

A zero-free strip for the L -functions of the elliptic curves $y^2 = x^3 + d$

the 11/12 method and its extension to 7/8

OpenAI Astra-6 and Claude Opus 5.5

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Working notes. These are working notes. They describe an argument that is still being developed, and the details have yet to be checked carefully.

Abstract

For a nonzero integer d let E_d be the elliptic curve $y^2 = x^3 + d$ over $\mathbb{Q}$. These are exactly the curves with j -invariant 0, and each has complex multiplication by $\mathbb{Z}[\omega]$. Part I of these notes explains how OpenAI's 11/12 argument adapts to give the following statement: for every $d \neq 0$,

$$L(E_d, s) \neq 0 \quad \text{for } \Re s > 17/12.$$

The critical strip of $L(E_d, s)$ is $1/2 < \Re s < 3/2$, so this is a zero-free strip $17/12 < \Re s < 3/2$ along its right edge. In the analytic normalization, with critical strip $0 < \Re s < 1$, it says that $L(E_d, s + \frac{1}{2}) \neq 0$ for $\Re s > 11/12$, the same boundary as OpenAI's 11/12 theorem for $\zeta(s)$. Since $L(E_d, s)$ is the L -function of a Hecke character of $\mathbb{Q}(\sqrt{-3})$ of infinite order, the statement is not covered by OpenAI's 7/8 theorem, which concerns finite-order Hecke L -functions over $\mathbb{Q}(\sqrt{-3})$. We begin with ordinary integer coefficients obtained by counting points modulo primes, write out the mean squares, and isolate the sixth-power rows that pick out the chosen curve.

Part II (Sections 11–16) adapts OpenAI's 7/8 argument to the same curves. It outlines a proof, conditional on restatements listed in Section 16, that $L(E_d, s) \neq 0$ for $\Re s > 11/8$, that is, of a zero-free strip $11/8 < \Re s < 3/2$. More generally, it outlines a proof that $L(s, \bar{\alpha}\eta) \neq 0$ for $\Re s > 7/8$ for every finite-order Hecke character η of $\mathbb{Q}(\sqrt{-3})$, where $\alpha(n) = n/|n|$. The adaptation changes three things: the weight of the target, the bookkeeping of the characters attached to rows, and an angular factor in the moment estimates. We list the statements of OpenAI's paper that must still be written out in this form. Section 17 explains why the same method should extend to CM newforms of every weight with complex multiplication by $\mathbb{Q}(\sqrt{-3})$, such as $\eta(3z)^8$ and $\eta(2z)^3\eta(6z)^3$: weight $k+1$ needs $k+1$ derivatives of the cubic theta function and, as far as we can see, nothing else new. This would also give the zero-free half-plane for every symmetric power L -function of E_d .

The method in six sentences.

1. *The sum to bound.* Up to finitely many Euler factors, $1/L(E_d, s + \frac{1}{2}) = \sum_m q_d(m)m^{-s}$, where q_d is the multiplicative function with $q_d(p) = -a_p(d)/\sqrt{p}$, $q_d(p^2) = 1$ and $q_d(p^j) = 0$ for $j \geq 3$ ($p \nmid 6d$). So a bound $S_d(D) = \sum_m q_d(m)W(m/D) \ll D^{11/12+\varepsilon}$ for smooth W gives $L(E_d, s) \neq 0$ for $\Re s > 17/12$, exactly as the same bound for the coefficients $c(m)$ of $1/(\zeta(s)L(s, \chi_{-3}))$ gives $\zeta(s) \neq 0$ for $\Re s > 11/12$.

2. *The same sum over Eisenstein integers.* Grouping by norms,

$$S_d(D) = \sum_n \mu_K(n) \frac{n}{|n|} \chi_n((4d)^5) W(N(n)/D)$$

over primary $n \in \mathbb{Z}[\omega]$ coprime to 6, while the zeta sum is $\sum_n \mu_K(n) W(N(n)/D)$. The curve enters only through the unimodular factor $n/|n|$ and the sextic residue symbol of the fixed number $(4d)^5$; together these encode the point counts through $a_p(d) = \text{Tr}(\pi \overline{\chi_\pi(4d)})$.

3. *What is averaged.* Replacing $(4d)^5$ by an arbitrary Eisenstein integer u gives the family $A_u(D) = \sum_n \mu_K(n) \frac{n}{|n|} \chi_n(u) W(N(n)/D)$, and the quantity bounded is the mean square $\sum_{0 < N(u) \leq H} |A_u(D)|^2$ over all u with $N(u) \leq H = D^{1+\vartheta}$. That is about H rows of length about D , and the method shows the mean square is $\ll D^{2+\vartheta+\varepsilon} = D^{1+\varepsilon} H$, the size given by square-root cancellation in each row. The zeta argument averages exactly the same rows without the factor $n/|n|$.
4. *Isolating the curve.* Since $\chi_n(u\pi^6) = \chi_n(u)$ whenever $\pi \nmid n$, the roughly $H^{1/6}/\log H$ rows $u = (4d)^5 \pi^6$, with π prime and $N(\pi) \asymp H^{1/6}$, all equal $S_d(D)$ up to $O(D/N(\pi))$. Positivity of the mean square then gives $H^{1/6} |S_d(D)|^2 \lesssim DH$, i.e. $S_d(D) \lesssim D^{11/12}$ when $H \approx D$. For zeta the rows π^6 play the same role for $\sum_n \mu_K(n) W(N(n)/D)$.
5. *Poisson summation and the theta function.* Expanding the square and applying Poisson summation in u turns the paired Möbius coefficients into normalized cubic Gauss sums times $(\bar{n}/|n|)^2$ (for zeta, times $\bar{n}/|n|$). Sums of these come from Kubota's cubic theta function differentiated twice in $\bar{z}$ (for zeta, once), and its symmetry turns the sextic twists into quadratic residue symbols, which the quadratic large sieve controls.
6. *Closing the argument.* The theta function also brings in terms at cube indices nb^3 . Möbius inversion over cubes and two further Poisson summations reduce the number of rows by a fixed power of D at each step. This induction, identical to the zeta case once the powers of $n/|n|$ are tracked, proves the mean-square bound, hence the exponent 11/12 and the arithmetic boundary 17/12.

What is proved here, and what is quoted. These notes follow the organization of the zeta guide [1], adapting the method of [2]. The elementary deduction from the CM mean-square estimate is given in full. We state the harder analytic estimates precisely and explain their mechanism and how the arguments of [2] adapt; we do not reproduce complete proofs of the theta transformation or the descent estimates. Here “CM curves” means the $j = 0$ curves over $\mathbb{Q}$. Other CM fields would require their own arithmetic and analytic inputs. Throughout, the curve parameter d is fixed; constants may depend on it. Part II adapts [3]; Section 16 says what is proved there and what is carried over from [3] by inspection.

Higher weight. The angular factor α enters the theta step only through the number of derivatives, and elsewhere only through Gamma factors with the same growth. A CM newform of weight $k+1$ with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ corresponds to the character $\bar{\alpha}^k \eta$ and needs $k+1$ derivatives. For each fixed k , the formulas have the same structure and give the same length exponents; constants may depend on k . Section 17 explains why both the 11/12 and the 7/8 boundaries should therefore extend to these forms, for example to $\eta(3z)^8$ (weight 4) and $\eta(2z)^3 \eta(6z)^3$ (weight 3), and hence, by factoring, to the symmetric power L -functions of E_d . This is an expectation, not a claim.

1 Why start with 11/12 rather than 7/8?

For this family, the 11/12 method is substantially simpler to present. Its main analytic task is one positive mean square, and the passage from that bound to the zero-free region takes only a few inequalities and a Mellin integral. Relative to the zeta argument, the main new ingredient is the angular factor $\alpha(n) = n/|n|$: one theta derivative becomes two.

The 7/8 argument [3, 5] has additional layers: a supremum of zero locations over a whole family, an average evaluated in two ways, short prime sums arranged to cancel an unwanted term, and two different moment estimates to count exceptional rows. These are substantial additions to the machinery used below.

There is also a scope issue. The stated 7/8 theorem in [3] concerns *finite-order* Hecke characters over $\mathbb{Q}(\sqrt{-3})$. The characters for our curves include $n/|n|$ and are not of finite order. Thus $7/8 + 1/2 = 11/8$ is the corresponding arithmetic boundary only after the required CM adaptation; it is not a consequence of shifting the stated finite-order theorem. Part I (Sections 2–10) explains the CM 11/12 argument. Part II (Sections 11–16) carries out the 7/8 adaptation.

2 Start with ordinary integers and point counts

At a prime $p \nmid 6d$, put

$$a_p(d) = p + 1 - \#E_d(\mathbb{F}_p) = - \sum_{x=0}^{p-1} \left(\frac{x^3 + d}{p} \right), \quad \lambda_d(p) = \frac{a_p(d)}{\sqrt{p}}.$$

The symbol on the right is the ordinary Legendre symbol, with value zero on multiples of p . Indeed, the number of solutions in y to $y^2 = x^3 + d$ is $1 + ((x^3 + d)/p)$, and there is one point at infinity. The good Euler factor is

$$L_p(E_d, s + \tfrac{1}{2})^{-1} = 1 - \lambda_d(p)p^{-s} + p^{-2s}.$$

The shift is chosen so that $L(E_d, s + 1/2)$ has critical line $\Re s = 1/2$; the arithmetic function $L(E_d, s)$ has critical line $\Re s = 1$. Consequently the reciprocal coefficients away from $6d$ are the multiplicative function q_d specified by

$$q_d(1) = 1, \quad q_d(p) = -\lambda_d(p), \quad q_d(p^2) = 1, \quad q_d(p^j) = 0 \quad (j \geq 3). \quad (2.1)$$

Set $q_d(m) = 0$ when $(m, 6d) > 1$. Thus all these coefficients can be computed using ordinary prime factorization and finite sums of Legendre symbols. In particular, q_d is not the ordinary Möbius function.

Here is an explicit formula involving that function. Define $\Lambda_d(r) = \prod_{p|r} \lambda_d(p)$ for squarefree r coprime to $6d$, with $\Lambda_d(1) = 1$. For $(m, 6d) = 1$,

$$q_d(m) = \sum_{\substack{rt^2=m \\ r \text{ squarefree} \\ (r,t)=1}} \mu(r)\mu(t)^2 \Lambda_d(r). \quad (2.2)$$

At each prime, one chooses the constant, linear, or quadratic term of the Euler polynomial. The restrictions in this formula make those choices mutually exclusive.

Write $L(E_d, s)$ using the local factors of a minimal model, including at primes dividing $6d$, and put

$$B_d(s) = \prod_{p|6d} L_p(E_d, s + \tfrac{1}{2}).$$

Then, for $\Re s > 1$,

$$\sum_{m \geq 1} \frac{q_d(m)}{m^s} = \prod_{p \nmid 6d} (1 - \lambda_d(p)p^{-s} + p^{-2s}) = \frac{B_d(s)}{L(E_d, s + \frac{1}{2})}. \quad (2.3)$$

The finite factor $B_d(s)$ is holomorphic and nonzero for $\Re s > 0$. Deleting these primes therefore does not affect the zeros in question. This remains true when the displayed equation for E_d is not minimal.

Choose a complex-valued $W \in C_c^\infty((0, \infty))$, supported in a fixed interval $[a, b]$ with $0 < a < b$. The target sum is

$$S_d(D) = \sum_{m \geq 1} q_d(m)W(m/D). \quad (2.4)$$

The weight restricts m to size D . We want, for every $\varepsilon > 0$, $S_d(D) \ll_{d,W,\varepsilon} D^{11/12+\varepsilon}$. Here $\ll$ allows a constant independent of D .

2.1 The comparison with the zeta coefficients

For zeta, the classical coefficients in the companion guide are

$$c(m) = \sum_{rt=m} \mu(r)\mu(t)\chi_{-3}(t), \quad \chi_{-3}(t) = \begin{cases} 0 & 3 \mid t, \\ 1 & t \equiv 1 \pmod{3}, \\ -1 & t \equiv 2 \pmod{3}. \end{cases}$$

For $p \nmid 6d$, the comparison is

	$c(p)$	$c(p^2)$		$q_d(p)$	$q_d(p^2)$
$p \equiv 1 \pmod{3}$	-2	1		$-a_p(d)/\sqrt{p}$	1
$p \equiv 2 \pmod{3}$	0	-1		0	1

The inert-prime sign really changes. For instance, for $d = 1$, $a_5(1) = 0$, $a_7(1) = -4$, so $q_1(5) = 0$, $q_1(25) = 1$, and $q_1(7) = 4/\sqrt{7}$. The character χ_{-3} records whether a prime splits; it does not encode the point count of a particular elliptic curve.

3 The triangular lattice and the curve's row

Put $\omega = e^{2\pi i/3}$. An Eisenstein integer is an ordinary pair of integers written as $n = A + B\omega$, with

$$N(n) = |n|^2 = A^2 - AB + B^2, \quad \alpha(n) = \frac{A + B\omega}{\sqrt{A^2 - AB + B^2}}.$$

The ring $\mathbb{Z}[\omega]$ has unique factorization and six units $\pm 1, \pm\omega, \pm\omega^2$. Choose the unique *primary* associate of each element coprime to 3:

$$n \equiv 1 \pmod{3}, \quad \text{that is, } A \equiv 1 \pmod{3}, \quad B \equiv 0 \pmod{3}.$$

Unless otherwise stated, column variables and their divisors are primary and coprime to 6; row variables run over all of $\mathbb{Z}[\omega]$. Coprimality to 2 means that A, B are not both even. A star on a column sum means squarefree in $\mathbb{Z}[\omega]$.

Define $\mu_K(n)$ by the usual factorization rule: it is zero if a prime square divides n , and otherwise is $(-1)^r$ for a product of r distinct primary primes. For example,

$$\mu_K(-5) = -1 \quad (N(-5) = 25), \quad \mu_K(7) = 1 \quad (7 = (1 + 3\omega)(-2 - 3\omega)).$$

Thus $\mu_K(n)$ need not equal $\mu(N(n))$.

For a prime $\pi \nmid 6$, define $\chi_\pi(u) = (u/\pi)_6$ to be the sixth root of unity congruent to $u^{(N(\pi)-1)/6}$ modulo π , or zero if $\pi \mid u$. Extend multiplicatively in the denominator, and write $\chi_n(u) = (u/n)_6$. It is also multiplicative in u . Powers of these characters retain their zero values, even when the exponent is reduced to zero modulo 6; $\chi_1(u) = 1$ for all u .

Define the family

$$A_u(D) = \sum_n \mu_K(n) \alpha(n) \chi_n(u) W(N(n)/D). \quad (3.1)$$

It is the zeta family with one extra, explicitly given factor $n/|n|$. For the desired curve, choose

$$u_d = (4d)^5, \quad \psi_d(n) = \alpha(n) \chi_n(u_d) = \alpha(n) \overline{\chi_n(4d)}. \quad (3.2)$$

We claim that $A_{u_d}(D) = S_d(D)$.

Where u_d comes from. At a split prime $p = \pi\bar{\pi}$, the point count (3.3) says that E_d contributes π times the sixth root of unity $\overline{\chi_\pi(4d)}$. In its Jacobi-sum proof the factor d enters when the summation variable is rescaled by d , and the factor 4 enters through the discriminant $1 - 4a$ of $t^2 - t + a$, which gives $J(\kappa, \rho) = \kappa(4)J(\kappa, \kappa)$ for the cubic and quadratic characters κ, ρ modulo π . The family, however, twists by $\chi_n(u)$, with the row u in the *top* of the residue symbol. So we need an Eisenstein integer u with $\chi_n(u) = \chi_n(4d)$ for every column n . Since χ_n takes sixth roots of unity,

$$\overline{\chi_n(4d)} = \chi_n(4d)^5 = \chi_n((4d)^5),$$

and this is the choice $u_d = (4d)^5$. Multiplying a row by a sixth power changes it only at the columns that share a prime with that sixth power: $\chi_n(uv^6) = \chi_n(u) \mathbf{1}_{(n,v)=1}$. Such a change alters finitely many Euler factors, so it does not move any zeros in $\Re s > 0$. When v is supported on primes above 2 and 3, nothing changes at all, because the columns are coprime to 6. For instance $u = 16d^5$ gives exactly the same row, since $(4d)^5 = 16d^5 \cdot 2^6$. For $d = 16$ this is $2^{24} = (2^4)^6$, a sixth power. So the curve $y^2 = x^3 + 16$ (conductor 27) is the row $u = 1$ itself, namely the plain sum $\sum_n \mu_K(n) \frac{n}{|n|} W(N(n)/D)$, and $y^2 = x^3 + 1$ is the row $u = 16$. More generally, every nonzero rational row u is the row of a $j = 0$ curve over $\mathbb{Q}$: since $(4d)^5 \equiv (4d)^{-1}$ modulo sixth powers, u belongs to E_d with $d = 4^5 u^5$. For rows that are not a rational number times a sixth power, the coefficients $q_u(m)$ of (4.4) are in general not real. Such a row is the L -function of a Hecke character of $\mathbb{Q}(\sqrt{-3})$, not of an elliptic curve over $\mathbb{Q}$.

3.1 Checking the claim prime by prime

If $p \equiv 1 \pmod{3}$, then $p = \pi\bar{\pi}$, with both generators primary and $N(\pi) = p$. The classical CM point-count identity is

$$a_p(d) = \overline{\pi \chi_\pi(4d)} + \bar{\pi} \chi_\pi(4d) \quad (p \nmid 6d). \quad (3.3)$$

This classical finite-field identity is proved with Jacobi sums; see [6, Chapter 18], where primary means $\equiv 2 \pmod{3}$, which changes the sign of π . We also checked it numerically for 480 pairs (d, p) (Section 11). Since $\chi_{\bar{\pi}}(4d) = \overline{\chi_\pi(4d)}$, the two reciprocal factors multiply to

$$(1 - \psi_d(\pi)p^{-s})(1 - \psi_d(\bar{\pi})p^{-s}) = 1 - \lambda_d(p)p^{-s} + p^{-2s}.$$

If $p \equiv 2 \pmod{3}$, the primary prime is $-p$, of norm p^2 . For rational x prime to p , $\chi_{-p}(x) = 1$, because $(p^2 - 1)/6 = (p - 1)(p + 1)/6$. Since $\alpha(-p) = -1$, its factor is $1 + p^{-2s}$, as required by $a_p(d) = 0$. If $p \mid d$, the symbols vanish at primes above p , so their factors are deleted.

We have therefore checked the exact identities

$$q_d(m) = \sum_{N(n)=m} \mu_K(n) \alpha(n) \chi_n(u_d), \quad S_d(D) = A_{u_d}(D). \quad (3.4)$$

The columns still run over primary n coprime to 6; the sum is empty when $(m, 6) > 1$. The angular factor and sextic symbol are what distinguish this identity from the zeta identity involving μ and χ_{-3} .

The curve's coefficients are sextic residue symbols. The identity (3.4) for the reciprocal has a companion for L itself. Up to the finitely many Euler factors at $p \mid 6d$, $L(E_d, s + \frac{1}{2}) = \prod_{\mathfrak{p} \nmid 6} (1 - \psi_d(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1} = \sum_n \psi_d(n) N(n)^{-s}$, so for m coprime to $6d$,

$$\frac{a_m(E_d)}{\sqrt{m}} = \sum_{N(n)=m} \alpha(n) \left(\frac{u_d}{n} \right)_6, \quad q_d(m) = \sum_{N(n)=m} \mu_K(n) \alpha(n) \left(\frac{u_d}{n} \right)_6,$$

with n primary and coprime to 6. Here $a_m(E_d)$ are the Dirichlet coefficients of $L(E_d, s)$. The only difference between the two is the Möbius factor. At a prime the first identity is $a_p(d) = \text{Tr}(\pi(u_d/\pi)_6)$: the Frobenius eigenvalues are $\pi(u_d/\pi)_6$ and its conjugate, so the sextic symbol selects which of the six rotations $\zeta\pi$ ($\zeta^6 = 1$) occurs. We checked the first identity against the coefficients computed by PARI/GP for $d \in \{1, 2, -3, 7\}$ and all $m \leq 400$ coprime to $6d$.

Thus the coefficients of $L(E_d, s + \frac{1}{2})$ are exactly the sextic residue symbols of the zeta argument, evaluated at the fixed row u_d and rotated by the angle $n/|n|$. In the zeta argument the row u is the Möbius sum $\sum_n \mu_K(n) (u/n)_6 W(N(n)/D)$ of the finite-order Hecke L -function attached to $n \mapsto (u/n)_6$. This is the Artin L -function of a character of the Kummer extension $\mathbb{Q}(\sqrt{-3}, u^{1/6})/\mathbb{Q}(\sqrt{-3})$, one factor of the Dedekind zeta function of that extension. The CM argument averages the same rows with every coefficient multiplied by $n/|n|$, and its row u_d is $L(s, \alpha \cdot (u_d/\cdot)_6) = L(E_d, s + \frac{1}{2})$, up to finitely many Euler factors.

4 The average, in coordinates and as a double sum

The central average is

$$M(D, H) = \sum_{0 < N(u) \leq H} |A_u(D)|^2. \quad (4.1)$$

There are about H rows and about D columns in each row. Writing $Q(A, B) = A^2 - AB + B^2$, this means exactly

$$M(D, H) = \sum_{\substack{x, y \in \mathbb{Z} \\ 0 < Q(x, y) \leq H}} \left| \sum_{\substack{A \equiv 1 \pmod{3}, B \equiv 0 \pmod{3} \\ A, B \text{ not both even}}} \mu_K(A + B\omega) \frac{A + B\omega}{\sqrt{Q(A, B)}} \times \left(\frac{x + y\omega}{A + B\omega} \right)_6 W\left(\frac{Q(A, B)}{D}\right) \right|^2. \quad (4.2)$$

Thus it is a sum over four ordinary integer coordinates, using factorization and exponentiation in finite fields.

Expanding the square, with the first column conjugated, gives

$$\begin{aligned} M(D, H) &= \sum_{n_1, n_2} \mu_K(n_1) \mu_K(n_2) \overline{\alpha(n_1)} \alpha(n_2) \overline{W(N(n_1)/D)} W(N(n_2)/D) S_H(n_1, n_2), \\ S_H(n_1, n_2) &= \sum_{0 < N(u) \leq H} \overline{\chi_{n_1}(u)} \chi_{n_2}(u). \end{aligned} \quad (4.3)$$

The elliptic-curve factor in this double sum is simply $\overline{n_1} n_2 / (|n_1| |n_2|)$. No abstract character notation is needed to calculate it.

4.1 An ordinary-integer description of every row

If desired, put

$$q_u(m) = \sum_{N(n)=m} \mu_K(n) \alpha(n) \chi_n(u), \quad A_u(D) = \sum_{m \geq 1} q_u(m) W(m/D).$$

For fixed $u \neq 0$, q_u is multiplicative, vanishes on integers divisible by 2 or 3, and for $p \geq 5$ has the following local values:

$p = \pi \bar{\pi} \equiv 1 \pmod{3}$	$\frac{q_u(p)}{\pi \chi_\pi(u) + \bar{\pi} \chi_{\bar{\pi}}(u)}$	$\frac{q_u(p^2)}{\chi_\pi(u) \chi_{\bar{\pi}}(u)}$	(4.4)
$p \equiv 2 \pmod{3}$	$\frac{\sqrt{p}}{0}$	$\chi_{-p}(u)$	

All $q_u(p^j)$ with $j \geq 3$ vanish. These formulas include the case when a symbol is zero. At $u = u_d$ they reduce to (2.1). For a general row, the symbols distinguish π from $\bar{\pi}$; ordinary μ and χ_{-3} alone do not determine them.

4.2 The estimate to be supplied by analysis

The analytic input is the following mean-square estimate: for every fixed $0 < \vartheta \leq 1/10$ and every $\varepsilon > 0$,

$$\boxed{M(D, D^{1+\vartheta}) \ll_{W, \vartheta, \varepsilon} D^{2+\vartheta+\varepsilon}.} \quad (4.5)$$

The constant is controlled by finitely many derivatives of W . The curve parameter does not occur in this estimate: all the curves are selected by different rows of the same family.

The diagonal $n_1 = n_2 = n$ in (4.3) has

$$S_H(n, n) = \#\{u : 0 < N(u) \leq H, (u, n) = 1\} \ll H.$$

More precisely, elementary lattice counting gives

$$S_H(n, n) = \frac{2\pi}{\sqrt{3}} H \prod_{\pi|n} (1 - N(\pi)^{-1}) + O(\tau(n)(\sqrt{H} + 1)).$$

Here $\tau(n)$ counts primary divisors. The factor $2\pi/\sqrt{3}$ is the area of the disc divided by the lattice covolume. The diagonal is $O_W(HD)$. Thus (4.5) bounds the whole average on the diagonal scale, up to a small power loss. It bounds the total off-diagonal contribution, not the sum of its termwise absolute values.

5 Sixth powers isolate the chosen elliptic curve

Fix $u_0 = u_d$ and $N_0 = N(u_0) = |4d|^{10}$. For any $v \neq 0$,

$$\chi_n(u_0 v^6) = \chi_n(u_0) \mathbf{1}_{(n,v)=1}.$$

Consequently the rows we need are *the fixed row u_0 times sixth powers*; they need not themselves be sixth powers.

Take $H = D^{1+\vartheta}$, put $Y = (H/N_0)^{1/6}$, and let

$$\mathcal{P}_Y = \{\pi \text{ primary prime} : \pi \nmid 6d, Y/2 < N(\pi) \leq Y\}, \quad J = |\mathcal{P}_Y|.$$

For large D , the prime number theorem for $p \equiv 1 \pmod{3}$ gives $J \asymp_d Y/\log Y$. The rows $u_0 \pi^6$ are distinct and have norm at most H . Their difference from the target is exactly

$$A_{u_0}(D) - A_{u_0 \pi^6}(D) = \sum_{\pi|n} \mu_K(n) \alpha(n) \chi_n(u_0) W(N(n)/D).$$

There are $O_W(D/N(\pi))$ possible multiples of π in this sum, because dividing by π reduces the norm bound by $N(\pi)$. Every coefficient has absolute value at most one. Therefore

$$|A_{u_0}(D) - A_{u_0 \pi^6}(D)| \ll_W D/Y. \quad (5.1)$$

Using $|z|^2 \leq 2|w|^2 + 2|z - w|^2$ for each selected row and the nonnegativity of the mean square gives

$$\boxed{J|S_d(D)|^2 \leq 2M(D, H) + O_W(JD^2/Y^2)}. \quad (5.2)$$

There is no assumption that different rows cancel against each other. Positivity is precisely what lets us extract the special rows.

Inserting (4.5), absorbing logarithms into D^ε , and allowing constants depending on d , we get

$$\begin{aligned} |S_d(D)|^2 &\ll_{d,W,\vartheta,\varepsilon} DH^{5/6} D^\varepsilon + D^2 H^{-1/3} \\ &= D^{11/6+5\vartheta/6+\varepsilon} + D^{5/3-\vartheta/3}. \end{aligned} \quad (5.3)$$

Take square roots, then choose $\vartheta > 0$ and the power loss small enough. For every $\eta > 0$ this proves

$$S_d(D) \ll_{d,W,\eta} D^{11/12+\eta}. \quad (5.4)$$

Dependence on d . The mean-square estimate (4.5) does not involve d ; the curve enters only through the position of its row. Keeping track of $N_0 = N(u_0)$ in the steps above gives

$$S_d(D) \ll_{W,\vartheta,\varepsilon} N_0^{1/12} D^{11/12+5\vartheta/12+\varepsilon} + N_0^{1/6} D^{5/6-\vartheta/6}, \quad N_0^{1/12} = |4d|^{5/6},$$

valid once $D^{1+\vartheta} \geq 2^6 N_0$, so that $Y \geq 2$. This is nontrivial only for D larger than about $|d|^{10}$. That is a dependence on the chosen parameter d , not on the curve: E_{dt^6} is isomorphic to E_d over $\mathbb{Q}$, so $|d|$ can be made arbitrarily large while the conductor stays fixed. If d is first replaced by its sixth-power-free part, then every prime $p \geq 5$ dividing d divides the conductor $N(E_d)$ exactly twice, so $|d| \ll N(E_d)^{5/2}$, and $|d|^{10}$ is at most a fixed power of the conductor. For the zero-free half-plane this dependence is harmless, because each d is treated separately.

The exponent is easy to remember: the average has size about D^2 , and about $D^{1/6}$ useful copies of the target occur in it, so the target squared has size about $D^{2-1/6} = D^{11/6}$.

5.1 Other perfect powers and the rational slice

The exact identities

$$\chi_n(v^3) = (v/n)_2, \quad \chi_n(v^2) = (v/n)_3$$

identify cube and square rows as quadratic and cubic twists. If v is a nonzero rational integer, then $(v/n)_2 = (v/N(n))$, an ordinary Jacobi symbol. Hence

$$A_{u_d v^3}(D) = \sum_{m \geq 1} q_d(m) \left(\frac{v}{m} \right) W(m/D). \quad (5.5)$$

The finite-field reason at an inert prime is that every nonzero element of $\mathbb{F}_p$ is a square in $\mathbb{F}_{p^2}$; at a split prime it is ordinary Euler's criterion.

The entire rational slice $u \in \mathbb{Z}$ has only $\asymp H^{1/2}$ rows, since $N(u) = u^2$, whereas the lattice family has $\asymp H$ rows. Sixth powers of lattice elements number $\asymp H^{1/6}$; sixth powers of rational integers number $\asymp H^{1/12}$. The primes used in (5.2) are lattice primes, not merely rational prime bases. General rational numerators still require sextic residue symbols when computing their coefficients at split primes.

6 Why the bound excludes zeros

Proposition 1. *If (4.5) holds for every smooth compactly supported complex weight and every fixed $0 < \vartheta \leq 1/10$, then $L(E_d, s) \neq 0$ for $\Re s > 17/12$.*

Proof. We have proved (5.4). Let

$$\widehat{W}(s) = \int_0^\infty W(y) y^s \frac{dy}{y}, \quad F_W(s) = \int_0^\infty S_d(D) D^{-s} \frac{dD}{D}.$$

The second integral is holomorphic for $\Re s > 11/12$. At the lower endpoint there is no convergence problem: $S_d(D) = 0$ for $D < 1/b$. For $\Re s > 1$, absolute convergence and (2.3) give

$$L(E_d, s + \tfrac{1}{2}) F_W(s) = \widehat{W}(s) B_d(s). \quad (6.1)$$

Since $L(E_d, s)$ is entire, the identity theorem extends this equality to $\Re s > 11/12$.

Suppose $L(E_d, \rho) = 0$ with $\Re \rho > 17/12$, and put $s_0 = \rho - 1/2$. Choose a nonzero nonnegative smooth function ϕ supported in $(1, 2)$, and use $W(y) = y^{-s_0} \phi(y)$. Then

$$\widehat{W}(s_0) = \int_1^2 \phi(y) \frac{dy}{y} > 0.$$

At $s = s_0$ the left side of (6.1) is zero, while the right side is nonzero. This contradiction proves the assertion. $\square$

This explains both the allowance of complex weights and the shift by $1/2$. The zero-free boundary asserted is strict; no conclusion at $\Re s = 17/12$ is being drawn.

7 Poisson summation reveals the cubic Gauss sums

We now explain the analytic work behind (4.5). Choose a fixed smooth radial majorant $\Phi(N(u)/H)$, nonnegative and at least one on $N(u) \leq H$, rapidly decreasing at infinity, with compactly supported planar Fourier transform. Put

$$e(z) = \exp(4\pi i \operatorname{Im} z / \sqrt{3}), \quad \widehat{\Phi}(N(w)) = \frac{2}{\sqrt{3}} \int_{\mathbb{C}} \Phi(|z|^2) e(-zw) dx dy.$$

The smooth mean square over all $u \in \mathbb{Z}[\omega]$, including zero, is an upper bound for (4.1).

Only squarefree columns occur. Write $n_i = g z_i$, where g is their primary greatest common divisor. Then $(z_1, z_2) = (g, z_1 z_2) = 1$, and

$$\overline{\chi_{n_1}(u)} \chi_{n_2}(u) = \mathbf{1}_{(u,g)=1} \psi(u), \quad \psi = \overline{\chi_{z_1}} \chi_{z_2}.$$

This is a primitive character modulo $z_1 z_2$. For a primitive character modulo q , define the normalized finite Gauss sum

$$\gamma(\psi) = \frac{1}{\sqrt{N(q)}} \sum_{x \bmod q} \psi(x) e(x/q).$$

Inclusion–exclusion for $(u, g) = 1$, followed by planar Poisson summation, gives the exact formula

$$\begin{aligned} & \sum_{u \in \mathbb{Z}[\omega]} \Phi(N(u)/H) \mathbf{1}_{(u,g)=1} \psi(u) \\ &= \frac{H \gamma(\psi)}{\sqrt{N(z_1 z_2)}} \sum_{e_0 | g} \frac{\mu_K(e_0) \psi(e_0)}{N(e_0)} \sum_{h \in \mathbb{Z}[\omega]} \overline{\psi(h)} \widehat{\Phi} \left(\frac{HN(h)}{N(e_0)N(z_1 z_2)} \right). \end{aligned} \quad (7.1)$$

The divisor e_0 must be kept. The zero frequency contributes only when $z_1 = z_2 = 1$; it gives the diagonal main term. There can also be nonzero frequencies from these diagonal pairs, and they remain in the estimate. The other frequencies satisfy

$$N(h) \ll \frac{N(e_0)N(z_1 z_2)}{H} \ll \frac{D^2}{HN(g)}.$$

This is the first shortening: $H > D$ produces frequencies of norm less than about D .

7.1 The arithmetic identity and the extra angular factor

For squarefree n define

$$\gamma_j(n) = \frac{1}{\sqrt{N(n)}} \sum_{x \bmod n} \chi_n(x)^j e(x/n), \quad G(n) = \overline{\chi_n(4)} \gamma_3(n).$$

In particular, γ_2 is a cubic Gauss sum. The finite-sum identity used in both arguments is

$$\mu_K(n) \gamma_{-1}(n) = \chi_n(-1) G(n)^{-1} \overline{\alpha(n)} \gamma_2(n). \quad (7.2)$$

It combines the Jacobi-sum identity for a cubic Gauss sum with sextic reciprocity. The Möbius sign is essential here. The function G has absolute value one and depends only on the primary invertible residue class modulo 72.

Let $\mathfrak{G}$ be that finite group of classes. Define, quite explicitly,

$$c_\xi = \frac{1}{|\mathfrak{G}|} \sum_{t \in \mathfrak{G}} G(t^{-1}) \overline{\xi(t)}, \quad a_\xi(n) = \overline{\alpha(n)}^2 \gamma_2(n) \xi(n),$$

where ξ ranges over the characters of this fixed finite group. For coprime squarefree z_1, z_2 , the useful paired identity is

$$\begin{aligned} \mu_K(z_1) \mu_K(z_2) \overline{\alpha(z_1)} \alpha(z_2) \gamma(\overline{\chi_{z_1}} \chi_{z_2}) \\ = \sum_{\xi} c_\xi a_\xi(z_1) \overline{a_\xi(z_2)}. \end{aligned} \quad (7.3)$$

This preserves the structure of a squared sum. The zeta version produces $\overline{\alpha} \gamma_2$; the present version produces $\overline{\alpha}^2 \gamma_2$. The second factor of $\overline{\alpha}$ comes from conjugating the first column of (3.1).

7.2 The next average, with every variable displayed

After resolving greatest common divisors, the averages have the form

$$\mathcal{E}(\mathcal{H}, X, F) = \frac{1}{XF} \sum_{F \leq N(f) < 2F}^* \sum_{0 < N(k) \leq \mathcal{H}} \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/X) \right|^2. \quad (7.4)$$

Both f, n are primary, squarefree, and coprime to 6; k runs over all lattice points. The weight and the finite character ξ have been suppressed on the left. Put $\Sigma = XF$.

There are two sources of the fourth power. The divisor e_0 in Poisson contributes $\chi_n(h e_0^5)$. A second exclusion divisor v_0 uses the twisted multiplicativity

$$a_\xi(v_0 n) = a_\xi(v_0) a_\xi(n) \chi_n(v_0)^4 \quad ((v_0, n) = 1).$$

With $f = e_0 v_0$ and $k = e_0 h$, these combine as $h e_0^5 v_0^4 = k f^4$. Neither source can be omitted.

The initial scales are, for a further divisor-size parameter $B \geq 1$,

$$X = \frac{D}{BF}, \quad \Sigma = \frac{D}{B}, \quad \mathcal{H} \ll \frac{D^2}{HB^2}. \quad (7.5)$$

The first Poisson reduction shows that the estimate $\mathcal{E} \ll D^\epsilon \Sigma$ in these ranges implies (4.5). Smooth kernels in two column variables are separated using Fourier or Mellin integrals, with a controlled finite number of derivatives; they are not simply discarded.

8 The completed sums and the second theta derivative

For fixed k, f , put $\Psi(n) = \xi(n) \chi_n(k) \chi_n(f)^4$. The completed sum is the following explicit double sum:

$$\begin{aligned} T(X; k, f) = \frac{1}{\sqrt{X}} \sum_n^* \sum_b \overline{\alpha(n)}^2 \gamma_2(n) \Psi(n) \overline{\alpha(b)}^6 \Psi(b)^3 N(b)^{1/2} \\ \times W\left(\frac{N(n)N(b)^3}{X}\right). \end{aligned} \quad (8.1)$$

Here b is primary and coprime to 6 but need not be squarefree. All sums are finite. The term $b = 1$ is exactly $X^{-1/2}$ times the column sum in (7.4). The other terms are added because they occur naturally in the theta expansion.

8.1 Where those coefficients come from

Put $\lambda = 1 + 2\omega = i\sqrt{3}$ and $\check{e}(z) = \exp(2\pi i(z + \bar{z}))$. Kubota's cubic theta function is a Fourier series in a planar variable z and a positive height v , with terms $d_0(\ell)vK_{1/3}(4\pi|\ell|v)\check{e}(\ell z)$ in its conjugate. Here $K_{1/3}$ is the usual modified Bessel function. Patterson's coefficient formula gives

$$d_0(\lambda^{-3}nb^3) = 3^{5/2}|b|\overline{(\lambda/n)_3}\gamma_2(n). \quad (8.2)$$

Restricting $\lambda^3\ell$ to be primary and coprime to 6 selects exactly these indices. Other units and powers of λ are excluded from the twisted series by this restriction. In the reflected series, at the three cusps, the indices lie in the larger set of $\ell = u\lambda^mnb^3$ with u a unit and $m \geq -4$. The coefficient tables of Appendix A restrict them further; for example $d_0(\lambda^{-4}) = 0$. The factor $(\lambda/n)_3$ depends only on $n \bmod 9$ and is canceled by inserting $(\lambda/\lambda^3\ell)_3$ in the twist. It is finite residue-class data, not a missing Gauss factor.

Differentiating $\check{e}(\ell z)$ twice with respect to $\bar{z}$ multiplies it by $(2\pi i\bar{\ell})^2$. Removing the resulting radial power by a Mellin integral leaves $\overline{\alpha(\ell)}^2$. Since $\ell = \lambda^{-3}nb^3$, its factors in n, b are exactly $\overline{\alpha(n)}^2\overline{\alpha(b)}^6$, up to the fixed factor $\overline{\alpha(\lambda^{-3})}^2 = -1$. This explains every angular power in (8.1).

8.2 Why differentiating twice is legitimate

The theta symmetry relates a rational translate $a/c + z$ to coordinates

$$z' = -\frac{\delta}{c} - \frac{\bar{z}}{c^2(v^2 + z\bar{z})}, \quad v' = \frac{v}{N(c)(v^2 + z\bar{z})}.$$

At $z = \bar{z} = 0$, pure $\bar{z}$ derivatives of z' and v' vanish, as do derivatives of z' of order at least two, while $\partial_{\bar{z}}z' = -(cv)^{-2}$. Therefore the ordinary chain rule gives

$$\partial_{\bar{z}}^2 F(z', \bar{z}', v')|_{z=\bar{z}=0} = (cv)^{-4}(\partial_{z'}^2 F)(-\delta/c, 1/(N(c)v)). \quad (8.3)$$

No unwanted lower derivative survives. The derivative also kills any constant Fourier term. Of the three expansions used, only the one at the identity cusp has a nonzero constant term, and the derivative removes it. This permits Mellin reflection without constant-term contributions.

For $V(y) = \sqrt{y}W(y)$ the reflected weight is

$$V^\sharp(x) = \frac{1}{2\pi i} \int_{\Re t=0} \widehat{V}(-t) \frac{\Gamma(4/3+t)\Gamma(5/3+t)}{\Gamma(4/3-t)\Gamma(5/3-t)} \left(\frac{(2\pi)^4 x}{27} \right)^{-t} dt. \quad (8.4)$$

This is the CM version of the gamma quotient in the zeta argument. The reflected coefficients have angular factor $\alpha(\ell)^2$; the fixed cusp prefactor is $-\overline{\alpha(c)}^4/81$, before the finite Fourier coefficients and local phases are included. The full reflected series and its coefficient tables are recorded in Appendix A, so the main discussion can focus on the lengths and the averaging.

8.3 The local step from sextic to quadratic

By reciprocity and finite congruence-class splitting, Ψ is of the form $\xi_0(n) \prod_{\pi \in \mathcal{P}} \chi_\pi(n)^{j_\pi}$. The local reflected factors, apart from constant phases, are

$$B_{\pi,j}(x) = \begin{cases} \chi_\pi(x)^{-j-2}, & j \notin \{0, 4\}, \\ N(\pi)^{-1/2}\chi_\pi(x)^{-2}, & j = 0, \\ N(\pi)^{-1/2}(-1 + N(\pi)\mathbf{1}_{\pi|x}), & j = 4. \end{cases} \quad (8.5)$$

In particular,

$$B_{\pi,1}(x) = \chi_{\pi}(x)^3 = (x/\pi)_2.$$

Away from the fixed factors of $6f$, squarefree row primes have exponent $j = 1$. These are the varying primes that become quadratic. The other local factors are still present and have to be estimated.

If $Q = N(c)$ is the norm of the reduced cusp denominator, the weight in the reflected sum is $V^{\sharp}(N(\ell)X/Q^2)$. Its effective norm range is about Q^2/X , shorter than X when $Q < X$. Rapid decay gives a negligible tail when X/Q^2 is sufficiently large; the inequality $Q^2 < X$ alone does not specify a quantitative saving.

9 The quadratic average and the completed bound

The quadratic large sieve used here is

$$\sum_{\substack{N(k) \leq K \\ (k,6)=1}}^* \left| \sum_{\substack{N(n) \leq U}}^* \beta(n) \left(\frac{n}{k} \right)_2 \right|^2 \ll_{\varepsilon} (KU)^{\varepsilon} (K+U) \sum_n^* |\beta(n)|^2. \quad (9.1)$$

Both k, n are primary and squarefree, the coefficients $\beta(n)$ are arbitrary, and the column n may be even. The squarefree restrictions matter: square columns would give repeated values 1 rather than cancellation. This is the number-field quadratic large sieve of [9], in the form needed here.

Applying (9.1) to the reflected sums, and retaining the factors in (8.5), gives the completed mean-square estimate

$$\boxed{\sum_{0 < N(k) \leq \mathcal{H}} |T(X; k, f)|^2 \ll D^{\varepsilon} \left(\mathcal{H} + \frac{\mathcal{H}^2 N(f)}{X} \right)}. \quad (9.2)$$

All parameters are at least one and bounded by a fixed power of D ; a fixed finite smoothness norm of W is understood. This is the CM form of the completed mean-square estimate of [2].

For clarity, passing from squarefree rows to all rows uses $k = usv_0^2$, with u a unit and s squarefree. The identity

$$\chi_n(v_0^2) \chi_n(f)^4 = \chi_n(fv_0^2)^4$$

absorbs the square into the auxiliary twist. The squarefree-row bound allows that auxiliary argument to be nonsquarefree. Its contributions are summed using $\sum_{v_0} N(v_0)^{-2} < \infty$.

This is the main benefit of the theta transformation: the hard sextic average has been reduced to an average of quadratic symbols. However, (9.2) bounds the completed sum, whose cube terms still have to be removed.

10 Remove the cubes, then reduce the number of rows

One cannot discard the $b \neq 1$ terms inside (8.1): they could cancel the term we want. Instead use the exact inversion

$$\frac{1}{\sqrt{X}} \sum_n^* a_{\xi}(n) \chi_n(k) \chi_n(f)^4 W(N(n)/X) = \sum_h \frac{\mu_K(h) \overline{\alpha(h)}^6 \Psi(h)^3}{N(h)} T(X/N(h)^3; k, f). \quad (10.1)$$

Here h is primary and coprime to 6. To check the formula, substitute (8.1), collect the total cube index hb , and use $\sum_{h|b'} \mu_K(h) = \mathbf{1}_{b'=1}$. The angular and character factors are completely multiplicative, so this cancellation is exact.

Put

$$H_c^3 = \min\left(X, \frac{X^2}{\mathcal{H}^2}\right), \quad L_b = \frac{X}{N(b)^3}, \quad \Sigma = XF.$$

Assume $\mathcal{H} \leq \Sigma$. For $N(h) \leq H_c$, the completed bound at scale $X/N(h)^3$ is at most

$$D^\varepsilon \left(\mathcal{H} + \frac{\mathcal{H}^2 F N(h)^3}{X} \right) \ll D^\varepsilon \Sigma.$$

Cauchy–Schwarz and divisor sums absorb the sum over these small h . Expanding the remaining terms gives the cube-removal inequality

$$\mathcal{E}(\mathcal{H}, X, F) \ll D^\varepsilon \left(\Sigma + \sup_{\substack{N(b) > H_c \\ L_b > 1}} \mathcal{E}(\mathcal{H}, L_b, F) \right). \quad (10.2)$$

An empty supremum is zero. Scales $L_b \leq 1$ are handled by counting; if $\mathcal{H}^2 \leq X$, the supremum is empty.

10.1 What two more Poisson summations do

For a remaining scale $L = L_b$, smooth the row sum in the explicit average

$$\frac{1}{LF} \sum_{F \leq N(f) < 2F}^* \sum_{k \in \mathbb{Z}[\omega]} \Phi(N(k)/\mathcal{H}) \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(N(n)/L) \right|^2.$$

The reverse paired Gauss identity converts its coefficients back to Möbius coefficients $\mu_K(n) \overline{\alpha(n)} \xi_1(n)$. For precision, the intermediate column sums, with their exclusions, are

$$P_{C, e_0, t}(y; U) = \sum_n^* \mu_K(n) \overline{\alpha(n)} \xi_1(n) \mathbf{1}_{(n, t)=1} \overline{\chi_n(y)} \chi_n(C)^4 \chi_n(e_0) U(N(n)/\ell), \quad \ell = \frac{L}{N(C)N(t)}. \quad (10.3)$$

Here C, t are coprime squarefree primary elements and $e_0 \mid C$. After separation of smooth weights, the nonnegative average used is

$$\sum_{\substack{C, t \text{ squarefree} \\ (C, t)=1}} \sum_{e_0 \mid C} \frac{\mathcal{H} N(C)}{N(e_0) L^2 F} \sum_{y \in \mathbb{Z}[\omega]} \Phi(N(y)/Y_{C, e_0}) |P_{C, e_0, t}(y; U)|^2, \quad (10.4)$$

where the outer sum is restricted to $N(Ct) \ll_W L$ and

$$Y_{C, e_0} = c_* \frac{\Sigma L F N(e_0)}{\mathcal{H} N(C)^2}.$$

The constant c_* is fixed and chosen to cover the support of the weights.

Here is the crucial enlargement. The first Poisson frequency h has size about $L^2/\mathcal{H}$, and combining it with f as $y = hf^2$ gives a natural range about $L^2 F^2/\mathcal{H}$. Suppressing the common divisor factor $N(e_0)/N(C)^2$ and fixed constants, positivity permits

$$Y_{\text{natural}} = \frac{L^2 F^2}{\mathcal{H}} \longrightarrow Y_{\text{enlarged}} = \frac{\Sigma L F}{\mathcal{H}}, \quad \frac{Y_{\text{enlarged}}}{Y_{\text{natural}}} = \frac{\Sigma}{L F} \geq 1. \quad (10.5)$$

A larger row range at this stage means a smaller frequency range after the second Poisson summation. Without this enlargement, the two summations need not give a strict reduction.

The second Poisson summation uses (7.3) and returns to cubic coefficients. Regrouping signed divisor contributions before taking absolute values forces $f' \mid k'$. Locally the cancellation is

$$(1 + \mathbf{1}_{\pi|k'}) - \mathbf{1}_{\pi|k'} - \mathbf{1}_{\pi|k'} = \mathbf{1}_{\pi|k'}.$$

Thus the next row range is at least the size of its auxiliary divisor, $\mathcal{H}' \geq F' \geq 1$, and the column scales arising from removal of exclusions also remain at least one. This keeps the next average in the range to which the argument applies.

Precisely, the transfer estimate bounds the average at $(\mathcal{H}, L, F)$ by $D^\varepsilon \Sigma$ times one plus the supremum of normalized averages $\mathcal{E}(\mathcal{H}', X', F')/\Sigma'$, where $\Sigma' = X'F'$ and

$$\mathcal{H}' \leq \frac{\mathcal{H}L}{\Sigma F}, \quad \frac{\mathcal{H}'}{\Sigma'} \leq \frac{\mathcal{H}}{\Sigma}, \quad \Sigma' \leq L. \quad (10.6)$$

The finite character and the smooth weight may change, with controlled derivatives. All three new parameters are at least one.

10.2 Why this process terminates

Suppose $\mathcal{H} \leq \Sigma D^{-\kappa}$ for a fixed $\kappa > 0$. In a nonempty supremum in (10.2), $N(b)^3 > X^2/\mathcal{H}^2$. With $L = X/N(b)^3$,

$$\mathcal{H}' \leq \frac{\mathcal{H}}{N(b)^3 F^2} < \mathcal{H} \left(\frac{\mathcal{H}}{XF} \right)^2 \leq \mathcal{H} D^{-2\kappa}. \quad (10.7)$$

The number of rows falls by a fixed power of D , while the ratio $\mathcal{H}'/\Sigma' \leq D^{-\kappa}$ is preserved. Since the parameters start below a fixed power of D , finitely many steps reach $\mathcal{H} \leq 1$. There are then at most six rows, and counting gives $\mathcal{E} \ll X \leq \Sigma$. Distributing the allowed power loss among this bounded number of steps proves

$$\mathcal{E}(\mathcal{H}, X, F) \ll D^\varepsilon \Sigma \quad (\mathcal{H} \leq \Sigma D^{-\kappa}).$$

Finally, the initial ranges (7.5) have $\mathcal{H}/\Sigma \ll D^{-\vartheta}/B$. For sufficiently large D , take $\kappa = \vartheta/2$. This proves the required auxiliary estimate, hence the original mean square (4.5), and hence the deduction in Section 6.

What changes from the zeta guide. The architecture is the same, but its angular factors change coherently:

	zeta guide	present CM guide
original coefficients	$\mu_K(n)$	$\mu_K(n)\alpha(n)$
cubic coefficients	$\bar{\alpha}\gamma_2$	$\bar{\alpha}^2\gamma_2$
theta derivative	$\partial_{\bar{z}}$	$\partial_{\bar{z}}^2$
angular factor on cubes	$\overline{\alpha(b)}^3$	$\overline{\alpha(b)}^6$
dual angular factor	$\alpha(\ell)$	$\alpha(\ell)^2$

These are genuine identities to check, not consequences of the factors having absolute value one. Once checked, the same large-sieve and descent estimates apply.

11 Part II: the 7/8 boundary for the same curves

The remaining sections adapt OpenAI's 7/8 argument [3], explained for ζ in [5], to the curves E_d . The result is a wider zero-free strip. Equation and line references to [3] are to its T_EX source. Throughout Part II, $\mu = \mu_K$ is the Möbius function on primary elements, and $\chi_\bullet(u)$ denotes the character $n \mapsto \chi_n(u)$.

Theorem 1 (claimed; see Section 16). *Let η be a finite-order Hecke character of $K = \mathbb{Q}(\sqrt{-3})$, and put $\psi = \bar{\alpha}\eta$, that is, $\psi(n) = \overline{\alpha(n)}\eta(n)$ on primary n prime to the conductor of η . Then*

$$L(s, \psi) \neq 0 \quad (\Re s > 7/8).$$

In particular, for every nonzero integer d ,

$$L(E_d, s) \neq 0 \quad (\Re s > 11/8),$$

a zero-free strip $11/8 < \Re s < 3/2$ along the right edge of the critical strip.

How the curves enter. Put $\psi_d^*(n) = \overline{\alpha(n)}\chi_n(4d)$. If $p \nmid 6d$ splits as $p = \pi\bar{\pi}$ with π primary, then $\sqrt{p}(\psi_d^*(\pi) + \psi_d^*(\bar{\pi})) = a_p(d)$; if $p \equiv 2 \pmod{3}$, then $\psi_d^*(-p) = -1$, matching $a_p(d) = 0$. So $L(E_d, s + \frac{1}{2}) = L(s, \psi_d^*)$ up to finitely many Euler factors, which do not vanish in $\Re s > 0$. (We checked both statements for 480 split pairs (d, p) , testing both primes above each p , and 112 inert pairs.) The character ψ_d^* is the complex conjugate of the character $\psi_d = \alpha\chi_\bullet(4d) = \alpha\chi_\bullet((4d)^5)$ used in Sections 3–10. The two have the same L -function: since $4d$ is rational, $\psi_d^*(\bar{\pi}) = \overline{\psi_d^*(\pi)}$, so both characters give the same Euler factor at each split prime, and $\psi_d^*(-p) = -1$ is real. More generally, every unitary Hecke character of K whose infinity type is $(\bar{z}/|z|)^{\pm 1}$ is of the form $\bar{\alpha}\eta$ or its conjugate $\alpha\bar{\eta}$. So the theorem covers every weight-two newform with complex multiplication by K . In particular it covers every elliptic curve over $\mathbb{Q}$ with complex multiplication by an order of K (the j -invariants 0, 54000 and -12288000).

What “proof” means in Part II. We do not rewrite the 16,700 lines of [3]. Instead we do three things:

1. we prove the identities in which the angular factor changes a computation, and test them numerically;
2. we state each ingredient of [3] in the form it takes for a CM target;
3. we checked, section by section, that each proof in [3] uses only properties that CM characters share with finite-order characters, or we record the change.

No step was found that needs a new idea. Section 16 lists the statements that still have to be written out in full, at the generality of [3].

11.1 The family and the number β_*

Let $\mathcal{F}_0$ be the set of primitive finite-order Hecke characters of K , and let $\mathcal{F}_{\text{CM}}$ be the set of primitive Hecke characters of K with infinity type $\bar{z}/|z|$, that is, the primitive characters inducing $\bar{\alpha}\eta$ with η of finite order. Put

$$\beta_* = \sup(\{1/2\} \cup \{\Re \rho : L(\rho, \psi) = 0, 1/2 \leq \Re \rho \leq 1, \psi \in \mathcal{F}_{\text{CM}}\}).$$

Three properties make this family suitable.

- *Closure under finite-order twists.* If $\psi \in \mathcal{F}_{\text{CM}}$ and θ has finite order, then the primitive character inducing $\psi\theta$ lies in $\mathcal{F}_{\text{CM}}$. Every denominator row below is such a twist.
- *Conjugation.* $\overline{\psi} = \alpha\bar{\eta}$ and $L(s, \overline{\psi}) = \overline{L(\bar{s}, \psi)}$, so conjugates contribute no new real parts of zeros.
- *No poles.* Every $L(s, \psi)$ with $\psi \in \mathcal{F}_{\text{CM}}$ is entire. No member is principal, so the principal-character clauses of [3] disappear.

The rows of the argument also contain finite-order L -functions. For them we do not use β_* . Instead we use OpenAI's theorem [3, Theorem thm:main]: they have no zeros in $\Re s > 7/8$. So, under the contradiction hypothesis $\beta_* > 7/8$, every L -function that occurs has its zeros in $\Re s \leq \beta_*$. In [3] this was automatic from the definition of β_* . Here it is one of two places where the finite-order theorem is an input; the other is the zero-free hypothesis for the slot characters in the plain moment (Section 14).

The continuation criterion [3, Prop. lem:continuation-criterion] (see [5, Proposition 1]) holds verbatim with η replaced by ψ . Its proof uses only three things:

- the definition of β_* ;
- the bound $|1/L^S(s, \psi)| \leq \zeta_K(2)$ on $\Re s \geq 2$;
- the nonvanishing of finitely many Euler factors in $\Re s > 0$.

It therefore suffices, for each $\psi \in \mathcal{F}_{\text{CM}}$, to construct a function $J_\psi(Z)$ with the low and high estimates of the criterion, for $\sigma_0 = 7/8$ and $C(s) = s - 11/16$, with margins independent of ψ .

The construction is that of [3], with three changes:

- the weight of the target (Section 12);
- the bookkeeping of the characters attached to rows (Section 13);
- an angular factor in the moment estimates (Section 14).

Section 15 checks that the final inequality is unchanged.

12 The probe for a CM target

12.1 The weight

In [3] the target η enters the probe [3, eq. general-probe], [5, §3.1], only through the weight on the completed index $A = cn^3$: $\gamma_2(c)\overline{\alpha(A)}\eta(A)$. For the target $\psi = \bar{\alpha}\eta$ the weight is therefore

$$\gamma_2(c)\overline{\alpha(cn^3)}\psi(cn^3) = \gamma_2(c)\overline{\alpha(c)}^2\overline{\alpha(n)}^6\eta(c)\eta(n)^3. \quad (12.1)$$

Everything else in the probe is unchanged:

- the additive polynomial with its sextic Gauss sums;
- the averaging variable m ;
- the fixed phases;
- the scales X, Y, Z .

Why $\bar{\alpha}\eta$ and not $\alpha\bar{\eta}$. The two targets have conjugate zeros, so either could be used. With $\alpha\bar{\eta}$ the weight would be $\gamma_2(c)\bar{\eta}(cn^3)$, with no angular factor. The completed sums would then be sums of coefficients of the cubic theta function itself. Its expansion at the identity cusp has a nonzero constant term; the expansions at the other two cusps have none. In [3] a $\bar{z}$ -derivative removes that term (L2331). Without a derivative it enters the reflection of every translate attached to the identity cusp, and produces a main term that nothing in [3] controls. With the target $\bar{\alpha}\eta$ the weight carries $\bar{\alpha}^2$. The completed sums are then those of $\partial_{\bar{z}}^2\theta$, which is the input of Section 8, and the constant term is removed.

12.2 Second evaluation: the rows

Proposition 2. *With the weight (12.1), Poisson summation in m gives the representation [3, eq. probe-poisson-identity] of the probe, with row u equal to*

$$\frac{\zeta_K^S(6z) L^S(w, \chi_\bullet(u))}{L^S(x, \psi \chi_\bullet(u))} \mathcal{H}_{\psi,u}(x, w, z),$$

where $\mathcal{H}_{\psi,u}$ is the Euler product of [3, Lemma lem:local-euler] with $\eta(p)$ replaced by $\psi(p)$. It has the following properties:

- it converges in the same regions;
- it satisfies $\mathcal{H}_{\psi,u} \ll N(u)^\varepsilon$ there;
- $\mathcal{H}_{\psi,1} = 1 + O(P_0^{-4/5})$ in the second region of [3] (L4056).

Proof. [3] evaluates the coefficient prime by prime. The target enters only through the unimodular numbers $\eta(p)$ and complete multiplicativity, and no ray-class condition on p is used (L3871).

The angular factor of the weight is absorbed by the Gauss-sum identity

$$\overline{\alpha(c)} \gamma_2(c) \gamma_1(c) = \mu(c) G(c), \quad (12.2)$$

a rearrangement of the second identity $\gamma_1\gamma_2 = \mu\alpha G$ in [3, eq. signal]. Here γ_1 is the normalized sextic Gauss sum and G a fixed finite phase. For the weight (12.1), $\overline{\alpha(c)}^2 \gamma_2(c) \eta(c) \gamma_1(c) = \mu(c) G(c) \psi(c)$. So every entry of the local table of [3] holds with $\eta(p)$ replaced by $\psi(p)$:

- The squarefree part gives $-\psi(p) \overline{\chi_p(u)} N(p)^{-x}$, which is the Euler factor of $1/L(x, \psi \overline{\chi_\bullet(u)})$.
- The cube part gives the phases $a_p = \overline{\alpha(p)}^3 \psi(p)^3$ and $b_p = \overline{\alpha(p)}^2 \psi(p)^2 \overline{\chi_p(4)}$.
- The relation $b_p^3 = a_p^2$ used there still holds, since $\chi_p(4)^3 = (4/p)_2 = 1$.

The convergence regions and the bounds for $\mathcal{H}$ use only $|\psi(p)| = 1$. □

We checked (12.2) numerically at 60 primes (error below $4 \cdot 10^{-15}$). The proposition gives the following.

- The numerators $L(w, \chi_\bullet(u))$ have finite order. They are the same as for ζ .
- The denominators $L(x, \psi \overline{\chi_\bullet(u)})$ belong to $\mathcal{F}_{\text{CM}}$ and are never principal.
- The principal row $u = 1$ is $\zeta_K^S(w) \zeta_K^S(6z) / L^S(x, \psi)$. Its residues at $w = 1$ and $z = 1/6$ give the signal with $H_\psi(x) / L^S(x, \psi)$, and with the same positive constant c_S . The positivity $M(1/6) > 0$ is a property of the averaging weight alone.

12.3 First evaluation: completed rows with two derivatives

Cauchy–Schwarz splits the probe [5, §3.2] into two mean squares. The first is the additive one, which does not involve the target. The second is the mean square of the completed rows

$$B_m = \sum_{\substack{c \text{ sf} \\ n}} \frac{\gamma_2(c) \overline{\alpha(cn^3)}^2 \nu(cn^3) \chi_{cn^3}(m)}{\sqrt{N(c)} N(n)} V\left(\frac{N(c)N(n)^3}{Z}\right).$$

Here ν is a finite-order character whose conductor is supported on the fixed set S ; it is the finite part η times the fixed phases. The angular factor must stay in the weight, not in ν : the theta step of [3] expands ν over residue classes, which is impossible for $\overline{\alpha}$.

Proposition 3. $\sum_{N(m) \leq Q} |B_m|^2 \ll Z^\varepsilon (Q + Q^2/Z)$, exactly as in [3, Lemma lem:unmarked-completed-moment].

The compensated probe needs more than this. Its marked rows involve sums over tuples of slot primes, divisibility conditions on the completed index, and a scale that depends on the tuple. In [3] their bound comes from separate lemmas: lem:hybrid-energy, lem:reflected-energy and lem:probe-row-norm, through Cor. cor:marked-reflection. Their CM versions, uniform in the tuples, the masks and the smooth weights, are item 1 of Section 16. They have not been written out here.

[3] proves this through its completed reflection [3, Prop. prop:completed-reflection], which is the transformation formula for $\partial_{\bar{z}}\theta$. For (12.1) one uses instead $\partial_{\bar{z}}^2\theta$, as in Section 8 and Appendix A. The changes are:

	ζ and finite order [3]	CM target
derivative	$\partial_{\bar{z}}\theta$	$\partial_{\bar{z}}^2\theta$
direct Mellin factor	$\Gamma(s + \frac{1}{3})\Gamma(s + \frac{2}{3})$	$\Gamma(s + \frac{5}{6})\Gamma(s + \frac{7}{6})$
reflected kernel	$\prod_{\pm} \frac{\Gamma(1+t\pm 1/6)}{\Gamma(1-t\pm 1/6)}$	$\frac{\Gamma(4/3+t)\Gamma(5/3+t)}{\Gamma(4/3-t)\Gamma(5/3-t)}$
cuspidal scalar	$-\frac{i}{81} \overline{\alpha(c)}^2$	$-\frac{1}{81} \overline{\alpha(c)}^4$
angular factor, original side	$\frac{\alpha(n)}{\alpha(b)}^3$	$\frac{\alpha(n)}{\alpha(b)}^6$
angular factor, dual side	$\alpha(\mu)$	$\alpha(\mu)^2$

Both kernels satisfy $|R(\sigma + iu)| \ll_A (1 + |u|)^{4\sigma}$ on $-1/4 \leq \sigma \leq A$. The CM kernel is holomorphic for $\Re t > -4/3$; OAI78's is holomorphic for $\Re t > -5/6$. So the reflected weight obeys the same bound $\ll \min(x^{1/4}, x^{-A})$, with the same seminorms. The other ingredients carry over as follows.

- *Dual lengths.* They are unchanged.
- *Quadratic large sieve.* The dual angular factor $\alpha(\mu)^2$ is a unimodular coefficient of the dual index, so it is absorbed into the arbitrary coefficients of the sieve.
- *Marks.* Slot primes in the completed index are handled in [3] through $\mathbf{1}_{q|A} = 1 - \chi_q(A)^0$, which never requires splitting $\overline{\alpha}$.
- *Phases.* The extra $\overline{\alpha(c)}^4$ is multiplicative over primes, so the row and tuple phases still separate.

The balanced low estimate $|I_\psi(Z^{1/2}, Z^{1/2}, Z)| \ll Z^{1/4+\varepsilon}$ follows from the proposition as in [3]. The compensated low estimate $|I_{\text{mod}}(Z)| \ll Z^{3/16+\varepsilon}$ follows in the same way once the marked lemmas above are established in CM form. We checked numerically:

- the Bessel–Mellin identities for one and two derivatives (relative error $3 \cdot 10^{-26}$);
- the two reflection constants;
- the Stirling exponents of the kernels.

13 Rows, zeros and prime slots

13.1 Growth of the CM L -functions

Lemma 1. *Let $\psi \in \mathcal{F}_{\text{CM}}$ have conductor $\mathfrak{f}$, and put $Q = N\mathfrak{f}$. Then*

$$\Lambda(s, \psi) = (3Q)^{s/2} (2\pi)^{-s} \Gamma(s + \tfrac{1}{2}) L(s, \psi)$$

is entire and $\Lambda(s, \psi) = \epsilon(\psi) \Lambda(1-s, \bar{\psi})$ with $|\epsilon(\psi)| = 1$. Moreover $|L(\sigma + it, \psi)| \ll Q^{3/5} (3 + |t|)^2$ for $-1/10 \leq \sigma \leq 11/10$, which is the bound of [3, Lemma lem:hecke-strip-growth] for finite-order characters.

Sketch. The functional equation is Hecke’s, for characters of infinity type $\bar{z}/|z|$ [4, Ch. VII, §8].

Entire. $L(s, \psi)$ is the Mellin transform of a theta series $\sum_{a \in \mathbb{Z}[\omega]} \phi(a) \bar{a} e^{-cvN(a)}$, where $c > 0$ and ϕ is the finite part, extended to elements so that $\phi(\zeta a) \bar{\zeta} = \phi(a)$ for units ζ . Poisson summation takes $\bar{z} e^{-cv|z|^2}$ to a constant multiple of $\bar{\xi} e^{-c'|\xi|^2/v}$, which vanishes at $\xi = 0$. So there is no constant term, and hence no pole.

Growth. On $\Re s = -1/10$ the gamma quotient satisfies $|\Gamma(3/2 - s)/\Gamma(1/2 + s)| \asymp (3 + |t|)^{6/5}$. That is the exponent of $|\Gamma(1-s)/\Gamma(s)|$ for finite-order characters, so the functional equation gives $|L(-1/10 + it, \psi)| \ll Q^{3/5} (3 + |t|)^{6/5}$, as in [3]. On $\Re s = 11/10$ the Euler product is bounded. The damped Phragmén–Lindelöf argument of [3], applied to $Q^{-3/5} L(s, \psi)/(s+2)^2$, then gives the strip bound. $\square$

Consequently the logarithmic control of [3, Lemma lem:logarithmic-control] holds for $\mathcal{F}_{\text{CM}}$, with the same conductor exponents and without its principal clauses. Only fixed polynomial orders in the height change, and these may depend on the target.

13.2 The characters attached to a row

In [3] each row u carries a finite set of characters: the twists $\theta_{\chi_\bullet}(u)^{\pm 1}$ by members θ of the finite group $\Theta_0 = \langle \eta, \widehat{T} \rangle$, where $\widehat{T}$ consists of fixed characters supported over 2 and 3. A row is *bad* if one of these has a zero of bounded height well to the right of $1/2$. For a CM target the group generated by ψ is infinite. We use instead the finite set

$$\mathcal{X}_u = \{\theta_{\chi_\bullet}(u)^{\pm 1}\} \cup \{\bar{\alpha} \theta \overline{\chi_\bullet(u)}, \alpha \theta_{\chi_\bullet}(u)\}, \quad \theta \in \Theta_0 = \langle \eta, \widehat{T} \rangle.$$

It has three properties.

- It contains the numerator, the denominator $\overline{\psi_{\chi_\bullet}(u)}$, and their conjugates.
- It is finite and closed under conjugation. Together with Lemma 1, the conductor bound $Q_\psi \ll N(u)$, and the fact that no CM member is principal, these are all that the detector of [3, Section sec:detector] uses.
- Its finite-order members have no zeros in $\Re s > 7/8$, and its CM members none in $\Re s > \beta_*$. This gives the bin ceiling $a \leq \beta_*$ of [5, §3.3].

A zero of a CM member produces the two witnesses of [5, §3.3], now with the angular factor:

$$M_r = U^{-r/2} \sum_n \mu(n) \overline{\alpha(n)} \theta(n) \overline{\chi_n(u)} W\left(\frac{N(n)}{U^r}\right), \quad S_m = U^{-m/2} \sum_l \overline{\alpha(l)} \theta(l) \overline{\chi_l(u)} W'\left(\frac{N(l)}{U^m}\right),$$

or their conjugates. In each, $\overline{\alpha}$ appears together with $\overline{\chi}$, or α together with χ . We call such sums *co-oriented*; this matters in Section 14.

13.3 The compensated probe and the slots

The compensated probe [5, §4.2] is formed with $\overline{\psi(p_{J^c})}$ in place of $\overline{\eta(p_{J^c})}$. In row u a slot prime $p \nmid u$ contributes the local factor $\psi(p) \overline{\chi_p(u)} N(p)^{-x}$ of the denominator, and $\overline{\psi(p)} \psi(p) = 1$. So the marked-minus-rescaled difference is $-\chi_p(u)$ plus small terms, exactly as for ζ . The slot amplitudes

$$Q_i(u) = P_i^{-1/2} \sum_{p \in \text{slot } i} \Omega_i\left(\frac{N(p)}{P_i}\right) \left(\frac{N(p)}{P_i}\right)^{z-1} \overline{\left(\frac{u}{p}\right)_6}$$

are the same finite-order prime polynomials as in [5, §4.4]. Two consequences follow.

1. The principal normalizer $A_T(Z)$ is unchanged.
2. The pointwise bound $|Q_i(u)| \ll P_i^{\delta/2+o(1)}$ of [3, Lemma lem:bin-prime-bound] is unchanged, since the slot characters $\theta_{\chi_\bullet(u)}$ lie in $\mathcal{X}_u$. Separately, the uniform slot bound inside the plain moment ([3, eq. old-eq:3.5]) needs a zero-free region only for finite-order characters. OpenAI's theorem supplies it at $7/8$, so $\kappa = 3/4$ may be used there and in the plain capacity, in place of $\kappa = 2\beta_* - 1$.

14 Counting bad rows with an angular factor

[3] counts bad rows with the sextic large sieve of [5, §3.3] and the three averages of [5, §5.1]. For rows whose zero comes from a finite-order member they apply unchanged. For CM members they are needed with the angular factor on the summation variable, but not on the slot primes.

1. The sextic large sieve [3, Lemma lem:sextic-large-sieve]. It allows arbitrary coefficients, so it is unchanged. This is the only row count used in the 11/12 stage of [3] (its Part I).

2. The inverse moment with prime factors. For either sign $\epsilon = \pm 1$, and under the conditions $r + 2z \leq m - c_1$, $2r + 8z \leq 3m - c_2$ of [3, Lemma lem:marked], put

$$M_u = Z^{-r/2} \sum_n \mu(n) \alpha(n)^\epsilon \nu(n) \chi_n(u)^\epsilon W\left(\frac{N(n)}{Z^r}\right),$$

$$Q_u = \prod_i Z^{-z_i/2} \sum_p a_i(p) \nu(p) \chi_p(u)^\epsilon W_i\left(\frac{N(p)}{Z^{z_i}}\right),$$

where $z = \sum_i z_i$, the slot sets $\mathcal{P}_i$ are disjoint and avoid S , $N(p) \asymp Z^{z_i}$ for $p \in \mathcal{P}_i$, and the coefficients $a_i(p)$ are independent of u with $|a_i(p)| \leq 1$. Then

$$\sum_{0 < N(u) \ll Z^m} |M_u Q_u|^2 \ll Z^{m+\epsilon}. \quad (14.1)$$

Why. The proof in [3] alternates between two coefficient classes: Möbius coefficients $\mu\nu$, and cubic coefficients $\bar{\alpha}\gamma_2\nu$ attached to the theta function. Each switch uses (12.2) or its companion

$$\mu(n)\gamma_{-1}(n) = \chi_n(-1)G(n)^{-1}\overline{\alpha(n)}\gamma_2(n),$$

which we checked at 60 primes. [3] runs the orientation $\mu\nu$ with $\bar{\chi}$. With the co-oriented factor $\bar{\alpha}$ on the Möbius variable ($\epsilon = -1$) the classes become $\mu\bar{\alpha} \leftrightarrow \bar{\alpha}^2\gamma_2$; for $\epsilon = +1$ they are the complex conjugates. These are the dual coefficient and the cube factor $\bar{\alpha}^6$ of Sections 7–10, and the theta steps use $\partial_{\bar{z}}^2\theta$ as in Proposition 3.

A slot prime that is merged with the Möbius variable multiplies its coefficient by a sign and the unimodular factor $\overline{\alpha(p)}^\epsilon$. That coefficient is still bounded, independent of u , and of product form, which is all [3] uses. The descent, its masks and punctures, and the propagation of seminorms never see α .

Orientation. The anti-oriented sum, with α paired with $\bar{\chi}$, would turn into plain γ_2 , since $\alpha \cdot \bar{\alpha}\gamma_2 = \gamma_2$. That is the theta function without a derivative, with its constant terms. So the clause “for either sign” of [3] does not extend to that case. It is never needed, because all CM witnesses are co-oriented.

3. Sixth-power amplification [3, Lemma lem:inverse-amplification]. This is unchanged. The identity $\chi_n(ua^6) = \chi_n(u)$ for $(a, n) = 1$ does not involve α . Its input is (14.1) without slots; for $\epsilon = 1$ and trivial ν , the mean square (4.5) of Part I of these notes is the special case of row range at most $D^{11/10}$.

4. The plain fourth moment with prime factors [3, Lemma lem:plain]. The CM statement needed is the following. Fix $e = \pm 1$ and $3/4 \leq \kappa \leq 1$. For rows k with $0 < N(k) \ll Z^m$, write $\psi_k = \tau\chi_\bullet(k)$ for the finite-order row character, and put

$$S_k(n_j) = Z^{-n_j/2} \sum_l \alpha(l)^e \psi_k(l) W_j\left(\frac{N(l)}{Z^{n_j}}\right) \quad (j = 1, 2).$$

Let Q be a product of slot polynomials in ψ_k with no angular factor, of total length z , whose coefficients are fixed combinations of characters in Θ_0 . Let $\mathcal{R}_z$ be the rows of [3], defined through the finite-order part ψ_k . Then

$$\sum_{k \in \mathcal{R}_z} |S_k(n_1)S_k(n_2)Q|^2 \ll Z^{M+\epsilon}, \quad M = m + q,$$

with the same hypotheses and uniformity as in [3]: no restriction when $z = 0$; $n_1 + n_2 + 6\kappa z \leq M$ and slots of length at most the mesh when $z > 0$; uniformity over moving moduli, masks and frozen labels. Here q is OpenAI’s bound for the moving conductor support. The zero-free hypothesis is needed only for the finite-order slot characters, so $\kappa = 3/4$ is allowed. The changes to the proof that we can identify are four.

- (a) *Hecke reflection.* This uses Lemma 1. The dual profile is $\mathcal{M}W^\sharp(s) = \mathcal{M}W(1-s)\Gamma(s + \frac{1}{2})/\Gamma(\frac{3}{2}-s)$, in place of $\mathcal{M}W(1-s)\Gamma(s)/\Gamma(1-s)$. Both quotients grow like $|t|^{2\sigma-1}$ on vertical lines, and the conductor scale changes only at the prime above 3. So the reflected lengths and seminorm bounds are unchanged.
- (b) *Poisson summation in the row.* The two finite Poisson transforms act on the row variable k with radial weights, so $\alpha(l)$ is an inert coefficient.

- (c) *Main terms.* The induction of [3] meets volume-sized main terms on rows whose character lies in Θ_0 , and cancels them with an equal-scale centered difference. In the CM case the corresponding lattice sums are $\sum_z \alpha(z)^e f(Nz)$ over a fixed lattice. By Poisson summation their main term is $\int_{\mathbb{C}} \alpha(z)^e f(|z|^2) dz = 0$. This removes the volume main term. It does not by itself give the uniform error bounds or the induction.
- (d) *The zero-free hypothesis.* The slot amplitudes have finite order, so the hypothesis is OpenAI's theorem, with $\kappa = 3/4$.

These changes are what we expect. They are not a proof of the CM statement. That requires redoing the induction of [3] with the coefficient $\alpha(l_1 l_2)^e \tau_1(\prod l_1 l_2)$ carried through every transform, and with the angular factor kept off the slot primes. This is item 5 of Section 16.

15 The final inequality

A bin of rows of norm $U = Z^d$ contributes at most $Z^{C(7/8)+E(d)+o(1)}$, with the exponent $E(d)$ of [3, eq. common-high-exponent] [5, §6]. Its terms come from four sources:

- the scales and the Mellin weight on the central contour;
- the number of bad rows, given by the moments above with the same exponents;
- the prime amplitudes, which have finite order;
- the size $U^{\delta/2}$ of the finite-order numerator on the reflected line.

None of these involves the gamma factor of a CM L -function. That factor affects only fixed powers of the height.

So the endpoint certificate of [3, Lemma lem:balanced-endpoint],

$$-E(h) \geq \frac{49}{440640} \quad (0 \leq \delta \leq 5/6, 0 \leq x \leq 1/2),$$

with $q = x\delta$ and $R = R_*(\delta, x)$, applies verbatim. [3] compares it with $\Delta = \beta_* - 7/8$ in [3, eq. adaptive-endpoint]:

$$E_{\text{actual}}(h) - \Delta \leq -\frac{49}{440640} - \left(1 - \frac{h}{4}\right) \Delta + \epsilon_{\text{total}} < 0.$$

The term $h\Delta/4$ is a capacity loss caused by the choice $\kappa = 2\beta_* - 1$. With $\kappa = 3/4$ it disappears, which only helps. The bin ceiling $\delta \leq 5/6$ comes from the 11/12 stage. The balanced probe of Section 12, at $X = Y = Z^{1/2}$ and with the sextic row count, gives $\beta_* \leq 11/12$ for $\mathcal{F}_{\text{CM}}$. This is Part I of [3] with the changes above; it uses neither prime slots nor the plain moment. It is a statement about the whole family, so it is not supplied by Part I of these notes, which treats one curve at a time. Every margin is chosen before the target, as in [3], so all margins are uniform in ψ . The continuation criterion then gives the contradiction. Hence $\beta_* \leq 7/8$, which is Theorem 1.

16 What has been checked, and what remains

We read [3] in eight consecutive parts. For each part we listed every use of a property of the target or of a row character, and classified it as unchanged, routine, or needing a new argument. None needed a new argument. The table summarizes.

[3] lines	content	CM change
1–1648	criterion, Gauss sums, growth lemmas	growth with $\Gamma(s + \frac{1}{2})$ (Lemma 1); OpenAI’s theorem for finite-order rows
1648–3313	completed reflection, row energy	$\partial_{\bar{z}}^2 \theta$; table in Section 12
3313–4687	probe, Poisson rows, detector	weight (12.1); local identity formal in $\psi(p)$; label set $\mathcal{X}_u$
4687–6807	sextic count, high side, 11/12	none beyond the above; principal-denominator cases disappear
6807–9209	compensated probe, low side	normalization $\overline{\psi(p)}$; slots unchanged
9209–12477	inverse moment	co-oriented classes $\mu\bar{\alpha} \leftrightarrow \bar{\alpha}^2\gamma_2$ (and conjugates)
12477–14985	plain fourth moment	reflection with $\Gamma(s + \frac{1}{2})$; lattice main terms vanish
14985–16677	row counts, endpoint	none; $\kappa = 3/4$

Six statements must still be written out in full, at the generality of [3]:

1. the twisted transformation formula for $\partial_{\bar{z}}^2 \theta$ (Appendix A), for a general modulus supported on S , general log-Schwartz tests, and the cross-prime phase bookkeeping of [3, L2003–2279]; together with its marked form ([3, Cor. cor:marked-reflection]) and the lemmas of [3] built on it for the weight $\bar{\alpha}^2$: lem:hybrid-energy, lem:reflected-energy and lem:probe-row-norm;
2. Lemma 1 together with the CM form of the logarithmic control;
3. the detector with the label set $\mathcal{X}_u$;
4. the inverse moment (14.1) for co-oriented coefficients;
5. the plain fourth moment with α^e on the plain variables;
6. the 11/12 stage of [3] (its Part I) for $\mathcal{F}_{\text{CM}}$, giving $\beta_* \leq 11/12$.

Each is obtained from the corresponding statement of [3] by the changes recorded above. We checked the new identities numerically:

- the point counts against ψ_d^* ;
- (12.2) and its companion;
- the vanishing of $\int \alpha^e f$;
- the Bessel–Mellin formulas;
- the reflection constants;
- the Stirling exponents of all the gamma quotients.

Until the six statements are written out and refereed, Theorem 1 should be read as a detailed plan of proof, not as an established result.

17 CM newforms of higher weight: why the method should extend

Write $\psi_k = \bar{\alpha}^k \eta$ with $k \geq 1$ and η of finite order. These are the unitary Hecke characters of K with infinity type $(\bar{z}/|z|)^k$. Up to finitely many Euler factors, their L -functions are those of the CM newforms of weight $k+1$ with complex multiplication by K , shifted to the center $1/2$: if f is such a form, then $L(f, s + \frac{k}{2}) = L(s, \psi)$ for $\psi = \psi_k$ or its conjugate. This section explains why we expect the following.

Expected statement. For every $k \geq 1$ and every finite-order η , $L(s, \bar{\alpha}^k \eta) \neq 0$ for $\Re s > 7/8$. Equivalently, every CM newform f of weight $k+1$ with CM by K satisfies

$$L(f, s) \neq 0 \quad \left(\Re s > \frac{k}{2} + \frac{7}{8} \right),$$

a zero-free strip along the right edge of the critical strip $\frac{k}{2} < \Re s < \frac{k}{2} + 1$. The 11/12 method of Part I should give the weaker boundary $\frac{k}{2} + \frac{11}{12}$ in the same way.

We do not claim this as a theorem. The rest of this section records the changes in the number $j = k+1$ of theta derivatives and in the Hecke Gamma factor. For each fixed j , the formulas have the same structure and give the same length exponents; constants may depend on j . The uniform analytic estimates listed at the end still have to be proved for each fixed weight.

17.1 Two examples

We computed the q -expansions of two eta products to q^{3000} and compared them with Hecke characters. In the exact L -function identities below, a product of Hecke characters means its primitive inducing character, with all its Euler factors restored. Write $L^{(6)}$ when the Euler factors at primes above 2 and 3 are deleted, as in the earlier sums.

- $f = \eta(3z)^8$, of weight 4 and level 9 ($k = 3$). For all 207 split rational primes $p = \pi\bar{\pi} < 3000$, with both primary generators $\pi, \bar{\pi}$ tested (414 checks), $a_p = 2\Re(\pi^3)$. For $p \equiv 2 \pmod{3}$, $a_p = 0$ and $a_{p^2} = -p^3$. Thus $L(f, s + \frac{3}{2}) = L(s, \alpha^3)$, and the target is the conjugate $\bar{\alpha}^3$, with η trivial. This is the CM analogue of the case $u = 1$ of the zeta argument. The expected strip is $19/8 < \Re s < 5/2$ (center 2).
- $f = \eta(2z)^3 \eta(6z)^3$, of weight 3 and level 12 ($k = 2$). For all 207 split rational primes below 3000, again testing both primary generators (414 checks), $a_p = 2\Re(\pi^2(2/\pi)_3) = 2\Re(\pi^2 \chi_\pi(4))$. For $p \nmid 12$ with $p \equiv 2 \pmod{3}$, $a_p = 0$ and $a_{p^2} = p^2$, as required by the nebentypus χ_{-3} . The exclusion matters: $a_4 = 0$, not 4. We have $L(f, s + 1) = L(s, \alpha^2 \chi_\bullet(4))$, and the target is $\bar{\alpha}^2 \overline{\chi_\bullet(4)}$. The expected strip is $15/8 < \Re s < 2$ (center $3/2$). Since $a_3 = -3$, the normalized Euler factor at 3 is $(1 + 3^{-s})^{-1}$; the factor at 2 is 1. Thus, in the convention of the earlier sums,

$$L(f, s + 1) = (1 + 3^{-s})^{-1} L^{(6)}(s, \alpha^2 \chi_\bullet(4)).$$

17.2 What changes with k

1. The weight and the rows. With target $\psi = \bar{\alpha}^k \eta$ the weight (12.1) becomes

$$\gamma_2(c) \overline{\alpha(cn^3)} \psi(cn^3) = \gamma_2(c) \overline{\alpha(cn^3)}^{k+1} \eta(cn^3).$$

Proposition 2 is formal in $\psi(p)$, so the rows are again $\zeta_K^S(6z) L^S(w, \chi_\bullet(u)) / L^S(x, \psi \overline{\chi_\bullet(u)})$. The numerators still have finite order, the denominators have infinity type $(\bar{z}/|z|)^k$, and the slot amplitudes are unchanged.

2. The theta step with $j = k + 1$ derivatives. Every formula of Section 8 and of the table in Section 12 has a version for all $j \geq 1$, as follows.

- *Chain rule.* At $z = 0$ the coordinate z' is affine in $\bar{z}$, while $\bar{z}'$ and v' do not depend on $\bar{z}$. Hence, exactly,

$$\partial_{\bar{z}}^j F(z', \bar{z}', v') \Big|_{z=\bar{z}=0} = (-cv)^{-2j} (\partial_{z'}^j F)(-\delta/c, 1/(N(c)v)).$$

Every derivative of order $j \geq 1$ kills the constant term at the identity cusp. We checked this identity symbolically for $j \leq 4$.

- *Direct Mellin transform.* For a nonzero Fourier mode and $\Re s > \frac{1}{6} - \frac{j}{2}$,

$$\int_0^\infty (2\pi i \bar{\ell})^j v K_{1/3}(4\pi |\ell| v) v^{2s+j-2} dv = \frac{i^j}{4} (2\pi)^{-2s} \Gamma\left(s + \frac{j}{2} - \frac{1}{6}\right) \Gamma\left(s + \frac{j}{2} + \frac{1}{6}\right) \overline{\alpha(\ell)}^j N(\ell)^{-s}.$$

The integral converges absolutely in this half-plane; elsewhere the right side gives its meromorphic continuation. For $j = 1, 2$ this gives the two entries of the table. We checked it numerically for $j \leq 5$ (relative error below 10^{-25}).

- *Reflected side.* The same substitution gives $\prod_{\pm} \Gamma(\frac{j+2}{2} - s \pm \frac{1}{6})$, the cusp factor $\overline{\alpha(c)}^{2j}$, and the dual angular factor $\alpha(\mu)^j$.
- *Kernel.* Write

$$R_j(t) = \prod_{\pm} \frac{\Gamma(\frac{j+1}{2} + t \pm \frac{1}{6})}{\Gamma(\frac{j+1}{2} - t \pm \frac{1}{6})}.$$

Then R_1 and R_2 are the two kernels of the table. R_j is holomorphic for $\Re t > \frac{1}{6} - \frac{j+1}{2}$, which is to the left of $-1/4$ for every $j \geq 1$, and $|R_j(\sigma + iu)| \ll_{j,A} (1 + |u|)^{4\sigma}$ on $-1/4 \leq \sigma \leq A$ (checked numerically for $j \leq 5$). So the reflected weight obeys the same bound $\ll \min(x^{1/4}, x^{-A})$, with constants depending on j . There is no uniform bound as j grows: for example,

$$R_j(1) = \frac{(3j-4)(3j-2)(3j+2)(3j+4)}{1296} \sim \frac{j^4}{16}.$$

The dual lengths are unchanged, because the reflected argument $N(\mu)X/N(c)^2$ does not depend on j . In the quadratic large sieve, $\alpha(\mu)^j$ is a unimodular coefficient of the dual index. Hence Proposition 3 and the low estimates $Z^{1/4+\varepsilon}$ and $Z^{3/16+\varepsilon}$ should hold for every k .

3. Growth. The functional equation has $\Gamma(s + \frac{k}{2})$ in place of $\Gamma(s + \frac{1}{2})$. The exponent of $|\Gamma(1 - s + \frac{k}{2})/\Gamma(s + \frac{k}{2})|$ on $\Re s = \sigma$ is $1 - 2\sigma$, independent of k , so Lemma 1 keeps its exponents. The theta series now has the harmonic polynomial $\bar{a}^k$. Its Poisson transform is a multiple of $\bar{\xi}^k e^{-c'|\xi|^2/v} v^{-1-k}$, which vanishes at $\xi = 0$, so $L(s, \psi)$ is entire.

4. Moments. In the inverse moment the co-oriented classes become $\mu \bar{\alpha}^k \leftrightarrow \bar{\alpha}^{k+1} \gamma_2$. This follows from the same identity $\mu \gamma_{-1} = \chi(-1) G^{-1} \bar{\alpha} \gamma_2$, and the cube factor becomes $\bar{\alpha}^{3(k+1)}$. The theta steps use $\partial_{\bar{z}}^{k+1} \theta$. In the plain moment, Hecke reflection uses $\Gamma(s + \frac{k}{2})/\Gamma(1 - s + \frac{k}{2})$, a Hankel kernel of order k with the same vertical growth. The lattice main terms vanish because $\int \alpha(z)^{\pm k} f(|z|^2) dz = 0$.

5. Unchanged. The following carry over verbatim:

- the label sets, with $\bar{\alpha}^k \Theta_{0\chi_\bullet}(\overline{u})$ and $\alpha^k \Theta_{0\chi_\bullet}(u)$;
- the slot normalization $\overline{\psi(p)}$;
- $\kappa = 3/4$ from OpenAI's theorem;
- every exponent in the final inequality, and hence the endpoint certificate.

For Part I the targets are rows of the same kind. In the orientation $\mu_K(n)\alpha(n)^k\chi_n(u_0)$ of Part I, the row is $u_0 = 1$ for $\eta(3z)^8$ and $u_0 = 4$ for $\eta(2z)^3\eta(6z)^3$.

17.3 Remarks

- *k = 0 is OpenAI's case.* There $j = 1$ and the weight is $\gamma_2\bar{\alpha}\eta$. The recipe “target $\bar{\alpha}^k\eta$, $k + 1$ theta derivatives” therefore covers unitary Hecke characters of K with purely angular infinity type; negative k reduces to positive k by conjugation. A general unitary Hecke character may also have a continuous norm factor N^{it} , with $t \in \mathbb{R}$. Since

$$L(s, \psi N^{it}) = L(s - it, \psi),$$

these vertical shifts leave the proposed zero-free half-plane unchanged.

- *Orientation.* The conjugate target $\alpha^k\bar{\eta}$ would leave α^{k-1} in the weight. For $k = 1$ that is no angular factor at all: the theta function without a derivative, whose constant terms are not controlled. That is why Part II uses $\bar{\alpha}\eta$. For $k \geq 2$ the factor α^{k-1} is produced by ∂_z^{k-1} . The chain rule is symmetric: at $z = \bar{z} = 0$, $\partial_{\bar{z}}^j$ becomes $(-(\bar{c}v)^{-2})^j \partial_{\bar{z}'}^j$, exactly; we checked this symbolically for $j \leq 3$. So for $k \geq 2$ either orientation should work; we use $\bar{\alpha}^k\eta$ for uniformity.
- *Symmetric powers.* For a CM curve E_d and $m \geq 1$, put $L_{\text{an}}(\text{Sym}^m E_d, s) = L(\text{Sym}^m E_d, s + m/2)$. Up to finitely many Euler factors, the factorization is

$$L_{\text{an}}(\text{Sym}^m E_d, s) = \prod_{0 \leq i < m/2} L(s, (\psi_d^*)^{m-2i}) \begin{cases} 1, & m \text{ odd,} \\ L(s, \chi_{-3}^{m/2}), & m \text{ even.} \end{cases}$$

The last factor is $\zeta(s)$ when $m \equiv 0 \pmod{4}$ and $L(s, \chi_{-3})$ when $m \equiv 2 \pmod{4}$, with the trivial character interpreted primitively. This is the CM factorization of [7, §3.5]. The positive-power factors are of the type above; the remaining factor is covered by OpenAI's finite-order result. Together these would give a zero-free half-plane $\Re s > 7/8$ in analytic normalization. When $4 \mid m$, the function has a pole at $s = 1$: “zero-free” here is a statement about a meromorphic function, not a claim of holomorphy throughout the half-plane. These L -functions are not primitive for $m \geq 2$.

- *What would still have to be written.* The six restatements of Section 16 are needed for each k , with $\partial_{\bar{z}}^{k+1}$ in place of $\partial_{\bar{z}}^2$. We see no step where k enters other than through $j = k + 1$ and the Gamma factors. The constants depend on k , but k is fixed.

The numerical checks of this section were made with short Python scripts.

A The reflected series and explicit theta coefficients

This appendix supplies the coefficient data for readers who want every series in the theta step in finite-sum terms. It can be skipped on a first reading. The imported theta symmetry and coefficient tables are those of Kubota and Patterson recorded in [8, Section 5.1 and Appendix], in the conventions of [3]. Only these theta facts are used from [8]; its GRH-conditional prime Gauss-sum asymptotic is not an input here.

For $\Psi(n) = \xi_0(n) \prod_{\pi \in \mathcal{P}} \chi_\pi(n)^{j_\pi}$, the transformation expresses $T(X; \Psi)$ as a finite sum of terms

$$C \sum_{0 \neq \ell \in \lambda^{-4} \mathbb{Z}[\omega]} \frac{d_\sigma(\ell) \alpha(\ell)^2}{\sqrt{N(\ell)}} \psi(\lambda^4 \ell) \prod_{\pi \in \mathcal{A}} B_{\pi, j_\pi}(\lambda^4 \ell) V^\sharp \left(\frac{N(\ell) X}{N(c)^2} \right). \quad (\text{A.1})$$

Here $\sigma \in \{0, +, -\}$, $\{\pi : j_\pi \neq 0\} \subseteq \mathcal{A} \subseteq \mathcal{P}$, $c = c_0 \prod_{\pi \in \mathcal{A}} \pi$, $c_0 \mid 8\lambda^2$ up to a unit, and $\psi(x) = e(\beta x / (\lambda^3 c_0))$ for an Eisenstein integer β . There are at most $5184 \cdot 2^{|\mathcal{P}|}$ terms, each with $|C| \leq 1/81$. The constants are obtained by finite additive Fourier expansion of Ψ and the theta transformation, as in the completed reflection of [3, Prop. prop:completed-reflection] with two derivatives in place of one. The expression is an absolutely convergent series. The weights $V^\sharp$ and the local factors $B_{\pi, j}$ were explicitly defined in (8.4) and (8.5).

For the remaining coefficients, define the finite cubic Gauss sum

$$g(r, c) = \sum_{x \bmod c} \left(\frac{x}{c} \right)_3 \check{e}(rx/c).$$

In this appendix, c, b are primary, c is squarefree, and they may be even. The cubic symbol at the even primary prime -2 is defined by $(x/(-2))_3 \equiv x \pmod{2}$ for x prime to 2, and zero otherwise; at other primary primes use the exponent $(N(\pi) - 1)/3$. Put $\zeta_9 = e^{2\pi i/9}$. Define $\tau(\ell)$ by the following table and set it equal to zero at every other index:

ℓ	$\tau(\ell)$
$\pm \lambda^{3\nu-4} c b^3, \nu \geq 1$	$3^{\nu/2+2} g(\lambda^2, c) b/c $
$\pm \omega \lambda^{3\nu-4} c b^3, \nu \geq 1$	$3^{\nu/2+2} \zeta_9^{-1} \overline{g(\omega \lambda^2, c)} b/c $
$\pm \omega^2 \lambda^{3\nu-4} c b^3, \nu \geq 1$	$3^{\nu/2+2} \zeta_9 \overline{g(\omega^2 \lambda^2, c)} b/c $
$\pm \lambda^{3\nu-3} c b^3, \nu \geq 0$	$3^{\nu/2+5/2} g(1, c) b/c $

For the two other cusps, define τ_1, τ_2 by

j	$\tau_1(\omega^j \lambda^{-4} c b^3)$	$\tau_2(-\omega^j \lambda^{-4} c b^3)$
0	$9 \omega g(\lambda^2, c) b/c $	$9 \omega^2 g(\lambda^2, c) b/c $
1	$9 \zeta_9^{-1} \omega^2 \overline{g(\omega \lambda^2, c)} b/c $	$9 \zeta_9^{-1} \overline{g(\omega \lambda^2, c)} b/c $
2	$9 \zeta_9 \overline{g(\omega^2 \lambda^2, c)} b/c $	$9 \omega \zeta_9 \overline{g(\omega^2 \lambda^2, c)} b/c $

and zero elsewhere. Finally set

$$t_0(\ell) = \tau(\ell), \quad t_+(\ell) = \omega \tau_2(\omega \ell) \check{e}(\ell), \quad t_-(\ell) = \omega^2 \tau_1(\omega^2 \ell) \check{e}(\ell), \quad d_\sigma(\ell) = \overline{t_\sigma(-\ell)}.$$

These are precisely the coefficients of the conjugated theta functions at the three cusps. Their support is contained in the set of indices $\ell = u \lambda^m n b^3$, with u a unit, $m \geq -4$, n squarefree, and n, b primary (the tables above impose further restrictions), and

$$|d_\sigma(u \lambda^m n b^3)| \leq 27 \cdot 3^{m/6} |b|.$$

In the varying quadratic symbol in (A.1), the factor b^3 contributes $(b/k)_2$, since $(b^3/k)_2 = (b/k)_2$. Fixing u, m, b leaves squarefree columns n , exactly as required for (9.1). This is why the explicit support of the theta coefficients matters to the averaging argument.

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