1. Linear algebra preliminaries

1.1. **Some facts about bilinear forms.**

Notation. Let V be an m-dimensional vector space over $k = \mathbb{R}$ or $\mathbb{C}$. We denote V^* the *dual space of* V consisting of linear functions $\alpha : V \to k$. Furthermore, we denote $W \leq V$ any linear subspace W of V, and

$$W^\circ := \{\alpha \in V^* : \alpha(w) = 0, \text{ for all } w \in W\}$$

the *annihilator of* W in V^*. Note that

$$\dim W + \dim W^\circ = \dim V.$$

Let $\psi : V \times V \to k$ be a bilinear form. The *rank of* ψ is then defined as the rank of any matrix representation of ψ.

Definitions. The bilinear form ψ is said to be *non-degenerate* if and only if the following holds:

$$\psi(v, v') = 0, \forall v' \in V \Longleftrightarrow v = 0.$$

Moreover, the bilinear form ψ is said to be *diagonalisable* if there exists a basis of V with respect to which ψ is represented by a diagonal matrix.

Note.

(i) ψ is non-degenerate if and only if all its matrix representations re non-singular.

(ii) If ψ is symmetric, then it is diagonalisable. If $k = \mathbb{R}$ and ψ is symmetric *and* non-degenerate, then it is either positive-definite, negative-definite, or indefinite, in which case it is said to be of *signature* (r, s) where

$$r = \# \text{ of positive eigenvalues}$$

$$s = \# \text{ of negative eigenvalues}$$

(with $r + s = m$). In addition, if V^+ and V^- are the positive and negative eigenspaces of ψ, respectively, then

$$V = V^+ \oplus V^-$$

with $\psi|_{V^+}$ positive definite and $\psi|_{V^-}$ negative definite, where $\psi|_W$ denotes the restriction of ψ to $W \times W$.

(iii) If $k = \mathbb{R}$ and ψ is positive definite, then $\psi|_W$ is positive definite for all $W \leq V$. However, in general, if ψ is an arbitrary non-degenerate bilinear form, there may exist a subspace $W \leq V$ such that $\psi|_W$ is degenerate.

E.g. If $V = \mathbb{R}^2$ and ψ is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then ψ is non-degenerate. But $\psi|_W \equiv 0$, where $W = \langle (1, 1) \rangle$.

Definition. Given a bilinear form $\psi : V \times V \to k$, we define

$$\begin{aligned} \widetilde{\psi} : V &\to V^* \\ v &\mapsto \psi(v, -). \end{aligned}$$

Then, $\widetilde{\psi}$ is a linear map and ψ is non-degenerate if and only if $\widetilde{\psi}$ is an isomorphism.

From now on, we will assume that ψ is either symmetric or skew-symmetric.

Definition. L et $W \leq V$. We define

$$W^\perp := \{v \in V : \psi(v, w) = 0, \text{ for all } w \in W\}$$

to be the *orthogonal complement of* W.

Note. (i) If $U \leq W$, then $W^\perp \leq U^\perp$.

(ii) If $k = \mathbb{R}$ and ψ is positive definite, then $W \cap W^\perp = \{0\}$ and $V = W \oplus W^\perp$. But this may not be true if ψ is not positive definite.

E.g. If $V = \mathbb{R}^2$, ψ is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $W = \langle (1, 1) \rangle$, then $W^\perp = W$ so that $W \cap W^\perp \neq \{0\}$.

(iii) $W \cap W^\perp = \{0\}$ if and only if $\psi|_W$ is non-degenerate. In particular, ψ is non-degenerate if and only if $V^\perp = \{0\}$ (exercise).

Proposition. If $W \leq V$, then

$$\dim W + \dim W^\perp = \dim V + \dim W \cap V^\perp.$$

In particular, if ψ is non-degenerate, then

$$\dim W + \dim W^\perp = \dim V.$$

Proof. Consider the linear map

$$\begin{aligned} \varphi = \widetilde{\psi}|_W : W &\to V^* \\ w &\mapsto \psi(w, -). \end{aligned}$$

Then, by rank-nullity,

$$\dim W = \dim \ker \varphi + \dim \operatorname{Im} \varphi.$$

But,

$$\ker \varphi = \{w \in W : \psi(w, v) = 0, \text{ for all } v \in V\}.$$

And since $\psi(w, v) = \pm\psi(v, w)$ by (skew)-symmetry of ψ, we have

$$\ker \varphi = \{w \in W : \psi(v, w) = 0, \text{ for all } v \in V\} = W \cap V^\perp.$$

Furthermore, $(\operatorname{Im} \varphi)^\circ \subset (V^*)^* = V$ is given by

$$(\operatorname{Im} \varphi)^\circ = \{v \in V : \alpha(v) = 0 \text{ for all } \alpha \in \operatorname{Im} \varphi\}.$$

Since every $\alpha \in \operatorname{Im} \varphi$ is of the form $\psi(w, -)$ for some $w \in W$, this means that

$$(\operatorname{Im} \varphi)^\circ = \{v \in V : \psi(w, v) = 0 \text{ for all } w \in W\} = W^\perp.$$

Hence,

$$\dim \operatorname{Im} \varphi = \dim V - \dim(\operatorname{Im} \varphi)^\circ = \dim V - \dim W^\perp,$$

implying that

$$\dim W = \dim W \cap V^\perp + (\dim V - \dim W^\perp).$$

□

Corollary. Suppose that ψ is non-degenerate and consider $W \leq V$. Then, $\psi|_W$ is non-degenerate if and only if $V = W \oplus W^\perp$.

1.2. **Skew-symmetric bilinear forms.**

Proposition. Let ψ be a skew-symmetric bilinear form. Then, there exists a basis with respect to which ψ is given by the matrix

$$\begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}. \tag{1}$$

In particular, the rank of ψ is even.

Proof. The proof is by induction on the dimension of V. If $\psi \equiv 0$, we are done. Otherwise, there exist $v_1, v_2 \in V$ with $\phi(v_1, v_2) \neq 0$ (or else we would have $\psi(v_1, v_2) = 0$ for all $v_1, v_2 \in V$, implying that $\psi \equiv 0$). After possibly normalising the vectors, we can assume that $\psi(v_1, v_2) = 1$. Let $W = \langle v_1, v_2 \rangle \leq V$ be the subspace spanned by v_1 and v_2. Note that v_1 and v_2 are linearly independent because if $v_2 = cv_1$ for some $c \in k$, then

$$\psi(v_1, v_2) = \psi(v_1, cv_1) = c\psi(v_1, v_1) = c \cdot 0 = 0$$

by skew-symmetry of ψ. Therefore, $\dim W = 2$ and $\psi|_W$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore, $\psi|_W$ is non-degenerate and $V = W \oplus W^\perp$. However, by the induction hypothesis, there exists a basis $\{v_3, \dots, v_m\}$ of $W^\perp$ with respect to which $\psi|_{W^\perp}$ is represented by a matrix of the form (1). Hence, ψ is represented by a matrix of the form (1) with respect to the basis $\{v_1, v_2.v_3, \dots, v_m\}$. □

Corollary. If ψ is non-degenerate and skew-symmetric, then the rank of ψ is even, implying that the dimension of V is even.

Note. The proof of the proposition tells us that, if ψ is skew-symmetric, there exists a basis $\{e_1, \dots, e_n, f_1, \dots, f_n, h_1, \dots, h_s\}$ of V, with $m = 2n + s$ and $2n$ the rank of ψ, such that

$$\psi(e_i, e_j) = \psi(f_i, f_j) = \psi(e_i, h_l) = \psi(f_j, h_l) = 0, \ \forall i, j, l$$

$$\psi(e_i, f_i) = 1, \ \forall i$$

$$\psi(e_i, f_j) = 0, \ \forall i \neq j.$$

(Indeed, just take $e_1 = v_1, f_1 = v_2, e_2 = v_3, f_2 = v_4$, etc....)

Definition. A bilinear form $\psi : V \times V \to k$ on a vector space V is called *symplectic* if it is skew-symmetric and non-degenerate. Furthermore, the pair (V, ψ) is called a *symplectic vector space* if ψ is a symplectic form on V.

Corollary. If (V, ψ) is a symplectic vector space, then the dimension of V is even.

Note. (i) If (V, ψ) is a symplectic vector space, there exists a basis

$$\{e_1, \dots, e_n, f_1, \dots, f_n\}$$

of V, with $m = 2n$, such that

$$\psi(e_i, e_j) = \psi(f_i, f_j),\ \forall i, j, l$$
$$\psi(e_i, f_i) = 1,\ \forall i$$
$$\psi(e_i, f_j) = 0,\ \forall i \neq j.$$

Such a basis is called a *symplectic basis* of (V, ψ).

(ii) A bilinear form $\psi : V \times V \to k$ is called *alternating* if $\psi(v, v) = 0$ for all $v \in V$. If the characteristic of k is not 2, one can show that ψ is alternating if and only if ψ is skew-symmetric (exercise); thus, if ψ is a symplectic bilinear form on V and $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is a symplectic basis of (V, ψ),

$$\psi = \sum_{i=1}^{n} e_i^* \wedge f_i^*.$$

Definition. Let (V, ψ) be a symplectic vector space and $W \leq V$. Then, W is called:

(i) *symplectic* if $W \cap W^\perp = \{0\}$;
(ii) *isotropic* if $W \subset W^\perp$;
(iii) *co-isotropic* if $W^\perp \subset W$;
(iv) *Lagrangian* if $W = W^\perp$.

Note. Lagrangian subspaces are both isotropic and co-isotropic.

Examples. Let $V = \mathbb{R}^2$, ψ be the bilinear form given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $W \leq V$. Then,

- $W = V$ is symplectic.
- *All* proper subspaces W of V are Lagrangian.
  (Indeed, if $W = \langle (u_1.u_2) \rangle$, then for all $(a, b) \in \mathbb{R}^2$,

$$\psi((a, b), (u_1, u_2)) = (a, b) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \Leftrightarrow au_2 - bu_1 = 0$$
$$\Leftrightarrow au_2 = bu_1 \Leftrightarrow (a, b) = c(u_1, u_2), c \in \mathbb{R}.$$

  Thus, $W^\perp = W$.)
- $W = \{0\}$ is vacuously symplectic.

Remark. Suppose that (V, ψ) is a symplectic vector space and $W \leq V$. Then, ψ is non-degenerate, implying that

$$\dim W + \dim W^\perp = \dim V = 2n.$$

Therefore,

(i) If W is symplectic, then $V = W \oplus W^\perp$;
(ii) If W is isotropic, then $\dim W \leq n$;
(iii) If W is co-isotropic, then $\dim W \geq n$;
(iv) If W is Lagrangian, then $\dim W = n$.

(Exercise.)