x) = \chi_\pi(x)^3 = \left(\frac{x}{\pi}\right)_2. \quad (7.6)$$

Thus the squarefree part of the row twist becomes *quadratic*. If c is the denominator of the rational translate in lowest terms, put $Q = N(c)$. The reflected weight has argument $N(\ell)X/Q^2$, so the effective dual norm range is $N(\ell) \lesssim Q^2/X$. It is shorter than the original range X when $Q < X$. Rapid decay becomes useful when X/Q^2 is large: the nonzero frequencies $\ell \in \lambda^{-4}\mathbb{Z}[\omega]$ have $N(\ell) \geq 1/81$, so their weight arguments then grow uniformly. The inequality $Q^2 < X$ alone is not a quantitative negligibility estimate.

The quadratic large sieve says, in the form needed here,

$$\sum_{\substack{N(k) \leq K \\ (k,6)=1}}^* \left| \sum_{\substack{N(n) \leq U}}^* \beta(n) \left(\frac{n}{k}\right)_2 \right|^2 \ll_\varepsilon (KU)^\varepsilon (K+U) \sum_n^* |\beta(n)|^2. \quad (7.7)$$

Here the generators are primary, the column n may be even, and the coefficients $\beta(n)$ are arbitrary. Both the row and column squarefree restrictions are essential for this bound. In particular, if square columns $n = b_1^2$ were allowed, then $(b_1^2/k)_2 = 1$ whenever $(b_1, k) = 1$, so coefficients supported on squares would reinforce each other instead of cancelling. This is the substantial large-sieve input; it is a specialization of the number-field quadratic large sieve of [5].

Applying (7.7) to the reflected sum, accounting for the local factors at primes dividing f , including both active and inactive branches, and then treating the square factors of k , gives the completed estimate

$$\boxed{\sum_{0 < N(k) \leq \mathcal{H}} |T(X; k, f)|^2 \ll D^\varepsilon \left(\mathcal{H} + \frac{\mathcal{H}^2 N(f)}{X} \right)}. \quad (7.8)$$

Parameters are bounded by fixed powers of D , and a fixed finite smoothness norm of W is understood. The estimate is derived from the theta transformation and the large sieve, as in [1, Proposition 5.2 and Section 6].

For clarity, the passage to all rows uses $k = usv^2$, with u a unit and s squarefree. The identity $\chi_n(v^2)\chi_n(f)^4 = \chi_n(fv^2)^4$ absorbs the square into f . The corresponding squarefree-row estimate allows this more general second argument. Summing over v costs $\sum_v N(v)^{-2} < \infty$.

7.3 Why the twists are sextic

Twists of any order k could be used to amplify $A_1(D)$. For $k = 2, 3, 6$, replace $\chi_n(u)$ in (3.2) by $\chi_n(u)^{6/k}$, which is $(u/n)_2$, $(u/n)_3$ or $(u/n)_6$ by (4.7). The rows $u = v^k$ then reproduce the untwisted sum, as in (4.3), and there are about $H^{1/k}$ of them. A mean-square bound of the diagonal size HD would give, exactly as in Section 5 with $H \approx D$,

$$|A_1(D)|^2 \ll \frac{HD}{H^{1/k}} \approx D^{2-1/k}, \quad \text{so} \quad A_1(D) \ll D^{1-1/(2k)+\varepsilon}. \quad (7.9)$$

Quadratic twists would give $3/4$, cubic twists $5/6$, and sextic twists only $11/12$. So the order 6 is chosen for the available proof of the mean-square bound. The difference lies in what Poisson summation in u does to the Möbius coefficients. As in (6.8), the coefficient of a column z in the dual sum is $\mu_K(z)$ times the Gauss sum of the order- k character $\chi_z^{\pm 6/k}$.

- *Quadratic twists.* Poisson summation produces the quadratic Gauss sums $\gamma_3(n)$. These are explicit: by (6.6), $\gamma_3(n) = \chi_n(4)G(n)$. Here G depends only on n modulo 72, and $\chi_n(4) = (2/n)_3$ depends only on n modulo 2 by cubic reciprocity, so $\gamma_3(n)$ depends only on n modulo 72. So the dual coefficients are still $\mu_K(n)$, up to a periodic phase, and the dual sum is again a mean square of Möbius sums with quadratic twists. Its square frequencies $k = v^2$ carry untwisted Möbius sums once more, so this transformation by itself supplies no new estimate. The quadratic large sieve (7.7) cannot be applied to the original family either: it requires squarefree rows, and the rows $u = v^2$ that carry $A_1(D)$ are exactly the ones it must exclude.
- *Cubic twists.* Poisson summation produces the cubic Gauss sums $\gamma_{\pm 2}(n)$. The identities behind (6.7) give

$$\mu_K(n)\gamma_2(n) = \chi_n(-1)G(n)\alpha(n)\gamma_{-1}(n) \quad (7.10)$$

for squarefree n . The Möbius function is absorbed, but into a *sextic* Gauss sum. This points toward theta functions on a sixfold metaplectic cover. Such theta functions do have automorphy, and Kubota's Dirichlet series built from higher-order Gauss sums have functional equations [6]. For the sextic theta function, Bröker and Hoffstein [7] prove a formula for the coefficients at π^4 for certain primes π , and conjecture the coefficients at π^2 and the squares of those at π from numerical evidence; the coefficients at squarefree indices are not known. What is missing from the present argument is a coefficient description and a uniform estimate

for the transformed sums that play the roles of (7.1) and (7.8). The failure of Whittaker uniqueness in this higher-cover setting helps explain the difficulty [4, 7]; it does not imply that transformation formulas do not exist.

- *Sextic twists.* Poisson summation produces the sextic Gauss sums $\gamma_{-1}(n)$, and (6.7) turns $\mu_K(n)\gamma_{-1}(n)$ into $\overline{\alpha(n)}\gamma_2(n)$, up to a periodic phase. The reason is the factorization, from the Hasse–Davenport product formula,

$$\gamma_1(n) = \overline{\chi_n(4)} \gamma_2(n)^2 \gamma_3(n), \quad \gamma_2(n)^3 = \mu_K(n) \alpha(n). \quad (7.11)$$

A sextic Gauss sum is the square of a cubic one times a quadratic one. The quadratic factor is explicit, and the cube of the cubic factor supplies exactly the Möbius sign. The resulting coefficient is a Fourier coefficient of the cubic theta function, which Patterson determined. Its automorphy then turns the remaining sextic twist into a quadratic one, (7.6). Only at that stage, when the coefficients are Gauss sums rather than values of the Möbius function, is the quadratic large sieve applied.

Of these three routes, the sextic one supplies the coefficient conversion and transformed estimates used here. The price is the count $H^{1/6}$ of amplifying rows, which gives 11/12 rather than 3/4 or 5/6. Notice that the order of the twists and the degree of the auxiliary metaplectic cover are different: sextic twists lead to the *cubic* theta function, whereas the cubic-twist calculation above produces *sextic* Gauss sums.

Better control of higher metaplectic theta coefficients could therefore help improve this method, but it is one possible route, not a necessary gateway to a larger zero-free region. Different averages or amplification could also help. Formula (7.9) shows that simply increasing the order of the twists makes this particular amplification weaker: perfect-power rows become rarer.

8 Removing the added cubes and closing the argument

The purpose of descent is to bound the original average $\mathcal{E}(\mathcal{H}, X, F)$ in (6.9). In plain terms, the completed theta estimate handles small cube indices, while the remaining large cube indices lead to averages of exactly the same kind with fewer rows. The point requiring care is that all the other conditions, including the smooth weights, must survive this reduction.

One cannot obtain a bound for the $b_1 = 1$ part of (7.2) simply by discarding the other terms: cancellation may occur inside T . Instead use the exact Möbius inversion

$$\frac{1}{\sqrt{X}} \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/X) = \sum_h \frac{\mu_K(h) \overline{\alpha(h)}^3 \Psi(h)^3}{N(h)} T(X/N(h)^3; k, f). \quad (8.1)$$

In this formula h is primary and coprime to 6. Substituting (7.2) and collecting the total cube index uses $\sum_{h|b_1} \mu_K(h) = \mathbf{1}_{b_1=1}$, proving the identity.

8.1 Small cubes are controlled by the completed estimate

Choose

$$H_c^3 = \min \left(X, \frac{X^2}{\mathcal{H}^2} \right), \quad L_{b_1} = X/N(b_1)^3. \quad (8.2)$$

For $N(h) \leq H_c$, estimate (7.8) at scale $X/N(h)^3$ gives terms bounded by

$$\mathcal{H} + \frac{\mathcal{H}^2 F N(h)^3}{X} \ll \Sigma, \quad \Sigma = XF, \quad (8.3)$$

provided $\mathcal{H} \leq \Sigma$. Cauchy–Schwarz costs only divisor sums and logarithms, which fit into D^ε . Expanding the remaining completed sums in (8.1) leaves the same original average at smaller column scales. The resulting cube-removal estimate [1, Lemma 5.3] is

$$\mathcal{E}(\mathcal{H}, X, F) \ll D^\varepsilon \left(\Sigma + \sup_{\substack{N(b_1) > H_c \\ L_{b_1} > 1}} \mathcal{E}(\mathcal{H}, L_{b_1}, F) \right). \quad (8.4)$$

Scales $L_{b_1} \leq 1$ are bounded by counting. If $\mathcal{H}^2 \leq X$, the supremum is empty and the desired estimate already follows. Here and below, the suppressed dependence on W is through a fixed finite smoothness norm. More precisely, the Σ term in (8.4) is $\Sigma \|W\|_{C^{J_c(I)}}^2$ for a finite integer J_c , with the same W in each average in the supremum.

8.2 Large cubes force a smaller row range

Keep $\Sigma = XF$ from the original average fixed while replacing the column length X by $L = L_{b_1}$. In particular, do not reset Σ to LF . The extra room $LF \leq \Sigma$ is what permits the enlargement below. For the remaining terms, two more Poisson summations exchange cubic Gauss coefficients for Möbius coefficients and then back again. The paired identity (6.8) is used in both directions. The first summation gives frequencies of norm about $L^2/\mathcal{H}$, with additional divisor factors. Combining the frequency with the auxiliary variable as $y = hf^2$ gives a natural Möbius-row range of size $L^2F^2/\mathcal{H}$. A crucial step is to enlarge this range by positivity before the second Poisson summation. Suppressing the common divisor factor $N(d)/N(C)^2$ and fixed support constants, the two scales are

$$Y_{\text{natural}} = \frac{L^2F^2}{\mathcal{H}}, \quad Y_{\text{enlarged}} = \frac{\Sigma LF}{\mathcal{H}}, \quad \frac{Y_{\text{enlarged}}}{Y_{\text{natural}}} = \frac{\Sigma}{LF} \geq 1. \quad (8.5)$$

After separating the smooth weights, the row sums are nonnegative mean squares, so the larger smooth majorant gives a valid upper bound. The enlarged range makes the second Poisson frequency range smaller. For a remaining divisor r in a range $R \leq N(r) < 2R$, it gives

$$\mathcal{H}' = \frac{\mathcal{H}L}{\Sigma FR^2}. \quad (8.6)$$

Without enlargement the corresponding scale would be $\mathcal{H}/(F^2R^2)$, which need not decrease when $F = R = 1$.

After the second summation, regrouping the divisor terms *before* taking absolute values forces a divisibility condition $f' \mid k'$. At each prime the cancellation is

$$(1 + \mathbf{1}_{\pi|k'}) - \mathbf{1}_{\pi|k'} - \mathbf{1}_{\pi \nmid k'} = \mathbf{1}_{\pi|k'}. \quad (8.7)$$

This divisibility keeps the next average in the ranges needed for induction. The kernel after the second summation vanishes unless $0 < N(k') \leq \mathcal{H}L/(\Sigma FN(r)^2)$. Retain only those r for which a nonzero term occurs. If $F' \leq N(f') < 2F'$, then $f' \mid k'$ gives

$$F' \leq N(f') \leq N(k') \leq \frac{\mathcal{H}L}{\Sigma FN(r)^2} \leq \mathcal{H}', \quad (8.8)$$

so $\mathcal{H}' \geq F' \geq 1$. It also keeps the column scales at least 1. Put $X' = L/(RF')$. Removing an exclusion divisor $j \mid r$ produces the replacement

$$(X', F') \longmapsto (X'/N(j), F'N(j)).$$

It preserves both $\mathcal{H}'$ and the product $\Sigma' = X'F' = L/R$, and hence the ratio $\mathcal{H}'/\Sigma'$. Since $N(j) \leq N(r)$, $R \leq N(r)$, $F \geq 1$ and $\mathcal{H} \leq \Sigma$, (8.8) gives

$$\frac{X'}{N(j)} \geq \frac{\Sigma F N(r)^2}{\mathcal{H} R N(j)} \geq \frac{\Sigma F}{\mathcal{H}} \geq 1. \quad (8.9)$$

8.3 The precise transfer estimate

We now state the analytic input needed from [1], in the form sufficient for this guide. Write $\mathcal{E}(\mathcal{H}, X, F; \xi, W)$ when the twist and weight in (6.9) need to be shown. For an interval I , put $\|W\|_{C^m(I)} = \max_{0 \leq r \leq m} \sup_{x \in I} |W^{(r)}(x)|$.

Proposition 1 (Transfer estimate [1, Proposition 5.4]). *Fix $C_0 \geq 1$, $D \geq 2$, a compact interval $I \subset (0, \infty)$, and $W \in C_c^\infty(I)$. Suppose $1 \leq \mathcal{H}, L, F, \Sigma \leq D^{C_0}$, with $\max\{\mathcal{H}, LF\} \leq \Sigma$. Here Σ is an independent target scale. For every integer $m \geq 0$ and $\delta > 0$,*

$$\mathcal{E}(\mathcal{H}, L, F; \xi, W) \ll D^\delta \Sigma \|W\|_{C^{4m+12}(I)}^2 \left(1 + \sup \frac{\mathcal{E}(\mathcal{H}', X', F'; \xi', U)}{\Sigma'} \right).$$

The supremum runs over averages of the same form, with $\mathcal{H}', X', F' \geq 1$, $\Sigma' = X'F'$, and

$$\mathcal{H}' \leq \frac{\mathcal{H}L}{\Sigma F}, \quad \frac{\mathcal{H}'}{\Sigma'} \leq \frac{\mathcal{H}}{\Sigma}, \quad \Sigma' \leq L. \quad (8.10)$$

For $I = [u, v] \subset (0, \infty)$, the new weights satisfy $U \in C_c^\infty([u/16, 4v])$ and $\|U\|_{C^m} \leq 1$. The characters ξ' belong to the same fixed finite family of residue-class twists. An empty supremum is zero. The implied constant may depend on m, C_0, δ, I and the fixed arithmetic data, but not on D or the moving scales.

The source proves the stronger version with a smooth majorant of the row sum on the left. Its proof, in [1, Section 7], is the two-Poisson argument just outlined, including the bounds for the separated smooth kernels and zero frequencies. The full uniform estimate is a substantive analytic input: the three scale inequalities (8.10), by themselves, do not establish it.

8.4 Why the induction closes

This subsection follows the proof of [1, Proposition 5.1], given in [1, Section 5.5]. Fix $\kappa > 0$, $C_0 \geq 1$, and $\varepsilon > 0$, and assume

$$1 \leq \mathcal{H}, X, F, \quad \Sigma = XF \leq D^{C_0}, \quad \mathcal{H} \leq \Sigma D^{-\kappa}.$$

The inductive assertion, for each integer $j \geq 0$, is

$$\mathcal{E}(\mathcal{H}, X, F; \xi, W) \ll D^\varepsilon \Sigma \|W\|_{C^{J_j}(I)}^2 \quad \text{whenever } \mathcal{H} \leq D^{j\kappa}.$$

This assertion is made for every $\varepsilon > 0$ and every compact interval $I \subset (0, \infty)$. The finite derivative order J_j may depend on $j, \kappa, C_0, \varepsilon$, but not on D or I ; the implied constant may also depend on I . This formulation allows the support interval to enlarge at a transfer step.

Base case. If $j = 0$, then $\mathcal{H} \leq 1$, so there are at most six nonzero rows. There are $O(F)$ choices of f and $O_I(X)$ column terms, each of absolute value at most $\|W\|_\infty$. The normalization in (6.9) therefore gives

$$\mathcal{E}(\mathcal{H}, X, F; \xi, W) \ll_I \frac{F}{XF} (X\|W\|_\infty)^2 = X\|W\|_\infty^2 \leq \Sigma\|W\|_\infty^2.$$

Induction step. Assume the assertion at level j , and let $\mathcal{H} \leq D^{(j+1)\kappa}$. Use that assertion with loss $\varepsilon/3$, and let m be its required derivative order. Apply cube removal (8.4) and Proposition 1, each with loss $\varepsilon/3$. If the cube-removal supremum is empty, its Σ term already suffices. Otherwise $N(b_1)^3 > X^2/\mathcal{H}^2$. For each remaining $L = L_{b_1}$ we have $LF \leq \Sigma$, so the transfer estimate applies with the unchanged target $\Sigma = XF$. Every child average satisfies

$$\mathcal{H}' \leq \frac{\mathcal{H}}{N(b_1)^3 F^2} < \mathcal{H} \left(\frac{\mathcal{H}}{XF} \right)^2 \leq \mathcal{H} D^{-2\kappa}. \quad (8.11)$$

In particular, $\mathcal{H}' \leq D^{j\kappa}$. The other hypotheses survive as well:

$$\frac{\mathcal{H}'}{\Sigma'} \leq \frac{\mathcal{H}}{\Sigma} \leq D^{-\kappa}, \quad \Sigma' \leq L \leq \Sigma \leq D^{C_0}, \quad \mathcal{H}', X', F' \geq 1.$$

The induction hypothesis applies to the child weight U on $[u/16, 4v]$. Since $\|U\|_{C^m} \leq 1$, it gives the uniform bound

$$\sup \frac{\mathcal{E}(\mathcal{H}', X', F'; \xi', U)}{\Sigma'} \ll D^{\varepsilon/3}.$$

The transfer contributes a second factor $D^{\varepsilon/3}$ and cube removal a third. Taking $J_{j+1} \geq \max\{4m + 12, J_c\}$ proves the assertion at level $j + 1$. Thus both the derivative requirement and the total D^ε loss are controlled.

Finally take $j = \lceil C_0/\kappa \rceil$; this covers every allowed $\mathcal{H}$. The number of induction levels depends on C_0, κ , not on D . Suppressing the resulting fixed smoothness norm again, we obtain

$$\mathcal{E}(\mathcal{H}, X, F) \ll D^\varepsilon \Sigma \quad (\mathcal{H} \leq \Sigma D^{-\kappa}). \quad (8.12)$$

No estimate for zeros of ζ is assumed in this induction: its inputs are cube removal and the transfer estimate, and its starting case is counting. Establishing those analytic inputs is the work that must precede this formal closure.

Finally, the initial scales (6.11), with $H = D^{1+\vartheta}$, satisfy

$$\frac{\mathcal{H}}{\Sigma} \ll \frac{D^{-\vartheta}}{B}. \quad (8.13)$$

For sufficiently large D this is at most $D^{-\vartheta/2}$. Thus (8.12) applies with $\kappa = \vartheta/2$. Empty ranges and bounded column scales are handled by counting. The first Poisson reduction gives (3.8), and Section 5 then gives the zero-free half-plane.

The roles of the ingredients. The ordinary convolution $c = \mu * (\mu\chi_{-3})$ identifies the reciprocal zeta product. Sixth-power rows recover that convolution inside a larger nonnegative average. Poisson summation shortens the row range and exposes cubic Gauss sums. The theta transformation converts their twists into quadratic ones, to which a large sieve applies. Cube inversion and the contraction (8.11) transfer the resulting completed bound back to the original sums.

9 Numerical checks of Sections 1–8

The identities and parameter inequalities listed below, from Sections 1–8, numbered or not, were tested on finite samples. The checks use Python with exact arithmetic in $\mathbb{Z}[\omega]$, written as pairs $a + b\omega$, and the conventions of these notes: primary means $\equiv 1 \pmod{3}$, $\lambda = 1 + 2\omega$, $\chi_n(x) = (x/n)_6$, and $e(z) = \exp(4\pi i \operatorname{Im} z / \sqrt{3})$. Residue symbols are computed by Euler's criterion modulo each prime,

and Gauss sums by brute force over complete residue systems. In Sections 1 and 2, $c(m)$ is computed from its definition (1.2) with the ordinary Möbius function, and $\mu_K(n)$ from the factorization of $N(n)$ in $\mathbb{Z}$, so that the two sides of (2.3) are computed independently. Sums of complex numbers are in double precision, and the Mellin and Bessel integrals use 25–30-digit arithmetic. “Exact” below means that the two sides agreed identically, or to rounding error, in every case tried. The scripts, with a description of each check, are at https://github.com/mrubinst/L-function_zero_free_strips/tree/main/zeta_11_12_tests.

Display	What was checked	Result
<i>Section 1</i>		
(1.1)	$\chi_{-3}(m)$ equals the Kronecker symbol $(-3/m)$ for $m \leq 3000$ and is completely multiplicative (2000 random pairs)	exact
(1.2)	coefficientwise, $c * (1 * \chi_{-3}) = \delta$ for all $m \leq 10^5$, i.e. the series of c times $\zeta(s)L(s, \chi_{-3})$ is 1; the truncated series $\sum_{m \leq 10^5} c(m)m^{-s}$ against $1/(\zeta(s)L(s, \chi_{-3}))$ at $s = 3, 3 + 4i, 4$	exact; 10^{-13}
(1.3)	the table for all 9690 prime powers $p^j \leq 10^5$ with $p \neq 3$, and $c(3) = -1$, $c(3^j) = 0$ for $j \geq 2$; $c(m) = \prod_{p^e \parallel m} c(p^e)$ for all $m \leq 10^5$; the values $c(7)$, $c(49)$, $c(25)$	exact
(1.5)	coefficientwise for all $m \leq 10^5$; the local factors at 2 and 3 are $1 - 4^{-s}$ and $1 - 3^{-s}$; numerically at $s = 3 + 2i$	exact; $7 \cdot 10^{-14}$
<i>Section 2</i>		
(2.1)	$\omega^2 + \omega + 1 = 0$; $ a + b\omega ^2 = a^2 - ab + b^2$ for $ a , b \leq 40$; there are exactly six units, $\pm 1, \pm \omega, \pm \omega^2$	exact
(2.2)	$n \equiv 1 \pmod{3}$ in $\mathbb{Z}[\omega]$ if and only if $a \equiv 1, b \equiv 0 \pmod{3}$; each n coprime to 3 has exactly one primary associate, and a multiple of λ has none ($ a , b \leq 60$)	exact
splitting	for every prime $p < 3000$: if $p \equiv 1 \pmod{3}$, there are exactly two primary elements of norm p , they are conjugate, and their product is p ; if $p \equiv 2 \pmod{3}$, no element has norm p and $-p$ is primary; $3 = -\lambda^2$	exact
(2.3)	the number of primary n with $N(n) = m$ equals $\sum_{d m} \chi_{-3}(d)$ for all $m \leq 10^5$ with $3 \nmid m$ (unique factorization); $\sum_{N(n)=m} \mu_K(n) = c(m)$ for all 33 333 values $m \leq 10^5$ coprime to 6; the local factors $(1 - p^{-s})^2$ and $1 - p^{-2s}$	exact
(1.4), (2.4)	$A_1(D)$ computed both ways, with a complex smooth W , for $D = 300, 3000, 40\,000$	10^{-14}
<i>Section 3</i>		
(3.1)	$6 \mid N(\pi) - 1$ for the first 3000 primary primes $\pi \nmid 6$; χ_n is multiplicative in the numerator and in the denominator (random samples)	exact
(3.4)	the coordinate conditions $a \equiv 1, b \equiv 0 \pmod{3}$, (a, b) not both even are equivalent to “primary and coprime to 6” ($ a , b \leq 40$); the coordinate sum equals $M(D, H)$ for $D = 60, H = 90$ and a complex weight	exact
(3.5), (3.6)	the double sum over columns, with S_H computed directly, equals $M(D, H)$	$5 \cdot 10^{-16}$

Display	What was checked	Result
(3.7)	$A_u(D) = \sum_{(m,6)=1} c_u(m)W(m/D)$ for 40 rows; $c_1(m) = c(m)$ for all $m \leq 2000$; an example showing that $c_u(p)$ depends on the residue symbols	10^{-15} ; exact
(3.8)	sanity check of the size: $M(D, D^{1.1})/(D^{2.1}) = 0.35, 0.32, 0.31$ for $D = 100, 200, 400$	consistent
<i>Section 4</i>		
(4.1), (4.2)	lattice counts for four n and $H \leq 10^4$: $ \text{count} - \text{main term} \leq 0.23 \tau(n)(\sqrt{H} + 1)$	consistent
(4.3)	$\chi_n(v^6) = \mathbf{1}_{(v,n)=1}$ (random v, n); $A_{v^6}(D)$ for prime and composite v	exact; 10^{-14}
(4.5)	$A_{\pi^6} - A_1 = -\sum_{\pi n} \mu_K(n)W$ for eight π , $D = 1500$; the ratio to $D/N(\pi)$ stays bounded	10^{-16}
(4.4)	$J \log Y/Y = 0.50, 0.52, 0.51$ for $Y = 10^3, 10^4, 10^5$; the rows π^6 are distinct	consistent
(4.6)	$J A_1 ^2 \leq 2 \sum A_{\pi^6} ^2 + 2 \sum A_{\pi^6} - A_1 ^2$ on actual values ($D = 1500, Y = 200, J = 21$); the error term is $0.036 JD^2/Y^2$	holds
(4.7), (7.6)	$\chi_n(v^3)$ and $\chi_n(v^2)$ against the quadratic and cubic symbols computed independently by Euler's criterion	exact
(4.8)	$(v/n)_2 = (v/N(n))$ for $ v \leq 30$ and 150 values of n	exact
(4.9)	$c_{v^3}(m) = c(m)(v/m)$ for all $m \leq 2000$ and $v \in \{2, -1, 3, 5, -3, 7, -6, 10\}$	exact
(4.10)	the Euler product has coefficients $c(m)(v/m)$, $m \leq 2000$	exact
<i>Section 5</i>		
(5.1)–(5.3)	the exponent algebra	symbolic
(1.5), (5.4)	$\sum_{(m,6)=1} c(m)m^{-s} = 1/F(s)$ at $s = 3$ and $s = 2.5 + 4i$	within truncation error
(5.5)	the substitution $\int W(m/D)D^{-s} dD/D = m^{-s} \widehat{W}(s)$	10^{-35}
<i>Section 6</i>		
(6.1)	$\widehat{\Phi}$ for $\Phi(t) = e^{-\pi t}$ by quadrature, against $(2/\sqrt{3})e^{-4\pi t/3}$	10^{-21}
(6.2)	the factorization of $\overline{\chi_{n_1}}\chi_{n_2}$; ψ is primitive (128 local components)	exact
(6.4)	the exact Poisson formula with $\Phi(t) = e^{-\pi t}$, five cases including $g \neq 1$ and $z_1 = 1$	$5 \cdot 10^{-15}$
(3.5)+(6.4)	end to end: the whole smoothed mean square, computed directly, equals its Poisson-dual form ($D = 40, H = 60$)	$2 \cdot 10^{-15}$
(6.5), (6.11), (8.13)	the scale algebra	symbolic
(6.6), (6.7)	the Möbius–Gauss identity for 70 squarefree n , including composite n	$5 \cdot 10^{-15}$
(6.6)	$G(n)$ depends only on $n \bmod 4$	exact
(6.8)	22 coprime pairs; a control with the conjugation on the wrong factor fails (error 1.3)	$4 \cdot 10^{-15}$
(6.10)	$a_\xi(vm) = a_\xi(v)a_\xi(m)\chi_m(v)^4$ (22 pairs) and the character identity behind $he^5v^4 = kf^4$	$4 \cdot 10^{-15}$
<i>Section 7</i>		
(7.1)	against the coefficient table of the theta function as transcribed from [2], 104 indices; $(\lambda/n)_3$ depends only on $n \bmod 9$; the selection of indices $\lambda^{-3}nb_1^3$ (908 indices)	$4 \cdot 10^{-15}$; exact

Display	What was checked	Result
theta input	automorphy under $z \mapsto z + 1$ and the inversion	$2 \cdot 10^{-16}$
Section 7	the transformation with one $\bar{z}$ -derivative, end to end at a single cusp: the twisted coefficient sum against the reflected sum with the kernel $\prod_{\pm} \Gamma(1+t \pm \frac{1}{6})/\Gamma(1-t \pm \frac{1}{6})$, for 39 cusp configurations (all cusp types, all three values of the multiplier). Controls with the sign flipped, with the kernel $\Gamma(\frac{4}{3} \pm t)\Gamma(\frac{5}{3} \pm t)$, or with the wrong power of $\overline{\alpha(c)}$ fail at size 1	$5.5 \cdot 10^{-9}$
(7.3)–(7.5)	the additive expansions of χ_{π}^j ($1 \leq j \leq 5$) and of $\mathbf{1}_{\pi \nmid x}$, and the reflected Gauss sums $\sum_{h \neq 0} \chi_{\pi}(h)^{-j-2} e(\varepsilon h^{-1} x/\pi)$, for eight primes; the active $j = 0$ branch, summed directly for several ε , equals $-\gamma_2(\pi) \chi_{\pi}(\varepsilon)^{-2} N(\pi)^{-1/2} \chi_{\pi}(x)^{-2}$, so it has modulus $N(\pi)^{-1/2}$ when $\pi \nmid x$ and vanishes when $\pi \mid x$	$3 \cdot 10^{-14}$
text after (7.6)	$\lambda^4 = 9$, so $N(\ell) = N(m)/81$ for $\ell = \lambda^{-4}m$; over $ a , b \leq 30$ the least nonzero value is $1/81$, attained exactly at the six units divided by 9	exact
(7.7)	sanity check with $K = U = 400$ (109 squarefree rows, 138 squarefree columns, even columns included): the ratio of the left side to $(K + U) \sum \beta ^2$ is at most 0.254 for six coefficient choices	consistent
(7.8)	sanity check: $\sum_k T(X; k, f) ^2 \approx (0.2\text{--}0.35)\mathcal{H}$ for $X = 1500$, $\mathcal{H} \leq 1000$, $f \in \{1, 7 + 9\omega\}$	consistent
text before (7.10)	$\gamma_3(n)$ depends only on $n \bmod 4$ (90 squarefree n , 12 classes), and $\chi_n(4)$ only on $n \bmod 2$	exact
(7.10), (7.11)	for 70 squarefree n , including composite n	$6 \cdot 10^{-15}$
Section 8		
(8.1)	the cube inversion for three pairs (k, f) at $X = 4000$, with six nontrivial cubes	$2 \cdot 10^{-15}$
(8.2)–(8.11)	the inequalities between scales, on 200 000 random admissible parameter sets, and (8.8)–(8.9) on a further 200 000 sets with a retained r . A control that keeps only $F' \leq \mathcal{H}'$ instead of (8.8) violates $X'/N(j) \geq \Sigma F/\mathcal{H}$ in about 9 000 sets. (8.7) exactly	no violation

These checks confirm the identities and scale algebra on the samples tried. They do not prove the uniform analytic estimates (3.8), (7.7), (7.8), (8.4), or Proposition 1. Finite sanity checks cannot establish such bounds. Section 8 proves the implication from its stated analytic inputs to (8.12); the numerical checks do not replace the proofs of those inputs.

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