PMATH 764: Assignment 5

Due: Friday, 15 July, 2013.

1. Let $C = V(f)$ be an affine plane curve that is smooth at the point $p \in C$, so that $\mathcal{O}_p(C)$ is a DVR.

   (a) Prove that the order function $\mathrm{ord}_p^C : \mathcal{O}_p(C) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ does not depend on the choice of local parameter of $\mathcal{O}_p(C)$.

   (b) Consider the extension of the order function $\mathrm{ord}_p^C : \mathcal{O}_p(C) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ to the function field $k(C)$, which is defined as

   $$\begin{aligned} \mathrm{ord}_p^C : k(C) &\to \mathbb{Z} \cup \{\infty\} \\ f = \bar{a}/\bar{b} &\mapsto \mathrm{ord}_p^C(\bar{a}) - \mathrm{ord}_p^C(\bar{b}). \end{aligned}$$

   Prove the following:

   (i) ord_p^C is a well-defined, i.e., $\mathrm{ord}_p^C(f)$ does not depend on the presentation $\bar{a}/\bar{b}$ of f.

   (ii) $\mathrm{ord}_p^C(f) = 0$ if and only if f is a unit in $\mathcal{O}_p(C)$.

   (iii) $\mathrm{ord}_p^C(f) = \infty$ if and only if f identically zero on C.

   (iv) $\mathrm{ord}_p^C(f_1 f_2) = \mathrm{ord}_p^C(f_1) + \mathrm{ord}_p^C(f_2)$.

   (v) $\mathrm{ord}_p^C(f_1 + f_2) \geq \min\{\mathrm{ord}_p^C(f_1), \mathrm{ord}_p^C(f_2)\}$.

   (c) Verify that $\mathcal{O}_p(C) = \{f \in k(C) \mid \mathrm{ord}_p^C(f) \geq 0\}$ and $M_p(C) = \{f \in k(C) \mid \mathrm{ord}_p^C(f) > 0\}$.

2. *Intersection multiplicity.* Let C and D be an affine plane curves and let $p \in C \cap D$. Suppose that C is smooth at p. If D is given by the polynomial $g \in k[x, y]$, we define the *intersection multiplicity of C and D at p* to be the integer

   $$I(p, C \cap D) := \mathrm{ord}_p^C(\bar{g}). \qquad (*)$$

   Note that, in the definition, only one of the curves needs to be smooth at p.

   (a) Let L be the line through $p = (0, 0)$ in $\mathbb{A}^2$ given by the parametric equation $L := \{v_1 t, v_2 t) \mid t \in k\}$ and let $C = V(f)$ be an affine plane curve that contains p. Recall that we defined the *intersection multiplicity of the line L with C at p* to be the integer m_0 such that

   $$f(tv_1, tv_2) = t^{m_0} q(t), \ q(0) \neq 0.$$

   Prove that $m_0 = \mathrm{ord}_p^L(\bar{f})$. Since L is smooth at p, $\mathrm{ord}_p^L(\bar{f})$ is equal to the general definition $(*)$ of intersection multiplicity, showing that both definitions agree.

   (b) Assume that $\mathrm{char}(k) = 0$. Find the intersection multiplicity $I(p, C \cap D)$ of $C = V(x + y^3)$ and $D = V(y(y^2 - x))$ at $p = (0, 0)$.

   (c) Assume that $\mathrm{char}(k) = 0$. Find the intersection multiplicity $I(p, C \cap D)$ of $C = V(x^2 - 1 - y^3)$ and $D = V(x^2 - 1 + 2y^4)$ at $p = (1, 0)$.

3. (a) Let C be an affine plane curve and p be a point on C. Show that if there exist two distinct lines L_1 and L_2 such that $I(p, C \cap L_i) \geq 2$ for $i = 1, 2$, then C is singular at p.

   (b) Let C be an affine plane curve, given by a degree d polynomial f. Prove that C cannot have a point p of multiplicity $m_p(C)$ greater than d. In particular, prove that if C has a point p of multiplicity d, then C consists of d lines through p (which are not necessarily distinct).

4. *Smooth variaties.* Let X be an r-dimensional variety in $\mathbb{A}^n$, and suppose the ideal of X is generated by $f_1, \ldots, f_s$. We define the *Jacobian matrix of* $f_1, \ldots, f_s$ *at* p to be

$$\mathrm{Jac}(f_1, \ldots, f_s)(p) := \left[\frac{\partial f_i}{\partial x_j}(p)\right] = \begin{bmatrix} \nabla f_1(p) \\ \vdots \\ \nabla f_s(p) \end{bmatrix},$$

and the *Zariski tangent space of* X *at* p to be

$$T_p(X) := \ker(\mathrm{Jac}(f_1, \ldots, f_s)(p)).$$

We then say that X is *smooth at* p if and only if $\mathrm{Jac}(f_1, \ldots, f_s)(p)$ has rank $(n-r)$ or, equivalently, if $\dim_k(T_p(X)) = r = \dim X$.

(a) Show that if $I(X) = < f_1, \ldots, f_s > = < g_1, \ldots, g_t >$, then the Jacobian matrices $\mathrm{Jac}(f_1, \ldots, f_s)(p)$ and $\mathrm{Jac}(g_1, \ldots, g_t)(p)$ have the same rank at all $p \in X$. The definition of smoothness is therefore independent of the choice of generators of $I(X)$.

(b) Determine whether or not $V(u^2 - x^3, y - xu) \subset \mathbb{A}^3$ is a smooth variety.

(c) Show that $\dim_k(T_p(X)) \geq r$, for all $p \in X$.

Note: This problem is a generalisation to higher dimensions of the notion of smooth affine plane curve.